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Green's Theorem and Minimal Quasi-Ideals in Γ-Semigroups

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Abstract

The definition of Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}$ in Γ-semigroups [5] mimics the definitions of the usual Green's relations in plain semigroups and the Green's relation in semirings [3] and rings [4]. The main result of this paper is the analogue of Green's theorem for plain semigroups which we call Green's theorem for Γ-semigroups. After we define the concept of quasi-ideals in Γ-semigroups and using our Green's theorem for Γ-semigroups, we prove that any quasi-ideal of a Γ-semigroup without zero is minimal if and only if it is a Γ-subgroup. Further, we prove that "if a Γ-semigroup S has a cancellable element contained in a minimal quasi-ideal Q of S, then S is a Γ-group"–a claim that mimics a result for plain semigroups. Finally we prove that, if for elements a, b of a Γ-semigroup we have $a\mathcal{D}b$ and the principal quasi-ideal $(a)_q$ is minimal, then the principal quasi-ideal $(b)_q$ is minimal too.

Introduction

Let S and Γ be two nonempty sets. Any map from $S \times \Gamma \times S$ to S will be called a Γ-multiplication in S and denoted by $(\cdot)_\Gamma$. The result of this multiplication for $a, b \in S$ and $\gamma \in \Gamma$ is denoted by $a\gamma b$. According to Sen and Saha [6], a Γ-semigroup S is an ordered pair $(S, (\cdot)_\Gamma)$ where S and Γ are non empty sets and $(\cdot)_\Gamma$ is a Γ-multiplication on S which satisfies the following property

$$\forall (a, b, c, \alpha, \beta) \in S^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$$

The following two examples are the most well known ones.

Example 1. The Γ-semigroup S of all mappings of a set A to a set B, where Γ is the set of all mappings of B to A and the multiplication $f\gamma g$ is defined as the composite $f \circ \gamma \circ g$ for $f, g \in S$ and $\gamma \in \Gamma$.

Example 2. The Γ-semigroup S of all $m \times n$ matrices with entries from a field P, where Γ is the set of all $n \times m$ matrices with entries from P. The Γ-multiplication of two $m \times n$ matrices A, B with an $n \times m$ matrix C is the usual product of matrices ACB.

Note that every plain semigroup S can be made into a Γ-semigroup by taking as Γ a singleton $\{1\}$ where 1 is the unit element of S whenever S is a monoid, or is a symbol not representing an element of S, and the multiplication is defined by $a1b = ab$.

Similarly to the definition of relations $\mathcal{R}_{\text{plain}}, \mathcal{L}_{\text{plain}}, \mathcal{H}_{\text{plain}}$ in plain semigroups, Saha [5] has introduced their analogues $\mathcal{R}, \mathcal{L}, \mathcal{H}$ in a Γ-semigroup S which are called the Green's relations in the Γ-semigroup S. Some properties of $\mathcal{R}$ and $\mathcal{L}$ in Γ-semigroups are the same as the properties of their counterparts $\mathcal{R}_{\text{plain}}$ and $\mathcal{L}_{\text{plain}}$ in plain semigroups; for example $\mathcal{R}$ and $\mathcal{L}$ commute with each other, a property that allows one to define the relation $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ in Γ-semigroups. Our main result claims that the analogue of Green's Theorem for the Green's relation $\mathcal{H}_{\text{plain}} = \mathcal{R}_{\text{plain}} \cap \mathcal{L}_{\text{plain}}$ holds true for Γ-semigroups. We call this the Theorem of Green for Γ-semigroups. Further we will define quasi-ideals in Γ-semigroups and the relation "to generate the same principal quasi-ideal" for which we show that coincides with $\mathcal{H}$. Then we use our Green's Theorem for Γ-semigroups to prove that any quasi-ideal of a Γ-semigroup without zero is minimal if and only if it is a Γ-subgroup. As a corollary of this we get the analogue of a result for plain semigroups which states that "if a Γ-semigroup S has a cancellable element contained in a minimal quasi-ideal Q of S, then S is a Γ-subgroup". Lastly, we show that if for elements a, b of a Γ-semigroup we have $a\mathcal{D}b$ and the principal quasi-ideal $(a)_q$ is minimal, then the principal quasi-ideal $(b)_q$ is minimal too.

