**International Journal of Algebra, Vol. 5, 2011, no. 10, 471 - 481**

# Permuting $f$-Triderivations on Lattices

**Abdul Rauf Khan[1] and Muhammad Anwar Chaudhry[1*]**

1-Center for Advanced Studies in Pure and Applied Mathematics,
Bahauddin Zakariya University Multan, Pakistan.
khankts@gmail.com, chaudhry@bzu.edu.pk

$*$ Corresponding author

**Abstract**

In this paper, we introduce the notion of permuting $f$-triderivation on lattices and investigate some related properties.

**Mathematics Subject Classification:** 16D30, 16S99, 16Y99, 03E72

**Keywords:** Lattice, derivation, permuting $f$-triderivation, two sided $f$-generalized $d$-derivation, trace

## 1 Introduction

Lattices play an important role in information theory and cryptanalysis [2, 4]. Recently the notion of derivation introduced in $C^{\star}$-algebra and rings has been studied by various researchers in the context of lattices (see [8] and references therein).

Zhan and Liu [12] introduced the notions of left, right and regular $f$-derivations on BCI algebras and investigated some properties of such derivations. Yilmaz and Ozturk [11] introduced the notion of $f$-derivation on a lattice and discussed some related properties.

Ozturk [5], Ozden and Ozturk [6] introduced the notion of permuting triderivations in prime and semiprime rings and proved some results. Later on Ozturk, Yazarli and Kim [7] applied this notion to lattices.

In this paper the notion of permuting $f$-triderivations, which is more general than the notion of permuting triderivations on lattices [7], is introduced and some properties of this general notion are studied.

## 2 Preliminaries

In this section we describe some definitions and results which will be used in the sequel.

**Definition 2.1** [3] A nonempty set $L$ together with the operations $\wedge$ and $\vee$ is called a lattice if it satisfies the following conditions for all $x, y, z \in L$ :
(1) $x \wedge x = x,\ x \vee x = x$.
(2) $x \wedge y = y \wedge x,\ x \vee y = y \vee x$.
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z),\ (x \vee y) \vee z = x \vee (y \vee z)$.
(4) $(x \wedge y) \vee x = x,\ (x \vee y) \wedge x = x$.
The lattice $L$ is denoted by $(L, \wedge, \vee)$.

**Definition 2.2** [3] Let $(L, \wedge, \vee)$ be a lattice. A nonempty subset $M$ of $L$ is called a sublattice of $L$ if
$a, b \in M$ implies $a \vee b \in M$ and $a \wedge b \in M$.

**Definition 2.3** [3] A lattice $(L, \wedge, \vee)$ is called a distributive lattice if it satisfies
(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$,
and
(6) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$.
In any lattice, the above conditions are equivalent.

**Definition 2.4** [3] Let $(L, \wedge, \vee)$ be a lattice. A binary relation $\leq$ on $L$ is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

**Definition 2.5** [1] A lattice $(L, \wedge, \vee)$ is called a modular lattice if it satisfies the following condition for all $x, y, z \in L$
(7) If $x \leq y$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

**Definition 2.6** [3] Let $(L, \wedge, \vee)$ be a lattice. A non-void subset $I$ of $L$ is called an ideal if it satisfies the following condition for all $x, y \in L$
(i) $x \leq y,\ y \in I \Rightarrow x \in I$,
(ii) $x, y \in I \Rightarrow x \vee y \in I$.

The following Lemma is already known [8]

**Lemma 2.7** [8] Let $(L, \wedge, \vee)$ be a lattice. Let $\leq$ be as defined in definition 2.4. Then $(L, \leq)$ is a poset for any $x, y \in L$, $x \wedge y$ is the $g.l.b$ of $\{x, y\}$ and $x \vee y$ is the $l.u.b$ of $\{x, y\}$.

**Definition 2.8** [8] Let $(L, \wedge, \vee)$ be a lattice. A function $d : L \to L$ is called a derivation on $L$ if it satisfies the following condition:
$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$ for all $x, y \in L$.

**Definition 2.9** [7] Let $(L, \wedge, \vee)$ be a lattice. A mapping $D(.,.,.) : L \times L \times L \to L$ is called a permuting mapping if
$D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) = D(z, x, y) = D(z, y, x)$ for all $x, y, z \in L$.

