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Fast Algorithms for Solving the Inverse Problem of $AX = b$ in the Class of the ULS r-Circulant (Retrocirculant) Matrices

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Abstract

In this paper, fast algorithms for solving the unique solution of the inverse problem of $AX = b$ in the class of the ULS r-circulant(retrocirculant) matrices over a field $\mathbf{F}$ are given by the largest common factor of polynomial. Examples show the effectiveness of the algorithm.

Mathematics Subject Classification: 15A29; 65F15

Keywords: inverse problem, ULS r-circulant matrix

1 Introduction

Circulant matrices have important applications in various disciplines including signal processing[1-4], communications[5-6], image processing[7-8], computer vision[9], encoding[10], physics, probability and statistics, number theory, geometry, and the numerical solution of ordinary and partial differential equations, and they have been put on firm basis with the work of P. Davis[11] and Z. L. Jiang[12].

The circulant matrices, long a fruitful subject of research [11, 12], have in recent years been extended in many directions [13-18]. The $f(x)$-circulant matrices are another natural extension of this well-studied class, and can be found in [16].

The $f(x)$-circulant matrix has a wide application, especially on the generalized cyclic codes[16], where $f(x) \in \mathbf{F}[x]$ is a monic polynomial with no repeated roots in its splitting field over a field $\mathbf{F}$. The properties and structures of the $x^n - rx - r$-circulant matrices, which are called ULS r-circulant matrices, are better than those of the general $f(x)$-circulant matrices, so there are good algorithms for solving the inverse problem of $AX = b$ in the class of the ULS r-circulant (retrocirculant) matrices over a field $\mathbf{F}$. The fast algorithms presented avoid the problems of error and efficiency produced by computing a great number of triangular functions by means of other general fast algorithms. There is only error of approximation when the fast algorithm is realized by computers, and only the elements in the first row of the ULS r-circulant(retrocirculant) matrix and the constant term are used by the fast algorithm, so the result of the computation is accurate in theory. Specially, the result computed by a computer is accurate over the rational field.

2 Definition and Lemma

Definition 2.1. A sum of the upper and the lower(ULS) r-circulant matrix over a field $\mathbf{F}$, denoted by $ULScirc_r(a_0, a_1, \ldots, a_{n-1})$, is meant a square matrix of the form

$$\begin{bmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ ra_{n-1} & a_0 + ra_{n-1} & \ldots & a_{n-2} \\ ra_{n-2} & ra_{n-1} + ra_{n-2} & \ldots & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots \\ ra_2 & ra_3 + ra_2 & \ldots & a_1 \\ ra_1 & ra_2 + ra_1 & \ldots & a_0 + ra_{n-1} \end{bmatrix}_{n \times n}.$$

We define Θ_r as the basic ULS r-circulant matrix over $\mathbf{F}$, that is,

$$\Theta_r = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ r & r & 0 & \ldots & 0 & 0 \end{bmatrix}_{n \times n}.$$

In the following, let that $g(x) = x^n - rx - r$ has no repeated roots in its splitting field over $\mathbf{F}$. It is easily verified that the polynomial $g(x) = x^n - rx - r$ is both the minimal polynomial and the characteristic polynomial of the matrix Θ_r. In addition, Θ_r is nonderogatory and

$$\Theta_r^n = rI_n + r\Theta_r.$$

In view of the structure of the powers of the basic ULS r-circulant matrix Θ_r over $\mathbf{F}$, it is clear that

$$A = ULScirc_r(a_0, a_1, \ldots, a_{n-1}) = \sum_{i=0}^{n-1} a_i \Theta_r^i \tag{1}$$

Thus, A is a ULS r-circulant matrix over $\mathbf{F}$ if and only $A = f(\Theta_r)$ for some polynomial $f(x)$ over $\mathbf{F}$. The polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^i$ will be called the representer of the ULS r-circulant matrix A over $\mathbf{F}$.

By Definition 2.1 and Equation (1), it is clear that A is a ULS r-circulant matrix over $\mathbf{F}$ if and only if A commutes with Θ_r, that is, $A\Theta_r = \Theta_r A$.

In addition to the algebraic properties that can be easily derived from the representation (1), we mention that ULS r-circulant matrices have very nice structure. The product of two ULS r-circulant matrices is a ULS r-circulant matrix and A^{-1} is a ULS r-circulant matrix, too. Furthermore, let $\mathbf{F}[\Theta_r] = \{A | A = f(\Theta_r), f(x) \in \mathbf{F}[x]\}$. It is a routine to prove that $\mathbf{F}[\Theta_r]$ is a commutative ring with the matrix addition and multiplication.

