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Intuitionistic Fuzzy Bi-Ideals and Regularity in Near-Rings

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Abstract

We study the intuitionistic fuzzification of the notion of bi-ideals in near-rings. We give characterizations of intuitionistic fuzzy bi-ideals in near-rings. Also we give conditions for a near-ring with unity to be strongly regular in terms of intuitionistic fuzzy set.

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1 Introduction

The fundamental concept of fuzzy set was introduced by Zadeh [10]. The study of fuzzy algebraic structures has started with the introduction of the concept of fuzzy subgroups in the pioneering paper of Rosenfeld [7]. Abou-Zaid [1] introduced the concept of fuzzy subnear-rings and studied fuzzy ideals in near-rings. After the introduction of the concept of fuzzy sets by Zadeh, several researches were conducted on the generalisation of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by Atanassov [2] as a generalization of the notion of fuzzy set. Mason [4] introduced the notion of strong regularity of a near-ring.

In this paper we study the intuitionistic fuzzification of the notion of bi-ideals in near-rings. We show that every intuitionistic fuzzy bi-ideal of a near-ring is an intuitionistic fuzzy subnear-ring. We give characterizations of intuitionistic fuzzy bi-ideals in near-rings. We give conditions for a near-ring with unity to be strongly regular in terms of intuitionistic fuzzy set.

2 Preliminary Notes

Throughout this paper N stands for a right zero-symmetric near-ring. For basic terminology in near-ring we refer to Pilz [5].

Given two subsets A and B of N, the product $AB = \{ab |\ a \in A,\ b \in B\}$ as defined in 2.56 of Pilz [5].

A subgroup B of N is a bi-ideal [8] of N if $BNB \subseteq B$.

Let X be a non-empty set. A mapping $\mu : X \to [0, 1]$ is a fuzzy set in X. The complement of μ, denoted by μ^c, is the fuzzy set in X given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$. For any $I \subseteq X$, χ_I denote the characteristic function of I.

For any fuzzy set μ in X and $r \in [0, 1]$, we define two sets, $U(\mu, r) = \{x \in X |\ \mu(x) \geq r\}$ and $L(\mu, r) = \{x \in X |\ \mu(x) \leq r\}$, which are called an upper and lower r-level cut of μ respectively and can be used to the characterization of μ.

A fuzzy set μ in N is a fuzzy subnear-ring of N if for all $x, y \in N$,
(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ and
(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

A fuzzy set μ in N is a fuzzy bi-ideal of N if
(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in N$ and
(ii) $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in N$.

3 Intuitionistic fuzzy sets and bi-ideals

Definition 3.1. *[2] An intuitionistic fuzzy set A in a non-empty set X is an object having the form $A = \{(x,\ \mu_A(x),\ \lambda_A(x)) |\ x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\lambda_A : X \to [0, 1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$.*

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) |\ x \in X\}$.

Definition 3.2. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in N is an intuitionistic fuzzy subnear-ring of N if for all $x, y \in N$,*
(i) $\mu_A(x - y) \geq min\{\mu_A(x), \mu_A(y)\}$
(ii) $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$
(iii) $\lambda_A(x - y) \leq max\{\lambda_A(x), \lambda_A(y)\}$
(iv) $\lambda_A(xy) \leq max\{\lambda_A(x), \lambda_A(y)\}$.

Definition 3.3. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in N is an intuitionistic fuzzy bi-ideal of N if for all $x, y, z \in N$,*
(i) $\mu_A(x - y) \geq min\{\mu_A(x), \mu_A(y)\}$
(ii) $\mu_A(xyz) \geq min\{\mu_A(x), \mu_A(z)\}$
(iii) $\lambda_A(x - y) \leq max\{\lambda_A(x), \lambda_A(y)\}$
(iv) $\lambda_A(xyz) \leq max\{\lambda_A(x), \lambda_A(z)\}$.

Lemma 3.4. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in N is an intuitionistic fuzzy bi-ideal of N if and only if the fuzzy sets μ_A and λ_A^c are fuzzy bi-ideals of N.*

Proof. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy bi-ideal of N, then clearly μ_A is a fuzzy bi-ideal of N.
For all $x, y \in N$,

$$\begin{aligned} \lambda_A^c(x - y) &= 1 - \lambda_A(x - y) \\ &\geq 1 - \max\{\lambda_A(x), \lambda_A(y)\} \\ &= \min\{1 - \lambda_A(x), 1 - \lambda_A(y)\} \\ &= \min\{\lambda_A^c(x), \lambda_A^c(y)\}. \end{aligned}$$

For all $x, y, z \in N$,

$$\begin{aligned} \lambda_A^c(xyz) &= 1 - \lambda_A(xyz) \\ &\geq 1 - \max\{\lambda_A(x), \lambda_A(z)\} \\ &= \min\{1 - \lambda_A(x), 1 - \lambda_A(z)\} \\ &= \min\{\lambda_A^c(x), \lambda_A^c(z)\}. \end{aligned}$$

Thus λ_A^c is a fuzzy bi-ideal of N.

