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On the Closure of Quasi-λ-Koszul Modules[1]

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Abstract

In this paper, the notion of *quasi-λ-Koszul module* is introduced. For any given short exact sequence $\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$, the main aim of this paper is to find conditions for the third term of ξ to be quasi-λ-Koszul provided that any two terms of ξ are quasi-λ-Koszul.

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1 Introduction

Koszul algebras were originally introduced by Priddy in 1970 (see [17]), which admit a lot of applications in different branches of mathematics, such as algebraic geometry, Lie theory, quantum groups, algebraic topology and combinatorics, etc.. The rich structure and long history of Koszul algebras are clearly detailed in [16]. Since then, there existed a lot of creative and meaningful generalizations of Koszul algebras in the past 10 years (see [2]-[14]). For examples: motivated by the cubic Artin-Schelter regular algebras, Berger first introduced the notion of nonquadratic Koszul algebras in 2000 (see [2]); later, inspired by the quiver theory, Green et al generalized this class of algebras to the nonlocal case and first adopted the name "D-Koszul algebras" (see [3]), where $D \geq 2$ is an integer; as another extension of D-Koszul algebras, Lü first introduced the notion of λ-Koszul algebra in 2009 (see[8]), etc..

Note that Koszul objects, D-Koszul objects and piecewise-Koszul objects have been generalized to the nongraded case (see [4], [6], [9], [14] and [15]),

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now a natural question is whether we can extend λ-Koszul objects to the non-graded case? In this paper, we provide a positive answer to this question and introduce the notions of *quasi-λ-Koszul algebra* and *quasi-λ-Koszul module*. In particular, we mainly discuss the extension closure of the category of quasi-λ-Koszul modules and obtain

Main result *Let* $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be a short exact sequence of finitely generated R-modules, where R is a Noetherian semiperfect algebra and J the Jacobson radical of R. Then we have the following statements:*

1. *If we have $J^kK = J^kM \cap K$ for all $k \geq 0$, then N is a quasi-λ-Koszul module provided that K and M are quasi-λ-Koszul modules.*

2. *If we have $JK = JM \cap K$, then M is a quasi-λ-Koszul module provided that K and M are quasi-λ-Koszul modules.*

3. *If $J^j\Omega^i(K) = \Omega^i(K) \cap J^j\Omega^i(M)$ for all $j, i \geq 0$, then K is a quasi-λ-Koszul module provided that M and N are quasi-λ-Koszul modules.*

2 Preliminaries

In this section, we first recall the definitions of λ-Koszul algebras and modules and then give the notions of quasi-λ-Koszul algebras and quasi-λ-Koszul modules.

Definition 2.1. ([8]) Let A be a standard graded algebra (i.e., a positively graded $\mathbb{k}$-algebra $A = \bigoplus_{i\geq 0} A_i$ with (a) $A_0 = \mathbb{k} \times \cdots \times \mathbb{k}$, a finite product of $\mathbb{k}$; (b) $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$; and (c) $\dim_{\mathbb{k}} A_i < \infty$ for all $i \geq 0$) and M a finitely generated graded A-module. Let

$$\mathcal{Q} : \cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

be a minimal graded projective resolution of M. Then M is called a *λ-Koszul module* if each Q_n is generated in degree $\delta_\lambda(n)$ for all $n \geq 0$, where $\lambda : \mathbb{N}^* \to \mathbb{N}^*$ is a periodic function and $|\lambda|$ denotes the smallest positive period of λ with $\lambda(1) \geq 2$ and λ being strictly increasing on the interval $[1, |\lambda|]$; and $\delta_\lambda : \mathbb{N} \to \mathbb{N}$ is another set function satisfying

1. $\delta_\lambda(0) = 0,\ \delta_\lambda(1) = 1,\ \delta_\lambda(2) = d$, where $d = \lambda(1) + 1$, a fixed integer;

2. $\delta_\lambda(2n + 1) - \delta_\lambda(2n) = 1$ for all $n \geq 0$;

3. $\delta_\lambda(2n) - \delta_\lambda(2n - 1) = \lambda(n)$ for all $n \geq 1$.

In particular, the standard graded algebra A will be called a *λ-Koszul algebra* provided that A_0 is a trivial λ-Koszul A-module.

Let $\|A\| := |\lambda|$, the $|\lambda|$ of λ. We usually call $\|A\|$ the *number of jump-degree* of A.