1 Preliminaries

Here we give a few notions and present some auxiliary results that will be used throughout the paper. Some of the results may be found in [5] and [6] but for the reader's convenience we have included their proofs here.

Let S be a Γ-semigroup and A, B subsets of S. We define the set

$$A\Gamma B = \{a\gamma b | a, b \in S \text{ and } \gamma \in \Gamma\}.$$

For simplicity we write $a\Gamma B$ instead of $\{a\}\Gamma B$ and similarly we write $A\Gamma b$, and write $A\gamma B$ in place of $A\{\gamma\}B$.

Definition 1. [5] Let S be a Γ-semigroup. A nonempty subset S_1 of S is said to be a Γ-subsemigroup of S if $S_1 \Gamma S_1 \subseteq S_1$.

Definition 2. [6] A right [left] ideal of a Γ-semigroup S is a nonempty subset $R[L]$ of S such that $R\Gamma S \subseteq R$, $[S\Gamma L \subseteq L]$.

By analogy with the definition of quasi-ideal in plain semigroups [7] we give the following

Definition 3. A quasi-ideal of a Γ-semigroup S is a nonempty subset Q of S such that $Q\Gamma S \cap S\Gamma Q \subseteq Q$.

Similarly to the plain semigroup situation, one can prove this

Lemma 1.1. *A non empty subset Q of a Γ-semigroup S is a quasi-ideal of S if and only if it can be expressed as the intersection of right ideal with a left ideal of S.*

Let S be a Γ-semigroup and keep $\gamma \in \Gamma$ fixed. As in [5] define $a \circ b = a\gamma b$. It is obvious that $\circ$ is associative, hence we obtain a semigroup $(S, \circ)$ which is shortly denoted by S_γ.

A zero of Γ-semigroup S is an element $0 \in S$ such that for all $a \in S$ and $\gamma \in \Gamma$ we have $a\gamma 0 = 0 = 0\gamma a$.

Theorem 1.1. *Let S be any Γ-semigroup without zero and $\gamma \in \Gamma$ a fixed element. Then, S_γ is a group if and only if S has no proper quasi-ideals.*

Proof. Suppose that $(S, \circ) = S_\gamma$ is a group and let Q be a quasi-ideal of the Γ-semigroup S. Then, for any a in Q we have:

$$S = S \circ a = S\gamma a = a \circ S = a\gamma S = S\gamma a \cap a\gamma S \subseteq S\Gamma Q \cap Q\Gamma S \subseteq Q,$$

whence $S = Q$.

Conversely, assume that S contains no proper quasi-ideals. Then for any element $s \in S$, $S\gamma s$ and $s\gamma S$ are respectively left and right ideals of S, therefore

$$S = S\gamma s = S \circ s = S\gamma s = s \circ S,$$

that is, $(S, \circ) = S_\gamma$ is a group. □

Corollary 1.1. *Let S be a Γ-semigroup without zero. If S_γ is a group for some $\gamma \in \Gamma$, then it is a group for all $\gamma \in \Gamma$.*

Proof. If for some $\gamma \in \Gamma$, S_γ is a group, then we infer from Theorem 1.1 that S has no proper quasi-ideals. Employing Theorem 1.1 again we see that every S_γ is a Γ-group. □

There is a direct proof of Corollary 1.1 in [5].

Definition 4. [5] A Γ-semigroup S is called a Γ-group if S_γ is a group for some (hence for all) $\gamma \in \Gamma$.

Let $\overline{S}$ be a Γ-subsemigroup of a Γ-semigroup S. From Corollary 1.1 if S_γ is a group for some $\gamma \in \Gamma$, then it is a group for all $\gamma \in \Gamma$ and so it is a Γ-group. In this case we will call $\overline{S}$ a Γ-*subgroup* of the Γ-semigroup S.

For each element a of a Γ-semigroup S, Saha has defined in [5] the *principal right ideal* $(a)_r$ generated by a as the smallest right ideal containing a. The *principal left ideal* $(a)_l$ generated by a is defined dually. It is easy to prove that

$$(a)_r = a \cup a\Gamma S \tag{1}$$

and

$$(a)_l = a \cup S\Gamma a. \tag{2}$$

Saha has defined in [5] the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}$ in a Γ-semigroup as follows:

$$\forall (a, b) \in S^2, a\mathcal{R}b \Leftrightarrow (a)_r = (b)_r,$$

$$\forall (a, b) \in S^2, a\mathcal{L}b \Leftrightarrow (a)_l = (b)_l,$$

$$\forall (a, b) \in S^2, a\mathcal{H}b \Leftrightarrow (a)_r = (b)_r \text{ and } (a)_l = (b)_l.$$

It turns out that $\mathcal{R}, \mathcal{L}, \mathcal{H}$ are equivalence relations. The respective equivalence classes of $a \in S$ are denote by R_a, L_a, H_a. It is easy to prove the following lemma.

