International Journal of Algebra, Vol. 5, 2011, no. 10, 491 - 498

A Note on Dedekind Modules

Hanni Garminia, Pudji Astuti and Irawati

Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung, Indonesia
hani@math.itb.ac.id

Abstract

This article studies relations among a Dedekind module, an HNP (hereditary, Noetherian and prime) module, and an order of a module. The concept of a module order will be introduced as a generalization of the concept of a ring order. The obtained result will then be employed to investigate a relation between a Dedekind algebra and a strongy-coprime cohereditary coalgebra.

Mathematics Subject Classification: 13C99, 13F05

Keywords: HNP modules, maximal order, cohereditary coalgebras

1 Introduction

Let R be a commutative ring with unit. The concept of a Dedekind module was introduced and studied in Naoum and Al-Alwan [1] as a generalization of the Dedekind ring concept. Following this introduction and study, several researchers were interested in investigating the structure and properties of this class of modules. For example, the same authors, Naoum and Al-Alwan [2] showed the characterization of a multiplicative faithful module being Dedekind is a π-module. Alkan, Saraç and Tiraş [8], obtained some characterizations of Dedekind modules and Dedekind domains through the introduction of integrally closed modules concept. In another report, Saraç, Smith and Tiraş [3] gave a characterization of a class of finitely generated torsion free Dedekind modules. Note that most of those results dealt with a class of multiplicative modules and faithful modules.

Proceeding those investigations, in this article we study properties of a Dedekind module connected to a hereditary Noetherian and prime (HNP) module and an order of a module. McConnell and Robson [10] documented the following interrelation; a ring R is a Dedekind prime ring if and only if it is an

HNP ring and a maximal order. In this note we extend this result concerning Dedekind modules. The concept of a module order will be introduced as a generalization of the ring concept. Then, the obtained properties will be employed to investigate a relation between a Dedekind algebra and a strongy-coprime cohereditary coalgebra.

To begin with, notation M stands for an R-module. Let S be a set of regular elements of R, RS^{-1} be the total quotient ring, and

$$T = \{t \in S | tm = 0 \text{ for some } m \in M \text{ implies } m = 0\}.$$

It is clear that T is a multiplicative subset of S. Hence we have the quotient ring RT^{-1}. For any N, an R-submodule of M, let $N' = \{x \in RT^{-1} | xN \subset M\}$. Then N' is an R-submodule of RT^{-1} satisfying $R \subset N'$ and $N'N \subset M$. The R-submodule N is called invertible in M if $NN' = M$. According to Naoum and Al-Alwan [1], an R-module M is called a Dedekind module if each non-zero submodule of M is invertible in M.

2 DEDEKIND MODULES AND HNP MODULES

In this section we investigate properties of Dedekind modules related to HNP modules. First, we investigate the relation between Dedekind modules and prime modules. The module M is called prime if for every non zero submodule K of M, $ann(K) = ann(M)$. We obtain that every Dedekind module is a prime module. Thus primeness is a necessary condition for Dedekind property, of a ring as well as a module.

Theorem 2.1 *If M is a Dedekind R-module then M is a prime R-module.*

Proof Since M is a Dedekind module then $M = KK'$ for every K non zero submodule of M. It is clear that $ann(M) \subseteq ann(K)$. Now let $a \in ann(K)$. It means $aK = 0$. This implies that $aM = a(KK') = (aK)K' = 0$. Hence $ann(K) \subseteq ann(M)$ ■

Naoum and Al Alwan was studying the relation between M, a Dedekind R-module, and $R/ann_R(M)$ through $End(M)$. In this paper we studied the relation directly.

Corollary 2.2 *If M is a Dedekind R-module then $R/ann_R(M)$ is an integral domain.*

Proof Since M is a Dedekind R-module, according to Theorem 2.1 then M is a prime R-module. This implies that $R/ann_R(M)$ is a prime ring. Then, the commutative property of R results in $R/ann_R(M)$ being an integral domain. ■

Second, we shall investigate the relation between Dedekind modules and Noetherian modules. Alkan et. all. [8] showed that any invertible submodule of a finitely generated module is finitely generated. In this paper we work without finitely generated condition. To prove our result we shall use the following theorem from Naoum and Al-Alwan [1] which relates a Dedekind module with its underlying ring.

Theorem 2.3 *[1] Let M be a faithful projective R-module. If M is a Dedekind R-module then M is finitely generated and R is a Dedekind domain.*

Note that a prime ring is a generalization of a noncommutative domain. It can be shown that every prime ring is faithful. Further, our goal is generalizing part of the Dedekind prime rings properties to modules. Since a Dedekind prime ring is not always a domain, in this section we investigates Dedekind modules without faithful condition.

Now we recall some familiar properties on modules over a factor ring which are a minor modification of 5.2 in Passman [5].