The notions of quasi-λ-Koszul algebra and quasi-λ-Koszul module are motivated by the following result:

Proposition 2.2. ([8]) Let A be a λ-Koszul algebra with $\|A\| = T \geq 1$. For $M \in gr_l(A)$, let

$$\cdots \longrightarrow P_n \xrightarrow{f_n} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a minimal graded projective resolution of M. Then M is a λ-Koszul module if and only if

1. For $i \equiv\ 2n \pmod{2\mathrm{T}}$, $(\mathrm{n} \in [0,\ \mathrm{T}-1])$, we have $\ker f_i \subseteq JP_i$ and $J\ker f_i = J^2P_i \cap \ker f_i$;

2. For $i \equiv\ 2n-1 \pmod{2\mathrm{T}}$, $(\mathrm{n} \in [1,\ \mathrm{T}])$, we have $\ker f_i \subseteq J^{\lambda(n)}P_i$ and $J\ker f_i = J^{\lambda(n)+1}P_i \cap \ker f_i$.

Definition 2.3. Let R be a Noetherian semiperfect algebra with Jacobson radical J, M a finitely generated R-module and

$$\cdots \longrightarrow P_n \xrightarrow{f_n} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a minimal projective resolution of M. Then M is called a *quasi-λ-Koszul module* if and only if there exists an integer $T \geq 1$, such that

1. For $i \equiv\ 2n \pmod{2\mathrm{T}}$, $(\mathrm{n} \in [0,\ \mathrm{T}-1])$, we have $\ker f_i \subseteq JP_i$ and $J\ker f_i = J^2P_i \cap \ker f_i$;

2. For $i \equiv\ 2n-1 \pmod{2\mathrm{T}}$, $(\mathrm{n} \in [1,\ \mathrm{T}])$, we have $\ker f_i \subseteq J^{\lambda(n)}P_i$ and $J\ker f_i = J^{\lambda(n)+1}P_i \cap \ker f_i$.

Let $\mathcal{QK}^\lambda(R)$ denote the category of quasi-λ-Koszul modules. In particular, R is called a *quasi-λ-Koszul algebra* if $R/J \in \mathcal{QK}^\lambda(R)$.

Example 2.4. Quasi-Koszul algebras/modules (see [4]), quasi-D-Koszul algebras/modules (see [6]) and quasi-piecewise-Koszul algebras/modules (see [9] and [14]) are special quasi-λ-Koszul modules.

3 The proof of the main result

Throughout, R denotes a Noetherian semiperfect algebra with Jacobson radical J, unless specially stated.

We begin with

Lemma 3.1. *([13]) Let* $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence of finitely generated* R*-modules. Then the following statements are equivalent:*

1. $J^kK = K \cap J^kM$ *for* $k \geq 0$*;*
2. $R/J^k \otimes_R K \to R/J^k \otimes_R M$ *is a monomorphism for all* $k \geq 0$*;*
3. $0 \to J^kK \to J^kM \to J^kN \to 0$ *is exact for all* $k \geq 0$*;*
4. $0 \to J^kK/J^{k+1}K \to J^kM/J^{k+1}M \to J^kN/J^{k+1}N \to 0$ *is exact for all* $k \geq 0$*;*
5. $0 \to J^kK/J^mK \to J^kM/J^mM \to J^kN/J^mN \to 0$ *is exact for all* $m > k$.

Lemma 3.2. *([11]) Let* A *be a graded algebra and* $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence of finitely generated graded* A*-modules. Then* $JK = K \cap JM$ *if and only if we have the following commutative diagram with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & L_0 & \longrightarrow & Q_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

such that $P_0 \to K \to 0$, $L_0 \to M \to 0$ *and* $Q_0 \to N \to 0$ *are projective covers, respectively.*

Lemma 3.3. *Let* $\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be a short exact sequence of finitely generated* R*-modules with* $J^kK = J^kM \cap K$ *for all* $k \geq 0$. *Then* $N \in \mathcal{QK}^\lambda(R)$ *provided that* $K,\ M \in \mathcal{QK}^\lambda(R)$.