Lemma 1.2. *In a Γ-semigroup S we have:*

1. *For every three elements $a, b, c \in S$ and every $\gamma \in \Gamma$*

   $$a\mathcal{R}b \Rightarrow c\gamma a\mathcal{R}c\gamma b \text{ and } a\mathcal{L}b \Rightarrow a\gamma c\mathcal{R}b\gamma c.$$

2. *For every two elements $a, b \in S$, $a\mathcal{R}b$ if and only if either $a = b$ or there exist $\alpha, \beta \in \Gamma$ and $c, d \in S$ such that $a = b\alpha c$ and $b = a\beta d$.*

3. *For every two elements $a, b \in S$, $a\mathcal{L}b$ if and only if either $a = b$ or there exist $\alpha, \beta \in \Gamma$ and $c, d \in S$ such that $a = c\alpha b$ and $b = d\beta a$.*

4. *$\mathcal{R}$ and $\mathcal{L}$ commute, that is, $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$. One can then define a fourth Green's relation in S which is $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$.*

For every element $a \in S$, we denote by $(a)_q$ the smallest quasi-ideal of S containing a and call it the principal quasi-ideal of S generated by a. One can prove easily that

$$(a)_q = a \cup (a\Gamma S \cap S\Gamma a) = (a \cup a\Gamma S) \cap (a \cup S\Gamma a). \tag{3}$$

Now we define the relation $\mathcal{Q}$ by

$$\forall (a,b) \in S^2, a\mathcal{Q}b \Leftrightarrow (a)_q = (b)_q.$$

In view of (3) we can write

$$a\mathcal{Q}b \Leftrightarrow (a)_q = (a \cup a\Gamma S) \cap (a \cup S\Gamma a) = (b \cup b\Gamma S) \cap (b \cup S\Gamma b) = (b)_q. \tag{4}$$

The relation $\mathcal{Q}$ is an equivalence relation and moreover we have

Lemma 1.3. *The equivalence relations $\mathcal{H}$ and $\mathcal{Q}$ coincide in a Γ-semigroup S.*

Proof. If the elements $a, b \in S$ are $\mathcal{H}$-equivalent, then from (1) and (2) we have

$$a \cup a\Gamma S = b \cup b\Gamma S \text{ and } a \cup S\Gamma a = b \cup S\Gamma b,$$

thus

$$(a \cup a\Gamma S) \cap (a \cup S\Gamma a) = (b \cup b\Gamma S) \cap (b \cup S\Gamma b),$$

and then from (4) we have $a\mathcal{Q}b$. Conversely, let $a\mathcal{Q}b$ and then from (3) we have

$$a \cup (a\Gamma S \cap S\Gamma a) = b \cup (b\Gamma S \cap S\Gamma b).$$

Thus we either have $a = b$ or $a \in b\Gamma S \cap S\Gamma b$ and $b \in a\Gamma S \cap S\Gamma a$. In both cases, in view of Lemma 1.2 we have $a\mathcal{H}b$. Hence $\mathcal{Q} \subseteq \mathcal{H}$. The two inclusions $\mathcal{Q} \subseteq \mathcal{H}$ and $\mathcal{H} \subseteq \mathcal{Q}$ imply the equality $\mathcal{H} = \mathcal{Q}$. □

2 Green's Theorem for Γ-Semigroups

To prove the analogue of Green's Theorem for Γ-semigroups we need two lemmas. The first of them, which we may call the Green's Lemma, has been proved in [2].

Lemma 2.1. (*Lemma 3.6 [2]*) *Let a, b be two elements of a Γ-semigroup S such that $a\mathcal{R}b$. If $a\alpha s = b$ and $b\beta s' = a$ where $s, s' \in S$, $\alpha, \beta \in \Gamma$, then the mappings*

$$\begin{array}{rccc} \sigma: & L_a & \rightarrow & L_b \\ & x & \mapsto & x\alpha s \end{array}, \quad \begin{array}{rccc} \sigma': & L_b & \rightarrow & L_a \\ & y & \mapsto & y\beta s' \end{array}$$

are mutually inverse, $\mathcal{R}$-class preserving, one to one maps from L_a to L_b and from L_b to L_a respectively.