Definition 2.2. A sum of the upper and the lower(ULS) r-retrocirculant matrix over a field $\mathbf{F}$, denoted by $ULSretrocirc_r(a_0, a_1, \ldots, a_{n-1})$, is meant a square matrix of the form

$$\begin{bmatrix} a_0 & \ldots & a_{n-2} & a_{n-1} \\ a_1 & \ldots & a_{n-1} + ra_0 & ra_0 \\ a_2 & \ldots & ra_0 + ra_1 & ra_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-2} & \ldots & ra_{n-4} + ra_{n-3} & ra_{n-3} \\ a_{n-1} + ra_0 & \ldots & ra_{n-3} + ra_{n-2} & ra_{n-2} \end{bmatrix}_{n \times n}.$$

Lemma 2.3. Let $A = ULScirc_r(a_0, a_1, \ldots, a_{n-1})$ be a ULS r-circulant matrix over $\mathbf{F}$ and $B = ULSretrocirc_r(a_{n-1}, a_{n-2}, \ldots, a_1, a_0)$ an ULS r-retrocirculant matrix over $\mathbf{F}$. Then $BK = A$ or $B = AK$, where

$$K = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

Lemma 2.4[18]Let $\mathbf{F}[x]$ be the polynomial ring over $\mathbf{F}$, and let $f(x), g(x) \in \mathbf{F}[x]$.

Suppose that the polynomial matrix $\begin{pmatrix} f(x) & 1 & 0 \\ g(x) & 0 & 1 \end{pmatrix}$ is changed into the polynomial matrix $\begin{pmatrix} d(x) & u(x) & v(x) \\ 0 & s(x) & t(x) \end{pmatrix}$ by a series of elementary row operations, then $(f(x), g(x)) = d(x)$ and $f(x)u(x) + g(x)v(x) = d(x)$.

Lemma 2.5. $\mathbf{F}[x]/\langle x^n - rx - r\rangle \cong \mathbf{F}[\Theta_r]$.

ProofConsider the following $\mathbf{F}$-algebra homomorphism

$$\varphi: \quad \mathbf{F}[x] \to \mathbf{F}[\Theta_r]$$

$$f(x) \to A = f(\Theta_r)$$

for $f(x) \in \mathbf{F}[x]$. It is clear that φ is an $\mathbf{F}$-algebra epimorphism. So we have

$$\mathbf{F}[x]/\ker\varphi \cong \mathbf{F}[\Theta_r].$$

Since $\mathbf{F}[x]$ is a principal ideal integral domain, there is a monic polynomial $p(x) \in \mathbf{F}[x]$ such that $\ker\varphi = \langle p(x) \rangle$. Since $x^n - rx - r$ is the minimal polynomial of Θ_r, then $p(x) = x^n - rx - r$.

By Lemma 2.5, we have the following lemma.

Lemma 2.6. Let $A = ULScirc_r(a_0, a_1, \ldots, a_{n-1})$ be an ULS r-circulant matrix over $\mathbf{F}$. Then A is nonsingular if and only if $(f(x), g(x)) = 1$, where $f(x) = \sum_{i=0}^{n-1} a_i x^i$ is the representer of A and $g(x) = x^n - rx - r$.

Proof A is nonsingular if and only if $f(x) + \langle x^n - rx - r \rangle$ is an invertible element in $\mathbf{F}[x]/\langle x^n - rx - r \rangle$. By Lemma 2.5, if and only if there exists $u(x) + \langle x^n - rx - r \rangle \in \mathbf{F}[x]/\langle x^n - rx - r \rangle$ such that $f(x)u(x) + \langle x^n - rx - r \rangle = 1 + \langle x^n - rx - r \rangle$ if and only if there exist $u(x), v(x) \in \mathbf{F}[x]$ such that $f(x)u(x) + (x^n - rx - r)v(x) = 1$ if and only if $(f(x), x^n - rx - r) = 1$.

3 Fast algorithm for solving the inverse problem

In this section, Consider the linear system

$$AX = b \tag{3}$$

Where $X = (x_0, x_1, \ldots, x_{n-1})^T$ and $b = (b_0, b_1, \ldots, b_{n-1})^T$ are given. The key of the inverse problem is how to find A, for this purpose, we first prove the following results.