Lemma 2.4 *Let I be an ideal of R. If M is an R/I-module then M is an R-module. Further, if N is an R-submodule of M then N is an R/I module and if $f : M \to M'$ is an R/I-homomorphism then $f : M \to M'$ is an R-homomorphism.*

Lemma 2.5 *If M is an R-module then M is an $R/ann_R(M)$-module. Further, if N is an $R/ann_R(M)$-submodule of M then N is an R-module and if $f : M \to M'$ is an R-homomorphism then $f : M \to M'$ is an $R/ann_R(M)$-homomorphism.*

These two lemmas avail us studying properties of $R/ann_R(M)$-modules. That is, we can investigate faithful, projective, Dedekind and Noetherian $R/ann_R(M)$-modules.

Lemma 2.6 *An R-module M is a faithful $R/ann_R(M)$-module.*

Proof Let $\overline{R} = R/ann_R(M)$ and $\overline{r} = r + ann_R(M) \in ann_{\overline{R}}(M)$. For all $m \in M$, $0 = \overline{r}m = rm$. Hence $r \in ann_R(M)$ implies $\overline{r} = 0$. This means that $ann_{\overline{R}}(M) = 0$ ■

Lemma 2.7 *If M is a projective R-module then M a projective $R/ann_R(M)$ module.*

Proof Let $f : M \to B$ be an $R/ann_R(M)$-homomorphism and $g : A \to B$ be an $R/ann_R(M)$-epimorphism. According to Lemma 2.4, $f : M \to B$ is an R-homomorphism and $g : A \to B$ is an R-epimorphism. Since M is projective there exists $h : M \to A$ an R-homomorphism such that $gh = f$. Meanwhile, applying Lemma 2.5, we obtain $h : M \to A$ is an $R/ann_R(M)$-homomorphism. Hence M, as an $R/ann_R(M)$-module, is projective. ■

Similar to the above lemma, that is by employing the connection between R-modules and $R/ann_R(M)$-modules we obtain the following two lemmas.

Lemma 2.8 *If M is a Dedekind R-module then M is a Dedekind $R/ann_R(M)$-module.*

Lemma 2.9 *If M is a Noetherian $R/ann_R(M)$-module then M is a Noetherian R-module*

Recall that the module M is Noetherian if all of its submodules are finitely generated. The following theorem describes a relation between a Dedekind projective module and a Noetherian module. In case M is a finitely generated torsion-free module over an integrally closed ring R, Alkan et.al. ([8] proved that Dedekind implies Noetherian. Whereas Our result includes modules containing torsion elements, and over an arbitarary commutative ring.

Theorem 2.10 *If M is a Dedekind projective R-module then M is a Noetherian R-module.*

Proof Since M is a Dedekind projective R-module, we have M is a Dedekind projective $R/ann_R(M)$-module. Further, M over $R/ann_R(M)$ is also faithful. As results, M is a finitely generated $R/ann_R(M)$-module and $R/ann_R(M)$ is a Dedekind domain. Hence M is a Noetherian $R/ann_R(M)$-module. Thus, according to Lemma 2.9, M is a Noetherian R-module. ■

The third and last part of the result in this section relates a Dedekind module and a hereditary module. Hence, we finish the proof of the main result in this section if we can show the following result. A module M is called hereditary if all of its submodules are projective. Alkan et. al. was studied the relation between invertible submodule and projective submodule over domain. In our situation, we do for arbitrary commutative rings.

Theorem 2.11 *If M is a Dedekind projective R-module then*
1. M is a hereditary RT^{-1}-module.
2. M is a hereditary R-module.