Conversely, suppose that μ_A and λ_A^c are fuzzy bi-ideals of N, then clearly the conditions (i) and (ii) of Definition 3.3 are valid.
Now for all $x, y \in N$,

$$\begin{aligned} 1 - \lambda_A(x - y) &= \lambda_A^c(x - y) \\ &\geq \min\{\lambda_A^c(x), \lambda_A^c(y)\} \\ &= 1 - \max\{\lambda_A(x), \lambda_A(y)\}. \end{aligned}$$

Therefore $\lambda_A(x-y) \leq max\{\lambda_A(x), \lambda_A(y)\}$.
For all $x, y, z \in N$,

$$\begin{aligned} 1-\lambda_A(xyz) &= \lambda_A^c(xyz) \\ &\geq \min\{\lambda_A^c(x), \lambda_A^c(z)\} \\ &= 1-\max\{\lambda_A(x), \lambda_A(z)\}. \end{aligned}$$

Therefore $\lambda_A(xyz) \leq \max\{\lambda_A(x), \lambda_A(z)\}$.
Thus $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy bi-ideal of N. □

Theorem 3.5. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in N is an intuitionistic fuzzy bi-ideal of N if and only if $A = (\mu_A, \mu_A^c)$ and $A = (\lambda_A^c, \lambda_A)$ are intuitionistic fuzzy bi-ideals of N.*

Proof. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy bi-ideal of N, then $\mu_A = (\mu_A^c)^c$ and λ_A^c are fuzzy bi-ideals of N, from Lemma 3.4. Therefore $A = (\mu_A, \mu_A^c)$ and $A = (\lambda_A^c, \lambda_A)$ are intuitionistic fuzzy bi-ideals of N.

Conversely, if A and A are intuitionistic fuzzy bi-ideals of N, then the fuzzy sets μ_A and λ_A^c are fuzzy bi-ideals of N. Therefore $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy bi-ideal of N. □

Theorem 3.6. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in N is a fuzzy bi-ideal of N if and only if all the non-empty sets $U(\mu_A, r)$ and $L(\lambda_A, t)$ are bi-ideals of N for all $r \in Im(\mu_A)$ and $t \in Im(\lambda_A)$ respectively.*

Proof. Suppose that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy bi-ideal of N. For $x, y \in U(\mu_A, r)$, we have $\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq r$. Therefore $x - y \in U(\mu_A, r)$. Let $x, z \in U(\mu_A, r)$ and $y \in N$. Then $\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z)\} \geq r$ and so $xyz \in U(\mu_A, r)$. Hence $U(\mu_A, r)$ is a bi-ideal of N for all $r \in \mathrm{Im}(\mu_A)$. Similarly we can show that $L(\lambda_A, t)$ is also a bi-ideal of N for all $t \in \mathrm{Im}(\lambda_A)$.

Conversely suppose that $U(\mu_A, r)$ and $L(\lambda_A, t)$ are bi-ideals of N for all $r \in \mathrm{Im}(\mu_A)$ and $t \in \mathrm{Im}(\lambda_A)$ respectively. Suppose that $x, y \in N$ and $\mu_A(x-y) < \min\{\mu_A(x), \mu_A(y)\}$. Choose r such that $\mu_A(x-y) < r < \min\{\mu_A(x), \mu_A(y)\}$. Then we get $x, y \in U(\mu_A, r)$ but $x - y \notin U(\mu_A, r)$, a contradiction. Hence $\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y)\}$. A similar argument shows that $\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z)\}$ for all $x, y, z \in N$. Likewise we can show that $\lambda_A(x-y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$ and $\lambda_A(xyz) \leq \max\{\lambda_A(x), \lambda_A(z)\}$. Hence $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy bi-ideal of N. □

Theorem 3.7. *A non-empty set B of N is a bi-ideal of N if and only if $A = (\chi_B, \chi_B^c)$ is an intuitionistic fuzzy bi-ideal of N.*

Proof. Straightforward □

4 Intuitionistic fuzzy bi-ideals and regularity

A near-ring N is regular if for every $a \in N$ there is an $x \in N$ such that $a = axa$.