**Definition 2.10** [7] Let $(L, \wedge, \vee)$ be a lattice and $D(.,.,.) : L \times L \times L \to L$ be a permuting mapping. The mapping $d : L \to L$ defined by $d(x) = D(x, x, x)$ is called the trace of $D(.,.,.)$.

**Definition 2.11** [7] Let $(L, \wedge, \vee)$ be a lattice and $D(.,.,.) : L \times L \times L \to L$ be a permuting mapping. We call $D$ a joinitive mapping, if it satisfies
$D(x \vee w, y, z) = D(x, y, z) \vee D(w, y, z)$ for all $x, y, z, w \in L$.

**Definition 2.12** [7] Let $(L, \wedge, \vee)$ be a lattice and $D(.,.,.) : L \times L \times L \to L$ be a permuting mapping. We call $D$ a permuting triderivation on $L$, if it satisfies the following condition
$D(x \wedge w, y, z) = (D(x, y, z) \wedge w) \vee (x \wedge D(w, y, z))$ for all $x, y, z, w \in L$.

**Definition 2.13** [7] Let $(L, \wedge, \vee)$ be a lattice and $d$ be the trace of the permuting triderivation $D$. A mapping $G : L \to L$ is called a generalized $d$-derivation on $L$ if it satisfies the following condition
$G(x \wedge y) = (G(x) \wedge y) \vee (x \wedge d(y))$ for all $x, y \in L$.

**Definition 2.14** [11] Let $(L, \wedge, \vee)$ be a lattice and $f : L \to L$ a mapping. A mapping $d : L \to L$ is called an $f$-derivation on $L$ if it satisfies
$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y))$ for all $x, y \in L$.

We rem that if $f = 1$, the identity on $L$, then 1-derivation on $L$ is a derivation on $L$.

# 3 Permuting $f$-triderivations

In this section we describe the concept of permuting $f$-triderivations on lattices and prove our results regarding this notion.

**Definition 3.1** *Let $(L, \wedge, \vee)$ be a lattice and $D(.,.,.) : L \times L \times L \to L$ a permuting mapping. We call $D$ a permuting $f$-triderivation on $L$, if it satisfies the following condition*
$D(x \wedge w, y, z) = (D(x, y, z) \wedge f(w)) \vee (f(x) \wedge D(w, y, z))$ *for all $x, y, z, w \in L$.*

**Remark 3.2** *It is obvious that if $D$ is a permuting $f$-triderivation then it satisfies the relations $D(x, y \wedge w, z) = (D(x, y, z) \wedge f(w)) \vee (f(y) \wedge D(x, w, z))$ and $D(x, y, z \wedge w) = (D(x, y, z) \wedge f(w)) \vee (f(z) \wedge D(x, y, w))$ for all $x, y, z, w \in L$.*

We now give some examples

**Example 3.3** *Let* $(L, \wedge, \vee)$ *be a lattice and* $f : L \to L$ *a mapping satisfying* $f(x \wedge y) = f(x) \wedge f(y)$. *Let* $D(.,.,.) : L \times L \times L \to L$ *be defined by* $D(x, y, z) = (f(x) \wedge f(y)) \wedge f(z)$ *for all* $x, y, z \in L$. *Then it is easy to verify that* $D$ *is a permuting* $f$*-triderivation on* $L$.

**Example 3.4** *Let* $(L, \wedge, \vee)$ *be a lattice and* $f : L \to L$ *a mapping satisfying* $f(x \vee y) = f(x) \vee f(y)$. *Let* $D(.,.,.) : L \times L \times L \to L$ *be defined by* $D(x, y, z) = (f(x) \vee f(y)) \vee f(z)$ *for all* $x, y, z \in L$. *Then it is easy to verify that* $D$ *is not a permuting* $f$*-triderivation on* $L$.

**Example 3.5** *Let* $(L, \wedge, \vee)$ *be a lattice and* $f : L \to L$ *a mapping satisfying* $f(x \wedge y) = f(x) \wedge f(y)$. *Let* $D(.,.,.) : L \times L \times L \to L$ *be defined by* $D(x, y, z) = [(f(x) \wedge f(y)) \wedge f(z)] \wedge f(a)$ *for all* $x, y, z, a \in L$. *Then it is easy to verify that* $D$ *is a permuting* $f$*-triderivation on* $L$.