Proof 1. Since M is projective R-module then, there are $\{m_k\}_{k\in K} \subseteq M$ and $\{\varphi_k : M \to R\}_{k\in K} \subseteq Hom(M,R)$ such that $x = \Sigma_{k\in K}\varphi_k(x)m_k$ for all $x \in M$. Let N be a submodule of M. Counting on M being Dedekind, construct the following two subsets $\{n_k : n_k = t_k m_k \text{ for } t_k \in T\}_{k\in K} \subseteq N$ and $\{\phi_k : N \to RT^{-1}$ by $n \mapsto \varphi_k(n)t_k^{-1}\}_{k\in K} \subseteq Hom(N, RT^{-1})$. We shall obtain N being projective over RT^{-1} if for all $n \in N$, $n = \Sigma_{k\in K}\phi_k(n)n_k$. Thus so it is, since $\Sigma_{k\in K}\phi_k(n)n_k = \Sigma_{k\in K}\varphi_k(n)t_k^{-1}t_k m_k = n$. Hence M is a hereditary RT^{-1}-module.
2. Repeating the above process, let $\{m_k\}_{k\in K} \subseteq M$, $\{\varphi_k : M \to R\}_{k\in K} \subseteq Hom(M,R)$ such that $x = \Sigma_{k\in K}\varphi_k(x)m_k$ for all $x \in M$ and N be a submodule of M. Since N is invertible in M, we have $m_k = \Sigma_{i\in I}q_{ki}n_{ki}$ for some $q_{ki} \in N'$ and $n_{ki} \in N$. Let $H_k = \{n_{ki} | m_k = \Sigma_{i\in I}q_{ki}n_{ki}\}_{i\in I}$ and $G_k = \{\phi_k : M \to R$ by $n \mapsto \varphi_{ki}(q_{ki}n)\}_{i\in I} \subseteq Hom(M,R)$. Hence we have $H = \cup_{k\in K}H_k$ and $G = \cup_{k\in K}G_k$ such that for all $n \in N$, $n = \Sigma_{k\in K}\varphi_k(n)m_k = \Sigma_{k\in K}\varphi_k(n)\Sigma_{i\in I}q_{ki}n_{ki} = \Sigma_{k\in K}\Sigma_{i\in I}\varphi_k(n)q_{ki}n_{ki} = \Sigma_{k\in K}\Sigma_{i\in I}\phi_{ki}(n)n_{ki}$. Thus M is a hereditary R-module. ■

Summarizing those three theorems above we have a relation between Dedekind modules and HNP modules as follows.

Corollary 2.12 *Every Dedekind projective R-module is HNP.*

3 DEDEKIND MODULES AND MAXIMAL ORDERS

In this section, our goal investigates a relation between a Dedekind module and an order in a module. Let Q be a quotient ring such that R is an order in Q, and M be a Q-module. Saraç et. al. [3] introduced the concept of the order of a module in quotient ring as follows. For any R-module M, the order of M in Q is defined to be the set $O(M) = \{q \in Q | qM \subseteq M\}$. Thus the order of an R-module M in Q is an order of Q. This concept is quite difference with the concept of a ring order.

To enable us extending the above mentioned result concerning a Dedekind prime ring , an HNP ring, and a maximal order documented in McConnell and Robson [10], we need to generalize the concept of a ring order to a module order. The following are definitions of orders, equivalent orders and maximal orders for modules which can be considered as generalization of similar concepts for a ring.

Definition 3.1 *Let R be an order in Q. An R-module N is said to be an order of the R-module M if N is an R-submodule of M such that for all $m \in M$, $m = ns^{-1}$ for some $n \in N$ and $s \in S$.*

Definition 3.2 *Let R be an order in Q. Two order N_1, N_2 of M are said to be equivalent, denoted by $N_1 \sim N_2$, if there are units $q_1, q_2 \in Q$ such that $q_1 N_1 \subseteq N_2$ and $q_2 N_2 \subseteq N_1$.*

It can be shown that the binary relation $\sim$ is an equivalent relation for the collection of all orders of the module M.

Definition 3.3 *Let R be an order in Q. An R-module N_1 is said to be maximal if for any $N_1 \sim N_2$ and $N_1 \subseteq N_2$ implies $N_1 = N_2$.*

Thus, a maximal order of a module is a maximal element in the class of the module order according to the equivalent relation defined in Definition 3.2, which can be considered as a generalization of the concept of a ring maximal order.

To prove our main result we shall employ the following theorem from Saraç et.al [3] which relates a Dedekind module with its underlying ring.

Theorem 3.4 [3] *Let R be an integral domain and M be a torsionfree R-module. If M is a finitely generated Dedekind R-module then $O(M)$ is a Dedekind domain.*

Since Theorem 3.4 requires R being an integral domain as well as M being a torsionfree R-module, we confine our investigation on these conditions as shown in the following theorem.

Theorem 3.5 *Let R be an integral domain and M be a torsionfree R-module. If M is a Dedekind projective R-module than M is an HNP and a maximal order.*

Proof Since M is a Dedekind projective R-module, according to Theorem 2.10, an R-module M is Noetherian. Hence M is a finitely generated Dedekind R-module. Since R is an integral domain and M is a torsionfree R-module, according to Theorem 3.4, $O(M)$ is a Dedekind domain. It is clear that $O(M)$ is a maximal order in Q. Now, claim that M is a maximal order in R-module