A near-ring N is strongly regular if for every $a \in N$ there is an $x \in N$ such that $a = xa^2$.

Lemma 4.1. *[6] Let N be a strongly regular near-ring. If $a = xa^2$ for some $a, x \in N$, then $a = axa$ and $ax = xa$.*

Theorem 4.2. *Every intuitionistic fuzzy bi-ideal in a regular near-ring is an intuitionistic fuzzy subnear-ring of N.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy bi-ideal of N and let $a, b \in N$. Since N is regular, there exists $x \in N$ such that $a = axa$. Then $\mu_A(ab) = \mu_A((axa)b) = \mu_A(a(xa)b) \geq \min\{\mu_A(a), \mu_A(b)\}$ and $\lambda_A(ab) = \lambda_A((axa)b) = \lambda_A(a(xa)b) \leq \max\{\lambda_A(a), \lambda_A(b)\}$. Thus $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy subnear-ring of N. □

Lemma 4.3. *If N is strongly regular, then for each $a \in N$ there is some $y \in N$ such that $a = a^2ya^2$.*

Proof. Since N is strongly regular, for each $a \in N$ there is an $x \in N$ such that $a = xa^2$. Then by Lemma 4.1, $a = axa$ and $ax = xa$. Therefore $a = a(ax) = a^2x$ and hence $a = axa = (a^2x)x(xa^2) = a^2ya^2$. □

Proposition 4.4. *Let N be a strongly regular near-ring. Then for every intuitionistic fuzzy bi-ideal $A = (\mu_A, \lambda_A)$ in N, we have $\mu_A(x) = \mu_A(x^2)$ and $\lambda_A(x) = \lambda_A(x^2)$ for all $x \in N$.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy bi-ideal of N and let $x \in N$. Since N is strongly regular, there exists $y \in N$ such that $x = x^2yx^2$. Then $\mu_A(x) = \mu_A(x^2yx^2) \geq \min\{\mu_A(x^2), \mu_A(x^2)\} = \mu_A(x^2) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$ and
$\lambda_A(x) = \lambda_A(x^2yx^2) \leq \max\{\lambda_A(x^2), \lambda_A(x^2)\} = \lambda_A(x^2) \leq \max\{\lambda_A(x), \lambda_A(x)\} = \lambda_A(x)$. □

Proposition 4.5. *[8] If B is a bi-ideal of N, then Bn and $n'B$ are bi-ideals of N where $n, n' \in N$ and n' is a distributive element in N.*

Theorem 4.6. *Let N be a near-ring with identity 1. If every intuitionistic fuzzy bi-ideal $A = (\mu_A, \lambda_A)$ in N satisfies either $\mu_A(x) = \mu_A(x^2)$ or $\lambda_A(x) = \lambda_A(x^2)$ for all $x \in N$, then N is strongly regular.*

Proof. Suppose that every intuitionistic fuzzy bi-ideal $A = (\mu_A, \lambda_A)$ of N satisfies either $\mu_A(x) = \mu_A(x^2)$ or $\lambda_A(x) = \lambda_A(x^2)$ for all $x \in N$. Let $a \in N$. Then $B = Na^2$ is a bi-ideal of N. Therefore $A = (\chi_B, \chi_B^c)$ is an intuitionistic fuzzy bi-ideal of N, by Theorem 3.7. Since $a^2 \in B = Na^2$, $\chi_B(a^2) = 1$. But by hypothesis $\chi_B(a) = \chi_B(a^2)$. Therefore $\chi_B(a) = 1$ and so $a \in B = Na^2$. Thus $a = xa^2$ for some $x \in N$. Hence N is strongly regular. $\square$

A near-ring N is B-simple [8] if it has no proper bi-ideals. That is the only bi-ideals of N are $\{0\}$ and N itself.

Lemma 4.7. *Let N be a near-ring. If N is B-simple, then $Na = N$ and $a'N = N$ for all $a(\neq 0)$, $a'(\neq 0) \in N$, where a' is a distributive element in N.*

Theorem 4.8. *Let N be a distributive near-ring. If N is regular and B-simple, then every intuitionistic fuzzy bi-ideal of N is a constant on $N \backslash \{0\}$.*