**Proposition 3.6** *Let* $(L, \wedge, \vee)$ *be a lattice and* $f : L \to L$ *a mapping. Let* $d$ *be the trace of permuting* $f$*-triderivation* $D$, *then* $d(x) \leq f(x)$ *for all* $x \in L$.

*Proof.* Since $x \wedge x = x$ for all $x \in L$, we have $d(x) = D(x, x, x) = D(x \wedge x, x, x) = (D(x, x, x) \wedge f(x)) \vee (f(x) \wedge D(x, x, x)) = D(x, x, x) \wedge f(x) = d(x) \wedge f(x)$. Thus
$d(x) = d(x) \wedge f(x)$, which implies $d(x) \leq f(x)$.

**Proposition 3.7** *Let* $(L, \wedge, \vee)$ *be a lattice and* $f : L \to L$ *a mapping. Let* $D$ *be a permuting* $f$*-triderivation* $D$, *then* $D(x, y, z) \leq f(x)$, $D(x, y, z) \leq f(y)$ *and* $D(x, y, z) \leq f(z)$ *for all* $x, y, z \in L$.

*Proof.* Since $x \wedge x = x$ for all $x \in L$, we have $D(x, y, z) = D(x \wedge x, y, z) = (D(x, y, z) \wedge f(x)) \vee (f(x) \wedge D(x, y, z)) = f(x) \wedge D(x, y, z)$, therefore $D(x, y, z) \leq f(x)$. Similarly we can show that $D(x, y, z) \leq f(y)$ and $D(x, y, z) \leq f(z)$.

**Remark 3.8** $(L, \wedge, \vee)$ *be a lattice and* $f : L \to L$ *a mapping. Let* $D$ *be a permuting* $f$*-triderivation. It is obvious from Proposition 3.7 that* $D(x, y, z) \leq f(x) \wedge f(y)$, $D(x, y, z) \leq f(x) \wedge f(z)$ *and* $D(x, y, z) \leq f(y) \wedge f(z)$ *for all* $x, y, z \in L$. *By a similar argument* $D(x, y, z) \leq (f(x) \wedge f(y)) \wedge f(z)$ *for all* $x, y, z \in L$.

**Corollary 3.9** *Let* $(L, \wedge, \vee)$ *be a lattice and* $f : L \to L$ *a mapping. Let* $D$ *be a permuting* $f$*-triderivation of* $L$. *If* $L$ *has a least element 0 and greatest element 1 and* $f(0) = 0$, *then* $D(0, y, z) = 0$, $D(1, y, z) \leq f(y)$ *and* $D(1, y, z) \leq f(z)$*for all* $y, z \in L$.

*Proof.* Replacing $x$ by 0 and $y$ by $x$ in Proposition 3.7, we get
$D(0,y,z) \leq f(0) = 0$. So $D(0,y,z) = 0$. Similarly replacing $x$ by 1 in Proposition 3.7, we get
$D(1,y,z) \leq f(y)$ and $D(1,y,z) \leq f(z)$ for all $y,z \in L$.

**Theorem 3.10** *Let $(L,\wedge,\vee)$ be a lattice and $f : L \to L$ a mapping. Let $d$ be the trace of the joinitive permuting $f$-triderivation $D$, then $d(x \vee y) = d(x) \vee d(y) \vee D(x,x,y) \vee D(x,y,y)$ and $d(x) \vee d(y) \leq d(x \vee y)$ for all $x,y \in L$.*

*Proof.* Since $d(x\vee y) = D(x\vee y, x\vee y, x\vee y) = D(x, x\vee y, x\vee y) \vee D(y, x\vee y, x\vee y) = D(x,x,x\vee y) \vee D(x,y,x\vee y) \vee D(y,x,x\vee y) \vee D(y,y,x\vee y) = D(x,x,x) \vee D(x,x,y) \vee D(x,y,x) \vee D(x,y,y) \vee D(y,x,x) \vee D(y,x,y) \vee D(y,y,x) \vee D(y,y,y) = d(x) \vee d(y) \vee D(x,x,y) \vee D(x,y,y)$.
Thus $d(x \vee y) = d(x) \vee d(y) \vee D(x,x,y) \vee D(x,y,y)$, therefore
$d(x) \vee d(y) \leq d(x \vee y)$ for all $x,y \in L$.