Theorem 3.1. Let $X = (x_0, x_1, \ldots, x_{n-1})^T$, $b = (b_0, b_1, \ldots, b_{n-1})^T$, $D = ULScirc_r(x_{n-1} - rx_0, x_{n-2}, \ldots, x_1, x_0)$ and $B = ULScirc_r(b_{n-1} - rb_0, b_{n-2}, \ldots, b_1, b_0)$. Then the inverse problem of $AX = b$ has a unique solution in the class of the ULS r-circulant matrices of order n if and only if $DY = b$ has a unique solution.

Proof If the inverse problem of $AX = b$ has a unique solution in the class of the ULS r-circulant matrices of order n, then there exists a unique ULS r-circulant matrix $A = ULScirc_r(a_0, a_1, \ldots, a_{n-1})$ of order n such that $AX = b$, i.e.

$$ULScirc_r(a_0, a_1, \ldots, a_{n-1})(x_0, x_1, \ldots, x_{n-1})^T = (b_0, b_1, \ldots, b_{n-1})^T \tag{4}$$

Let $\beta = (a_{n-1}, \ldots, a_1, a_0 + ra_{n-1})^T$, we know that β is the unique solution of $DY = b$, by Equation (4). Conversely, if $DY = b$ has a unique solution $Y = (a_{n-1}, \ldots, a_1, a_0)^T$ and let $A = ULScirc_r(a_0 - ra_{n-1}, a_1, \ldots, a_{n-1})$ and

$B = ULScirc_r(b_{n-1} - rb_0, b_{n-2}, \dots, b_1, b_0)$, then $DA = B$ has a unique solution $A = D^{-1}B$. Since $D^{-1}X = (0, \dots, 0, 1)^T$, then $AX = D^{-1}BX = BD^{-1}X = b$. So $A = D^{-1}B$ is the unique solution of the inverse problem of $AX = b$ in the class of the ULS r-circulant matrices of order n.

Theorem 3.2. The inverse problem of $AX = b$ has a unique solution in the class of the ULS r-circulant matrices of order n if and only if $(d(x), g(x)) = 1$, where $d(x) = (x_{n-1} - rx_0) + \sum_{i=1}^{n-1} x_{n-1-i}x^i$, $g(x) = x^n - rx - r$, X and b are given in Theorem 3.1.

Proof By Theorem 3.1, we know that the inverse problem of $AX = b$ has a unique solution in the class of the ULS r-circulant matrices of order n if and only if $DY = b$ has a unique solution if and only if D is nonsingular if and only if $(d(x), g(x)) = 1$ by Lemma 2.6, where D is given in Theorem 3.1.

By Lemma 2.4, Theorem 3.1 and Theorem 3.2, we have the following fast algorithm for solving the unique solution of the inverse problem of $AX = b$ in the class of the ULS r-circulant matrices of order n.

Algorithm 3.1:

Step 1 From $X = (x_0, \dots, x_{n-1})^T$ and $b = (b_0, \dots, b_{n-1})^T$, we get the polynomial $d(x) = (x_{n-1} - rx_0) + \sum_{i=1}^{n-1} x_{n-1-i}x^i$, and $h(x) = (b_{n-1} - rb_0) + \sum_{i=1}^{n-1} b_{n-1-i}x^i$;

Step 2 Change the polynomial matrix $\begin{pmatrix} d(x) & h(x) \\ g(x) & 0 \end{pmatrix}$ into the polynomial matrix $\begin{pmatrix} u(x) & v(x) \\ 0 & s(x) \end{pmatrix}$ by a series of elementary row operations;

Step 3 If $u(x) = 1$, then The $D = ULScirc_r(x_{n-1} - rx_0, x_{n-2}, \dots, x_1, x_0)$ is nonsingular. So the inverse problem of $AX = b$ has a unique solution $A = D^{-1}B = v(\Theta_r)$ in the class of the ULS r-circulant matrices of order n, where $\Theta_r = ULScirc_r(0, 1, 0, \dots, 0)$.

By Lemma 2.3 and Theorem 3.1, we have the following theorem.

Theorem 3.3. Let $X' = (x_{n-1}, \dots, x_1, x_0)^T$. Then the inverse problem of $CX' = b$ has a unique solution in the class of the ULS r-retrocirculant matrices of order n if and only if $DY = b$ has a unique solution, where D, b are given in Theorem 3.1.

By Lemma 2.3 and Theorem 3.2, we have the following theorem.

