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M-SP-Projective Modules

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Abstract

In this paper we studied M-small principally projective modules (In short, M-sp-projective modules) which is the dual notion of M-sp-injective modules and generalization of M-projective modules. We provide an example of a M-sp-projective modules which is not M-projective. We also study some properties related to Summand Intersection Property(SIP) and Hopfian modules.

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1 Introduction:

Through out the paper rings are associative with identity and modules are unitary right R-modules. Let M and N be two R-modules. A module N is called M-generated, if there is an epimorphism $M^{(I)} \longrightarrow N$ for some index set I, if I is finite then N is called finitely M-generated. In particular, a submodule N of M is called an M-cyclic submodule of M, if it is isomorphic to M/L for some submodule L of M, or equivalently to say that there exists an epimorphism from M to N. A submodule K of an R-module M is said to be small in M, written as $K << M$, if for every submodule $L \subset M$ with

$K + L = M$ implies $L = M$. A non-zero R-module M is called hollow, if every proper submodule of it is small in M. In [5] Sanh et.al gave the idea of M-principally injective modules. A module N is called M-principally injective, if every R-homomorphism from an M-cyclic submodule of M to N can be extended to an R-homomorphism from M to N. A module M is called quasi principally (or semi) injective, if it is M-principally injective. In [6] Tansee and Wongwai studied the M-principally projective modules. In this paper we introduce the notion of M-small principally projective modules and quasi small principally projective modules which is abbreviated as M-sp-projective and quasi-sp-projective modules. In this paper we study M-sp-projective module and M-sp-projective rings. For undefined notation and terminology see [1].

2 M-Small Principally Projective Modules

Definition 2.1 *A module N is called M-small principally projective module, if for every small M-cyclic submodule K of M any homomorphism from N to K can be lifted to a homomorphism from N to M. If M is M-small principally projective module (In short M-sp-projective) then it is called quasi-small principally projective module and the ring R is called small principally projective ring, if R_R as a right small principally projective R-module.*

We now give some example of M-sp-projective modules.

Example 2.2 (1) *$\mathbb{Z}/4Z$ is $\mathbb{Z}$-sp-projective module but not $\mathbb{Z}$-projective.*
(2) *Every M-principally projective module is M-sp-projective module.*
(3) *Every semi-simple module is M-sp-projective module.*

Lemma 2.3 *Every direct summand of M-sp-projective module is an M-sp-projective module.*

Lemma 2.4 *Let $L \subseteq N \subseteq M$ be R-modules:*
(1) *If $L << N$, $N << M$ then $L << M$;*
(2) *If $L << N$ and $N \subseteq^{\oplus} M$ then $L << M$.*

Lemma 2.5 *Let $L \subseteq N \subseteq M$ be R-modules. If L is an N-cyclic submodules of N and N is an M-cyclic submodule of M, then L is an M-cyclic submodule of M.*

Proof : Straightforward.

Lemma 2.6 *Let $L \subseteq N \subseteq M$ be R-modules. If L is small N-cyclic submodules of N and N is small M-cyclic submodule of M then L is small M-cyclic submodule of M.*

Proof : Applying lemmas (2.4) and (2.5) we get the proof.

Proposition 2.7 (1) *Let A, B and X be an R-modules with $A \cong B$. If A is X-sp-projective module then B is X-sp-projective module;*
(2) *Let X, Y and M be R-modules with $X \cong Y$. If M is Y-sp-projective module then M is X-sp-projective module.*

Proof : Straightforward.

Corollary 2.8 *Let N and M be two R-modules. N is M-sp-projective module if and only if N is X-sp-projective for any small M-cyclic submodule X of M.*

Proposition 2.9 *$\oplus_{i=1}^{n} M_i$ is M-sp-projective module if and only if each M_i is M-sp-projective module.*

Corollary 2.10 *Every direct summand of M-sp-projective module is an M-sp-projective module.*

Remark 2.11 *Projective $\Rightarrow$ quasi-projective $\Rightarrow$ quasi-principally projective $\Rightarrow$ quasi-sp-projective.*

Here, we study some property of quasi-sp-projective modules.
A module M is said to have the Summand Intersection Property (SIP) if the intersection of any two direct summands of M is a direct summand of M.