The dual of Lemma 2.1 also holds true.

By Lemma 2.1 the following proposition is easily proved.

Proposition 2.1. *Let a, b be two elements of a Γ-semigroup S such that $a\mathcal{R}b$. If $a\alpha s = b$ and $b\beta s' = a$ where $s, s' \in S$, $\alpha, \beta \in \Gamma$, then the mappings:*

$$\begin{array}{rccc} f: & (a)_l \cap (a)_r & \rightarrow & (b)_l \cap (b)_r \\ & x & \mapsto & x\alpha s \end{array}, \quad \begin{array}{rccc} g: & (b)_l \cap (b)_r & \rightarrow & (a)_l \cap (a)_r \\ & y & \mapsto & y\beta s' \end{array}$$

are mutually inverse, one to one maps of $H_a$onto H_b and of H_b onto H_a respectively. It follows that two $\mathcal{H}$-classes contained in the same $\mathcal{R}$-class have the same cardinal number.

The dual of Proposition 2.1 for the relation $\mathcal{L}$ also holds true.

Lemma 2.2. *If the elements $h, h\gamma s, [s\gamma h]$ of a Γ-semigroup S $(\gamma \in \Gamma$, $s \in S)$ both belong to the same $\mathcal{H}$-class of S, then $H\gamma s = H, [s\gamma H = H]$, where*

$$H\gamma s = \{a\gamma s \in S | a \in H\}$$

$$[s\gamma H = \{s\gamma a \in S | a \in H\}].$$

Proof. The elements $h, h\alpha s$ are $\mathcal{R}$-equivalent, therefore there exist $\alpha \in \Gamma$ and $s' \in S$ such that $h = (h\gamma s)\alpha s'$. Hence from Proposition 2.1 the mappings

$$\begin{array}{rccc} f: & (h)_l \cap (h)_r & \rightarrow & (h\gamma s)_l \cap (h\gamma s)_r \\ & x & \mapsto & x\gamma s \end{array}$$

$$\begin{array}{rccc} g: & (h\gamma s)_l \cap (h\gamma s)_r & \rightarrow & (h)_l \cap (h)_r \\ & y & \mapsto & y\alpha s' \end{array}$$

are mutually inverse and one to one maps from $H = H_h$ to $H = H_{h\gamma s}$. Hence $H = f(H) = H\gamma s$. The other equality is proved similarly using the dual of Proposition 2.1. $\square$

We will use Lemma 2.1 and Lemma 2.2 to prove the main result of the paper.

Theorem 2.1. (*Green's Theorem*) *If the elements a, b and $a\gamma b$ of a Γ-semigroup S all belong to the same $\mathcal{H}$-class H of S, then H is a subgroup of the semigroup S_γ. Moreover, for any two elements $h_1, h_2 \in H$, the element $h_1\gamma h_2$ belongs to H.*

Proof. Since $a, a\gamma b \in H$, then from Lemma 2.2 we have that $H\gamma b = H$, hence $H \circ b = H$. For any $c, d \in H$ we have $c \circ d \in H \circ b = H$, hence both b and $c \circ b = c\gamma b$ belong to H. Applying Lemma 2.1 we get $c\gamma H = H$, hence $c \circ H = H$. The element c belongs to H therefore $c \circ d = c\gamma d \in H$. Since $c, c\gamma d \in H$, then again from Lemma 2.1 we have $H\gamma d = H$, hence $H \circ d = H$. The equalities

$$c \circ H = H = H \circ d$$

show that H is a subgroup of S_γ. To conclude the proof, let $h_1, h_2 \in H$. Then since H is a subgroup, we have $h_1\gamma h_2 = h_1 \circ h_2 \in H$. □

An element e of a Γ-semigroup S is called *idempotent* if there exists $\alpha \in \Gamma$ such that $e = e\alpha e$. From Theorem 2.1 we get immediately this:

Corollary 2.1. *If an $\mathcal{H}$-class H of a Γ-semigroup S contains an idempotent $e = e\alpha e$, $\alpha \in \Gamma$, then H is a subgroup of S_α.*

The reader can find a direct proof of the following theorem in [5].