Theorem 3.4. The inverse problem of $CX' = b$ has a unique solution in the class of the ULS r-retrocirculant matrices of order n if and only if $(d(x), g(x)) = 1$, where $d(x), g(x)$ are given in Theorem 3.2 and X', b are given in Theorem 3.3.

By Lemma 2.4, Theorem 3.3 and Theorem 3.4, we have the following fast algorithm for solving the unique solution of the inverse problem of $CX' = b$ in the class of the ULS r-retrocirculant matrices of order n .

Algorithm 3.2:

Step 1 From $X' = (x_{n-1}, \dots, x_1, x_0)^T$, $b = (b_0, \dots, b_{n-1})^T$, we get the polynomial $d(x) = (x_{n-1} - rx_0) + \sum_{i=1}^{n-1} x_{n-1-i}x^i$, and $h(x) = (b_{n-1} - rb_0) + \sum_{i=1}^{n-1} b_{n-1-i}x^i$;

Step 2 Change the polynomial matrix $\begin{pmatrix} d(x) & h(x) \\ g(x) & 0 \end{pmatrix}$ into the polynomial matrix $\begin{pmatrix} u(x) & v(x) \\ 0 & s(x) \end{pmatrix}$ by a series of elementary row operations;

Step 3 If $u(x) = 1$, then the $D = ULScirc_r(x_{n-1} - rx_0, x_{n-2}, \ldots, x_1, x_0)$ is nonsingular. So the inverse problem of $CX' = b$ has a unique solution $C = v(\Theta_r)K$ in the class of the ULS r-retrocirculant matrices of order n, where $\Theta_r = ULScirc_r(0, 1, 0, \ldots, 0)$ and K is given in Equation (2).

4 Numerical Examples

Example 4.1. Find the solution of the inverse problem of $AX = b$ in the class of the ULS 2-circulant matrices of order 4, where $X = (1, -1, 1, 3)^T$ and $b = (1, 2, -1, 1)^T$.

From the $X = (1, -1, 1, 3)^T$ and $b = (1, 2, -1, 1)^T$, we get the polynomial $d(x) = 1 + x - x^2 + x^3$ and $h(x) = -1 - x + 2x^2 + x^3$. On the other hand, $g(x) = -2 - 2x + x^4$. Then

$$A(x) = \begin{pmatrix} d(x) & h(x) \\ g(x) & 0 \end{pmatrix} = \begin{pmatrix} 1 + x - x^2 + x^3 & -1 - x + 2x^2 + x^3 \\ -2 - 2x + x^4 & 0 \end{pmatrix}.$$

We transform the polynomial matrix $A(x)$ by a series of elementary row operations as follows:

$$A(x) = \begin{pmatrix} 1 + x - x^2 + x^3 & -1 - x + 2x^2 + x^3 \\ -2 - 2x + x^4 & 0 \end{pmatrix}$$

$$\xrightarrow{(2) - x(1)} \begin{pmatrix} 1 + x - x^2 + x^3 & -1 - x + 2x^2 + x^3 \\ -2 - 3x - x^2 + x^3 & x + x^2 - 2x^3 - x^4 \end{pmatrix}$$

$$\xrightarrow{(2) - (1)} \begin{pmatrix} 1 + x - x^2 + x^3 & -1 - x + 2x^2 + x^3 \\ -3 - 4x & 1 + 2x - x^2 - 3x^3 - x^4 \end{pmatrix}$$

$$\xrightarrow{4(1) + x^2(2)} \begin{pmatrix} 4 + 4x - 7x^2 & -4 - 4x + 9x^2 + 6x^3 - x^4 - 3x^5 - x^6 \\ -3 - 4x & 1 + 2x - x^2 - 3x^3 - x^4 \end{pmatrix}$$

$$\xrightarrow{4(1) - 7x(2)} \begin{pmatrix} 16 + 37x & 16 - 23x + 22x^2 + 31x^3 + 17x^4 - 5x^5 - 4x^6 \\ -3 - 4x & 1 + 2x - x^2 - 3x^3 - x^4 \end{pmatrix}$$

$$\xrightarrow{4(1)+37(2)} \begin{pmatrix} -47 & -27-18x+51x^2+13x^3+31x^4-20x^5-16x^6 \\ -3-4x & 1+2x-x^2-3x^3-x^4 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{47}(1)} \begin{pmatrix} 1 & \frac{-27-18x+51x^2+13x^3+31x^4-20x^5-16x^6}{-47} \\ -3-4x & 1+2x-x^2-3x^3-x^4 \end{pmatrix}.$$