Proposition 2.12 *If a quasi-sp-projective module M has the SIP, then for any direct summands A and B of M, $A + B$ is sp-projective module.*

Proof : Suppose that M is quasi-sp-projective and has the SIP and let A and B be any direct summands of M. Let $M = (A \cap B) \oplus K$ for some $K \subseteq M$. Then $A = (A \cap B) \oplus (A \cap K)$, $B = (A \cap B) \oplus (B \cap K)$ and $A + B = (A \cap B) \oplus (A \cap K) \oplus (B \cap K)$. By hypothesis $A \cap B$, $A \cap K$ and $B \cap K$ are direct summands of M and so they are sp-projective and hence $A + B$ is sp-projective.

Corollary 2.13 *If a quasi-p-projective module M has the SIP, then for any direct summands A and B of M, $A + B$ is p-projective module.*

Corollary 2.14 *If a quasi-projective module M has the SIP, then for any direct summands A and B of M, $A+B$ is projective module.*

The following proposition is the generalization of Proposition 1 of Varadarajan [7].

Proposition 2.15 *Let M be quasi-sp-projective module. Assume that either $dimM < \infty$ or $CodimM < \infty$. Then for every integer $n \geq 1$, M^n is Hopfian.*

Proof : Since dim M^n = n(dim M), Codim M^n = n(Codim M) and since M is quasi-sp-projective implies that M^n is quasi-sp-projective we see that M^n satisfies the hypothesis. It is suffices to prove that M is Hopfian.
Let $f : M \to M$ be any surjective endomorphism of M. Due to quasi-sp-projectivity of M there exists a map $s : M \to M$ with $f \circ s = Id_M$. Hence, $M \cong M \oplus kerf$. So, $dimM = dimM + kerf$ and $CodimM = CodimM + kerf$. If $dimM < \infty$ or $CodimM < \infty$ implies that $kerf = 0$ and so f is a monomorphism. Hence, M is Hopfian.

Corollary 2.16 *Let M be quasi-p-projective module. Assume that either $dimM < \infty$ or $CodimM < \infty$. Then for every integer $n \geq 1$, M^n is Hopfian.*

Corollary 2.17 *Let M be quasi-projective module. Assume that either $dimM < \infty$ or $CodimM < \infty$. Then for every integer $n \geq 1$, M^n is Hopfian.*

3 Semi co-Hopfian and Semi Hopfian Modules

In this section we study some properties of semi co-Hopfian and semi Hopfian modules related to quasi principally injective and quasi principally projective modules. In (2008) Aydogdu and Ozcan [4] gave the idea of semi co-Hopfian and semi Hopfian modules.
A module M is called **Semi co-Hopfian (resp. Semi Hopfian)** if any injective (surjective)endomorphism of M has direct summand image (resp. kernel).
A module M is called Epi-retractable, if every submodule of M is an M-cyclic submodule M.

Proposition 3.1 *Every quasi principally injective module is semi co-Hopfian.*

Proof: Since every quasi principally injective module has (C_2) and a module has this condition is semi co-Hopfian see [[4], Lemma(2.1)] M is semi co-Hopfian modules.

Proposition 3.2 *Every quasi principally projective module is semi Hopfian.*

Proof : Proof of this proposition is dual to the proof given in above proposition.

Proposition 3.3 *Let M be Epi-retractable, and every submodule of M is M-principally injective, then M is semi co-Hopfian.*

Proof : Let $f \in S = End(M_R)$. By hypothesis, $f(M)$ is M- principally injective. Therefore, it is a direct summand of M, it follows that M is semi co-Hopfian.

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References

[1] F. W. Anderson, K. R. Fuller, Rings and Categories of Modules, New York: Springer Verlag, 1993.

[2] V.Kumar, A.J.Gupta, B.M.Pandeya and M. K.Patel, M-SP-Injective Modules, communicated.

[3] S. H. Mohamed, B. J. Müller, Continuous and Discrete Modules, London Math. Soc. Lecture Notes Series 147, Cambridge 1990.

[4] P. Aydogdu and A. C. Ozcan, Semi co-Hopfian and Semi Hopfian Modules, East West J. Math., 10(1), 2008, 57–72.

[5] N. V. Sanh, K. P. Shum, S. Dhompongsa and S. Wongwai, On Quasi Principally Injective Modules, Algebra Coll. 6:3 (1999), 269–276.

[6] H. Tansee and S. Wongwai, A note on Semi projective modules, Kyungpook Math. J., 42(2002), 369–380.

[7] K. Varadarajan, Properties of Endomorphism Rings, Acta Math. Hungar., 74(1-2) (1997), 83–92.

[8] G. V. Wilson, Modules with the Summands Intersection Property, Comm. Algebra, 14(1986), 21–38.

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