M. Let N be an order in M such that $M \sim N$ and $M \subseteq N$. Then $aM \subseteq N$ and $bN \subseteq M$. For every $x \in O(M)$ we have $abxN = axbN \subseteq axM \subseteq aM \subseteq N$. This implies that $abx \in O(N)$. Hence $abO(M) \subseteq O(N)$. With the same proses, we have $abO(N) \subseteq O(M)$. Hence $aO(M) \subseteq b^{-1}O(N)$. This implies that $ab^2b^{-1}O(N) = abO(N) \subseteq O(M)$. It means that $O(M)\ b^{-1}O(N)$. Since $M \sim N$ and $M \subseteq N$, then $bN \subseteq M \subseteq N$. Hence $b \in O(N)$. For every $x \in O(M)$ we have $baxN = xbN \subseteq xM \subseteq M \subseteq N$, this implies that $bx \in O(N)$. Hence $bO(M) \subseteq O(N)$. Then $O(M) \subseteq b^{-1}O(N)$. It means that $O(M) = b^{-1}O(N)$, implies $b^{-1} \in O(M)$. Hence $N = b^{-1}bN \subseteq b^{-1}M \subseteq M$. ■

4 DEDEKIND ALGEBRA AND STRONGLY COPRIME COHEREDITARY COALGEBRA

Let C denote an R-coalgebra which, as an R-module, is locally projective. For basic properties of coalgebras we refer to Brzezinski and Wisbauer [11]. It is known that $C^* = Hom_R(C, R)$ is an R-algebra. In this section we study a relation between a Dedekind algebra and a strongly coprime cohereditary coalgebra. An R-algebra is called Dedekind if it is a Dedekind ring as well as a Dedekind R-module. The concepts of cohereditary and coprime coalgebras were introduced respectively in Nastasescu at. al. [4] and Wijayanti and Wisbauer [7].

Definition 4.1 *1. An R-coalgebra C is called a right cohereditary if every factor of C by a left C^*-submodule is injective as C^*-module.*
2. An R-coalgebra C is called a right strongly coprime if every proper C^*-submodule of C, L, is in $\sigma_{C^*}[C/L]$, where $\sigma_C^*(C/L)$ denoted the C^*-module category consisting of all C/L-subgenerated C^*-modules.

Concerning those two notions, the following properties can be found in Wijayanti and Wisbauer [7] and Garminia at. al. [6].

Theorem 4.2 [7] *Let R be self injective and cogenerator. Then, C is strongly coprime if and only if C^* ia a prime algebra.*

Theorem 4.3 [6] *The dual R-algebra C^* is hereditary if and only if the R-coalgebra C is cohereditary.*

Employing Theorem 4.2 dan Theorem 4.3 and also our result Corollary 2.12 we obtain the following result.

Theorem 4.4 *Let R be self injective and cogenerator. If C^* is a Dedekind R-algebra, then C is a strongly coprime cohereditary R-coalgebra.*

Proof It is clear that C^* is a Dedekind domain and a Dedekind R-module. Hence C^* is a hereditary prime R-algebra. According to Theorem 4.2 and Theorem 4.3 C is a strongly coprime cohereditary R-coalgebra. ∎

ACKNOWLEDGEMENTS. The authors are grateful to Prof. Robert Wisbauer for valuable discussions. This research supported by Dikti, Departemen Pendidikan Nasional, Indonesia according to Surat Perjanjian No: 467 /SP2H/PP/PP2M/VI/2009.

References

[1] A.G. Naoum and F.H. Al-Alwan, Dedekind modules, *Communications in Algebra*, **24**(2) (1996), 397-412.

[2] A.G. Naoum and F.H. Al-Alwan, Dense submodules of multiplication modules, *Communications in Algebra*, **24**(2)(1996), 413-424.

[3] B. Saraç, P.F. Smith and Y. Tiraş, On Dedekind modules, *Communications in Algebra*, **35**(5)(2007), 1533-1538.

[4] C. Natasescu, B. Torrecillas and Y.H. Zhang, Hereditary Coalgebras, *Communications in Algebra*, **24**, (1996), 1521-1528.

[5] D.S. Passman, *A course in ring theory*, Wadsworth and Brooks/ Cole Advanced Books and Software, Pacific Grove, CA, 1991.

[6] H. Garminia, P. Astuti, and Irawati, An Intertwinning of a Hereditary Algebra and a Cohereditary Coalgebra, *Journal of Fundamental Science*, **4**(1) (2008), 261-267.

[7] I. Wijayanti and R. Wisbauer, Coprime Modules and Comodules, *Communications in Algebra*, **37**(4) (2009), 1533-1538.

[8] M. Alkan, B. Saraç, Y. Tiraş, Dedekind Modules, *Communications in Algebra*, **33**(5)(2005), 1617-1625.

[9] M. Alkan and Y. Tiraş, On Invertible and Dense Submodules, *Communications in Algebra*, **32** (10)(2005), 3911 3919.

[10] M.J.C. McConnell and J.C. Robson, *Noncommutative Noetherian rings*, Gordon and Breach Science Publishers, 1987.

[11] T. Brzezinski and R. Wisbauer, *Corings and comodules*. London Mathematical Society Lecture Note Series, 309. Cambridge University Press, Cambridge, 2003.

Received: November, 2010