Theorem 2.2. *Let S be a regular Γ-semigroup. If the idempotent $e = e\alpha e$, $\alpha \in \Gamma$, together with $a, b \in S$ all belong to some $\mathcal{H}$-class H, then $e\alpha a = a\alpha e = a$ and $a\alpha b \in H$.*

Here we will prove this theorem in a much shorter way and most importantly, we remove the regularity condition. We prove this

Theorem 2.3. *Let S be an arbitrary Γ-semigroup. If the idempotent $e = e\alpha e$, $\alpha \in \Gamma$, together with $a, b \in S$ all belong to some $\mathcal{H}$-class H, then $e\alpha a = a\alpha e = a$ and $a\alpha b \in H$.*

Proof. Since H contains an idempotent $e = e\alpha e$, $\alpha \in \Gamma$, then Corollary 2.1 imply that H is a subgroup of S_α. The unit of H is e since $e \circ e = e\alpha e = e$. Since $a \circ e = e \circ a = a$, we have $a\alpha e = a = e\alpha e$. The elements a, b belong to H, therefore $a\alpha b = a \circ b \in H$. □

3 Minimal Quasi-Ideals in Γ-Semigroups

We will use Green's Theorem to prove some results concerning minimal quasi-ideals in Γ-semigroups without zero.

A right ideal [left ideal, quasi-ideal] of a Γ-semigroup S without zero is called minimal if it does not contain another right ideal [left ideal, quasi-ideal] of S.

Theorem 3.1. *A quasi-ideal Q of a Γ-semigroup S without zero is minimal if and only if it is the intersection of a minimal right ideal with a minimal left ideal of S.*

Proof. Assume that $Q = L \cap R$, where L is a minimal left ideal and R is a minimal right ideal. If $Q' \subseteq Q$ is another quasi-ideal of S, then $S\Gamma Q'$ and $Q'\Gamma S$ are respectively left and right ideals, and $S\Gamma Q' \subseteq S\Gamma L \subseteq L$ and $Q'\Gamma S \subseteq R\Gamma S \subseteq R$. By the minimality of L and R we have $S\Gamma Q' = L$ and $Q'\Gamma S = R$. Hence $Q = L \cap R = S\Gamma Q' \cap Q'\Gamma S \subseteq Q'$, that is, $Q = Q'$ which shows the minimality of Q.

Conversely, let a be an element of the minimal quasi-ideal Q of S. By Lemma 1.1 the intersection $S\Gamma a \cap a\Gamma S$ is a quasi-ideal of S such that $S\Gamma a \cap a\Gamma S \subseteq Q$. Since Q is minimal then $Q = S\Gamma a \cap a\Gamma S$. We show now that $S\Gamma a$ is a minimal left ideal of S. If $L \subseteq S\Gamma a$ is a left ideal of S, then $L \cap a\Gamma S \subseteq S\Gamma a \cap a\Gamma S = Q$. It is easy to show that $L \cap a\Gamma S$ is a quasi-ideal of S, therefore the minimality of Q implies $L \cap a\Gamma S = Q$, whence $Q \subseteq L$. So we have $S\Gamma a \subseteq S\Gamma Q \subseteq S\Gamma L \subseteq L$. The inclusions $L \subseteq Sa$ and $Sa \subseteq L$ imply that $S\Gamma = L$. Thus $S\Gamma a$ is minimal left ideal. The minimality of the right ideal $a\Gamma S$ can be proved dually. □

Theorem 3.2. *A quasi-ideal Q of a Γ-semigroup S without zero is minimal if and only if Q is an $\mathcal{H}$-class.*

Proof. A quasi-ideal Q of a Γ-semigroup S is minimal if and only if it is generated by any of its elements. Thus a minimal quasi-ideal is a $\mathcal{Q}$-class. By Lemma 1.3 this is equivalent for Q to be an $\mathcal{H}$-class. □

Theorem 3.3. *A quasi-ideal Q of a Γ-semigroup S without zero is minimal if and only if Q is a Γ-subgroup of S.*

Proof. If Q is a minimal quasi-ideal of the Γ-semigroup S, then by Theorem 3.2 all elements of Q are $\mathcal{H}$-equivalent. Thus, for two elements a, b of Q and every $\gamma \in \Gamma$, the elements $a, b, a\gamma b$ all belong to the same $\mathcal{H}$-class H of S. Now from the Green's Theorem we infer that H is a subgroup of every S_γ, $\gamma \in \Gamma$, therefore Q is also a Γ-subgroup of S since Q is a Γ-subsemigroup of S.