Proof. Suppose that N is regular and B-simple. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy bi-ideal of N. Consider the set

$$K = \{c(\neq 0) \in N |\, c = c^2\}.$$

Let $a(\neq 0) \in N$. Since N is regular, there exists $x \in N$ such that $a = axa$. Then $ax = (axa)x = (ax)^2$ and $xa = x(axa) = (xa)^2$. Hence $ax, xa \in K$ and hence $K \neq \phi$. Let $b \in K$. First we show that $\mu_A(b) = \mu_A(c)$ and $\lambda_A(b) = \lambda_A(c)$ for all $c \in K$. Choose c in K. Since N is B-simple, $Nb = N$ and $bN = N$ for all $b(\neq 0) \in N$. Since $c \in N$, $c \in Nb = bN$. Therefore $c = bx = yb$ for some $x(\neq 0)$, $y(\neq 0) \in N$. Hence $c^2 = b(xy)b$. Since A is an intuitionistic fuzzy bi-ideal of N,

$$\begin{aligned} \mu_A(c^2) = \mu_A(b(xy)b) \geq min\{\mu_A(b), \mu_A(b)\} = \mu_A(b) \\ \lambda_A(c^2) = \lambda_A(b(xy)b) \leq max\{\lambda_A(b), \lambda_A(b)\} = \lambda_A(b) \end{aligned} \tag{1}$$

Since $c \in K$, $c = c^2$. Hence $\mu_A(c^2) = \mu_A(c)$ and $\lambda_A(c^2) = \lambda_A(c)$. Therefore from (1), $\mu_A(c) \geq \mu_A(b)$ and $\lambda_A(c) \leq \lambda_A(b)$. Since $Nc = N = cN$, by a similar argument as above, we get $\mu_A(c) \leq \mu_A(b)$ and $\lambda_A(c) \geq \lambda_A(b)$. This shows that $A = (\mu_A, \lambda_A)$ is a constant on K. Since ax, $xa \in K$, by previous arguments we have $\mu_A(ax) = \mu_A(b) = \mu_A(xa)$ and $\lambda_A(ax) = \lambda_A(b) = \lambda_A(xa)$. Since

$(ax)a(xa) = (axa)xa = axa = a$,

$$\begin{aligned}\mu_A(a) &= \mu_A((ax)a(xa)) \\ &\geq \min\{\mu_A(ax), \mu_A(xa)\} \\ &= \min\{\mu_A(b), \mu_A(b)\} \\ &= \mu_A(b) \qquad\qquad \text{and} \\ \lambda_A(a) &= \lambda_A((ax)a(xa)) \\ &\leq \max\{\lambda_A(ax), \lambda_A(xa)\} \\ &= \max\{\lambda_A(b), \lambda_A(b)\} \\ &= \lambda_A(b).\end{aligned}$$

Since $b \in Na = aN$, $b = xa = ay$ for some $x(\neq 0)$, $y(\neq 0) \in N$. Therefore $b^2 = a(yx)a$. Hence $\mu_A(b^2) = \mu_A(a(yx)a) \geq \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a)$ and $\lambda_A(b^2) = \lambda_A(a(yx)a) \leq \max\{\lambda_A(a), \lambda_A(a)\} = \lambda_A(a)$. Since $b \in K$, we get $b = b^2$. Then it follows that $\mu_A(b) = \mu_A(b^2) = \mu_A(a)$ and $\lambda_A(b) = \lambda_A(b^2) = \lambda_A(a)$. Hence $A = (\mu_A, \lambda_A)$ is a constant on $N\backslash\{0\}$. □

References

[1] S. Abou-Zaid, On fuzzy subnear-rings and ideals, Fuzzy Sets and Sys. 44(1991), 139-146.

[2] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Syst. 20(1986), 87-96.

[3] P. Dheena and S. Kumaressane, Generalization of $(\epsilon, \epsilon \vee q)$-fuzzy subnear-rings and ideals, Iranian Journal of Fuzzy Systems, Vol. 5, No.1 (2008), 79-97.

[4] G. Mason, Strongly regular near-rings, Proc. Edinburgh Math. Soc., 23 (1980), 27-35.

[5] G. Pilz, Near-rings, North-Holland Publishing Company, Amsterdam, New york, Oxford, 1983.

[6] Y. V. Reddy and C. V. L. N. Murty, On strongly regular near-rings, Proc. Edinburgh Math. Soc., 27 (1984), 61-64.

[7] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35(1971), 512-517.

[8] T. Tamizh Chelvam and N. Ganesan, On bi-ideals of near-rings, Indian J. Pure and Appl. Math. 18(11) (1987) 1002-1005.

[9] Young Bae Jun, Intuitionistic fuzzy bi-ideals of odered semigroups, Kyungpook Math. J. 45(2005), 527-537.

[10] L. A. Zadeh, Fuzzy sets, Inform and control, 8(1965), 338-353.

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