Proof. By Lemma 3.2, we have the following commutative diagram since $JK = K \cap JM$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0^K & \longrightarrow & P_0^M & \longrightarrow & P_0^N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where P_0^K, P_0^M and P_0^N are projective covers of K, M and N respectively, which implies the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & JP_0^K & \longrightarrow & JP_0^M & \longrightarrow & JP_0^N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & JK & \longrightarrow & JM & \longrightarrow & JN & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Tensoring $R/J \otimes_R -$ to the above exact sequence, by Lemma 3.1, we have the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R/J \otimes_R \Omega^1(K) & \longrightarrow & R/J \otimes_R \Omega^1(M) & \longrightarrow & R/J \otimes_R \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R/J \otimes_R JP_0^K & \longrightarrow & R/J \otimes_R JP_0^M & \longrightarrow & R/J \otimes_R JP_0^N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R/J \otimes_R JK & \longrightarrow & R/J \otimes_R JM & \longrightarrow & R/J \otimes_R JN & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

By Lemma 3.1 again, we have $J\Omega^1(N) = \Omega^1(N) \cap J^2P_0^N$ and $J\Omega^1(K) = \Omega^1(K) \cap J\Omega^1(M)$.

Similarly, by Lemma 3.2, we have the following commutative diagram since $J\Omega^1(K) = \Omega^1(K) \cap J\Omega^1(M)$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^2(K) & \longrightarrow & \Omega^2(M) & \longrightarrow & \Omega^2(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1^K & \longrightarrow & P_1^M & \longrightarrow & P_1^N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where P_1^K, P_1^M and P_1^N are projective covers of $\Omega^1(K)$, $\Omega^1(M)$ and $\Omega^1(N)$ respectively, which implies the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^2(K) & \longrightarrow & \Omega^2(M) & \longrightarrow & \Omega^2(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^{\lambda(1)}P_1^K & \longrightarrow & J^{\lambda(1)}P_1^M & \longrightarrow & J^{\lambda(1)}P_1^N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^{\lambda(1)}\Omega^1(K) & \longrightarrow & J^{\lambda(1)}\Omega^1(M) & \longrightarrow & J^{\lambda(1)}\Omega^1(N) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Tensoring $R/J \otimes_R -$ to the above exact sequence, by Lemma 3.1, we have the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R/J \otimes_R \Omega^2(K) & \longrightarrow & R/J \otimes_R \Omega^2(M) & \longrightarrow & R/J \otimes_R \Omega^2(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R/J \otimes_R J^{\lambda(1)}P_1^K & \longrightarrow & R/J \otimes_R J^{\lambda(1)}P_1^M & \longrightarrow & R/J \otimes_R J^{\lambda(1)}P_1^N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R/J \otimes_R J^{\lambda(1)}\Omega^1(K) & \longrightarrow & R/J \otimes_R J^{\lambda(1)}\Omega^1(M) & \longrightarrow & R/J \otimes_R J^{\lambda(1)}\Omega^1(N) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

By Lemma 3.1 again, we have $J\Omega^2(N) = \Omega^2(N) \cap J^{\lambda(1)+1}P_0^N$ and $J\Omega^2(K) = \Omega^2(K) \cap J\Omega^2(M)$.

Repeat the above arguments, we obtain that $N \in \mathcal{QK}^\lambda(R)$, which completes the proof. □

Similar to the proof ideal of Lemma 3.3, we have the following two lemmas.

Lemma 3.4. *Let* $\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be a short exact sequence of finitely generated* R*-modules with* $JK = JM \cap K$*. Then* $M \in \mathcal{QK}^\lambda(R)$ *provided that* $K,\ N \in \mathcal{QK}^\lambda(R)$*.*

Lemma 3.5. *Let* $\xi : 0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ *be a short exact sequence of finitely generated* R*-modules with* $J^j\Omega^i(K) = \Omega^i(K) \cap\ J^j\Omega^i(M)$ *for all* $k, \geq 0$*. Then* $K \in \mathcal{QK}^\lambda(R)$ *provided that* $M,\ N \in \mathcal{QK}^\lambda(R)$*.*

Now putting Lemmas 3.3, 3.4 and 3.5 together, we have proved our main result.

We end this paper with an easy example satisfying the conditions of Lemmas 3.3, 3.4 and 3.5.

Example 3.6. Any split exact sequence satisfies the conditions: "$J^j\Omega^i(K) = \Omega^i(K) \cap J^j\Omega^i(M)$ for all $i \geq 0, j \geq 0$", where $\Omega^0(X) := X$.

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