Since $u(x) = 1$, then the inverse problem of $AX = b$ has a unique solution in the class of the ULS 2-circulant matrices of order 4. On the other hand, $v(x) = -(-27-18x+51x^2+13x^3+31x^4-20x^5-16x^6/47)$. Substituting x by Θ_2 in polynomial $v(x)$, we know that a unique solution of the inverse problem of $AX = b$ in the class of the ULS 2-circulant matrices of order 4 is

$$A = v(\Theta_2) = ULScirc_2(-\frac{35}{47}, -\frac{4}{47}, \frac{21}{47}, \frac{19}{47}) = \frac{1}{47}\begin{pmatrix} -35 & -4 & 21 & 19 \\ 38 & 3 & -4 & 21 \\ 42 & 80 & 3 & -4 \\ -8 & 34 & 80 & 3 \end{pmatrix}.$$

Example 4.2. Find the solution of the inverse problem of $AX = b$ in the class of the ULS 3-retrocirculant matrices of order 4, where $X = (2, 1, -1, 1)^T$ and $b = (1, -1, 1, 1)^T$.

From the $X = (2, 1-1, 1)^T$ and $b = (1, -1, 1, 1)^T$, we get the polynomial $d(x) = -1 + x - x^2 + x^3$ and $h(x) = -2 + x - x^2 + x^3$. On the other hand, $g(x) = -3 - 3x + x^4$. Then

$$A(x) = \begin{pmatrix} d(x) & h(x) \\ g(x) & 0 \end{pmatrix} = \begin{pmatrix} -1+x-x^2+x^3 & -2+x-x^2+x^3 \\ -3-3x+x^4 & 0 \end{pmatrix}.$$

We transform the polynomial matrix $A(x)$ by a series of elementary row operations as follows:

$$A(x) \xrightarrow{(2)-x(1)} \begin{pmatrix} -1+x-x^2+x^3 & -2+x-x^2+x^3 \\ -3-2x-x^2+x^3 & 2x-x^2+x^3-x^4 \end{pmatrix}$$

$$\xrightarrow{(1)-(2)} \begin{pmatrix} 2+3x & -2-x+x^4 \\ -3-2x-x^2+x^3 & 2x-x^2+x^3-x^4 \end{pmatrix}$$

$$\xrightarrow{3(2)-x^2(1)} \begin{pmatrix} 2+3x & -2-x+x^4 \\ -9-6x-5x^2 & 6x-x^2+4x^3-3x^4-x^6 \end{pmatrix}$$

$$\xrightarrow{3(2)+5x(1)} \begin{pmatrix} 2+3x & -2-x+x^4 \\ -27-8x & 8x-8x^2+12x^3-9x^4+5x^5-3x^6 \end{pmatrix}$$

$$\xrightarrow{8(1)+3(2)} \begin{pmatrix} -65 & -16+16x-24x^2+36x^3-19x^4+15x^5-9x^6 \\ -27-8x & 8x-8x^2+12x^3-9x^4+5x^5-3x^6 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{65}(1)} \begin{pmatrix} 1 & \frac{-16+16x-24x^2+36x^3-19x^4+15x^5-9x^6}{-65} \\ -27-8x & 8x-8x^2+12x^3-9x^4+5x^5-3x^6 \end{pmatrix}.$$

Since $u(x) = 1$, then the inverse problem of $AX = b$ has a unique solution in the class of the ULS 3-retrocirculant matrices of order 4. On the other hand, $v(x) = -(\frac{-16+16x-24x^2+36x^3-19x^4+15x^5-9x^6}{65})$. Substituting x by Θ_3 in polynomial $v(x)$, we know that a unique solution of the inverse problem of $AX = b$ in the class of the ULS 3-retrocirculant matrices of order 4 is

$$A = v(\Theta_3)K = ULScirc_3(\frac{73}{65}, -\frac{4}{65}, \frac{6}{65}, -\frac{9}{65})K$$

$$= \frac{1}{65}\begin{pmatrix} 73 & -4 & 6 & -9 \\ -27 & 46 & -4 & 6 \\ 18 & -9 & 46 & -4 \\ -12 & 6 & -9 & 46 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{65}\begin{pmatrix} -9 & 6 & -4 & 73 \\ 6 & -4 & 46 & -27 \\ -4 & 46 & -9 & 18 \\ 46 & -9 & 6 & -12 \end{pmatrix} = ULSretrocirc_3(-\frac{9}{65}, \frac{6}{65}, -\frac{4}{65}, \frac{73}{65}).$$

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