Conversely, let the quasi-ideal Q of a Γ-semigroup S be a Γ-subgroup of S. If $Q' \subseteq Q$ is another quasi-ideal of S, then by Theorem 1.1 Q' can not be a proper quasi-ideal of the Γ-subgroup Q, therefore $Q = Q'$. This means that Q is a minimal quasi-ideal of S. □

Definition 5. An element a of a Γ-semigroup S is called cancellable if for every two elements $b, c \in S$, and every $\gamma \in \Gamma$

$$a\gamma b = a\gamma c \Rightarrow b = c \text{ and } b\gamma a = c\gamma a \Rightarrow b = c.$$

The following theorem mimics a result for plain semigroups (Theorem 5.9 of [7]).

Theorem 3.4. *If a* Γ*-semigroup* S *without zero has a cancellable element contained in a minimal quasi-ideal* Q *of* S*, then* S *is a* Γ*-group.*

Proof. By Theorem 3.3, Q is a Γ-subgroup of S. Let $e = e\gamma e$ be the unit of Q_γ and $a \in Q$ a cancellable element of S. Since e is the unit, we have $e\gamma a = a$. For any element $s \in S$ we have $(s\gamma e)\gamma a = s\gamma(e\gamma a) = s\gamma a$, whence $s\gamma e = s$. Dually we obtain $e\gamma s = s$. Thus e is the unit of S_γ. Since $e \in Q$, we can write

$$s = s\gamma e = e\gamma s \in S\Gamma Q \cap Q\Gamma S \subseteq Q,$$

that is, $S = Q$. Thus $S_\gamma = Q_\gamma$ is a Γ-group, whence from Corollary 1.1, S is a Γ-group. □

For Γ-semigroups, similarly with plain semigroups [1] and rings [4], we have the following theorem.

Theorem 3.5. *Let* a, b *be two elements of a* Γ*-semigroup* S *without zero such that* $a\mathcal{D}b$*. Then the principal quasi-ideal* $(a)_q$ *is minimal if and only if the same holds for* $(b)_q$*.*

Proof. In view of the duality between $\mathcal{R}$ and $\mathcal{L}$, it is sufficient to prove the theorem for relation $\mathcal{R}$ in place of $\mathcal{D}$. If $b = a\gamma s$ then from Proposition 2.1 the mapping

$$\begin{array}{rccc} f: & (a)_l \cap (a)_r & \rightarrow & (b)_l \cap (b)_r \\ & x & \mapsto & x\gamma s \end{array}$$

is one to one and maps H_a onto H_b. Suppose that $(a)_q$ is a minimal quasi-ideal. From Theorem 3.1 and Theorem 3.2 we have

$$(a)_l \cap (a)_r = (a)_q = H_a.$$

Let y be an arbitrary element of the quasi-ideal $(b)_q = (b)_l \cap (b)_r$. There is an element $x \in (a)_q$ such that $y = x\gamma s$. So $x \in H_a$ and then $y = x\gamma s \in H_b$. Hence $(b)_q = H_b$ and then by Theorem 3.2 the quasi-ideal $(b)_q$ is minimal. □

References

[1] Clifford, A.H., *Remarks on 0-minimal quasi-ideals in semigroups*, Semigroup Forum 1978, 16, 183-196

[2] Dutta, T. K. and Charterjee, T. K., *Green's equivalences on* Γ*-semigroups*, Bull. Cal. Math. Soc. 80 (1987), 30-35

[3] Grillet, M. P., *Green's relations in a semiring* Port. Math. 1970, 29, 181-195

[4] Petro. P., *Green's relations and minimal quasi-ideals in rings*, Comm. Algebra Vol. 30, No. 10, 4677-4688, 2002

[5] Saha N. K., *On* Γ*-semigroups*, Bull. Cal. Math. Soc. 79 (1987), 331-335

[6] Sen M. K., Saha N. K., *On* Γ*-semigroups I*, Bull. Cal. Math. Soc. 78 (1986), 180-186

[7] Steinfeld O., *Quasi-ideals in semigroups and rings*, Akademiai Kiado, Budapest 1978

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