**Theorem 3.11** *Let $(L,\wedge,\vee)$ be a distributive lattice and $f : L \to L$ a mapping. Let $D$ be a permuting $f$-triderivation of $L$ with trace $d$. Then*
*$d(x\wedge y) = (f(x) \wedge d(y)) \vee (f(y) \wedge d(x)) \vee D(x,x,y) \vee D(x,y,y)$ for all $x,y \in L$.*

*Proof.* Using proposition 3.7, we get
$d(x\wedge y) = D(x\wedge y, x\wedge y, x\wedge y) = (D(x, x\wedge y, x\wedge y) \wedge f(y)) \vee (f(x) \wedge D(y, x\wedge y, x\wedge y)) = [\{(D(x,x,x\wedge y) \wedge f(y)) \vee (f(x) \wedge D(x,y,x\wedge y))\} \wedge f(y)] \vee [f(x) \wedge \{(D(y,x,x\wedge y) \wedge f(y)) \vee (f(x) \wedge D(y,y,x\wedge y))\}] = [\{(D(x,x,x\wedge y) \wedge f(y)) \vee D(x,y,x\wedge y)\} \wedge f(y)] \vee [f(x) \wedge \{D(y,x,x\wedge y) \vee (f(x) \wedge D(y,y,x\wedge y))\}] = \{(D(x,x,x\wedge y) \wedge f(y)) \wedge f(y)\} \vee \{D(x,y,x\wedge y) \wedge f(y)\} \vee \{f(x) \wedge D(y,x,x\wedge y)\} \vee \{f(x) \wedge (f(x) \wedge D(y,y,x\wedge y))\} = \{D(x,x,x\wedge y) \wedge (f(y) \wedge f(y))\} \vee \{D(x,y,x\wedge y) \wedge f(y)\} \vee \{f(x) \wedge D(y,x,x\wedge y)\} \vee \{(f(x) \wedge f(x)) \wedge D(y,y,x\wedge y)\} = \{D(x,x,x\wedge y) \wedge f(y)\} \vee \{D(x,y,x\wedge y) \wedge f(y)\} \vee D(y,x,x\wedge y) \vee \{f(x) \wedge D(y,y,x\wedge y)\} = [\{(D(x,x,x) \wedge f(y)) \vee (f(x) \wedge D(x,x,y))\} \wedge f(y)] \vee [\{(D(x,y,x) \wedge f(y)) \vee (f(x) \wedge D(x,y,y))\} \wedge f(y)] \vee \{(D(y,x,x) \wedge f(y)) \vee (f(x) \wedge D(y,x,y))\} \vee \{f(x) \wedge [(D(y,y,x) \wedge f(y)) \vee (f(x) \wedge D(y,y,y))]\} = ((D(x,x,x) \wedge f(y)) \wedge f(y)) \vee ((f(x) \wedge D(x,x,y)) \wedge f(y)) \vee ((D(x,y,x) \wedge f(y)) \wedge f(y)) \vee ((f(x) \wedge D(x,y,y)) \wedge f(y)) \vee D(y,x,x) \vee D(y,x,y) \vee (f(x) \wedge ((D(y,y,x) \wedge f(y))) \vee (f(x) \wedge (f(x) \wedge D(y,y,y))) = d(x) \wedge (f(y) \wedge f(y)) \vee (D(x,x,y) \wedge f(y)) \vee (D(x,y,x) \wedge f(y)) \vee (D(x,y,y) \wedge f(y)) \vee D(y,x,x) \vee D(y,x,y) \vee (f(x) \wedge D(y,y,x)) \vee (f(x) \wedge f(x)) \wedge D(y,y,y) = (d(x) \wedge f(y)) \vee D(x,x,y) \vee D(x,y,x) \vee D(x,y,y) \vee D(y,x,x) \vee D(y,x,y) \vee D(y,y,x) \vee (f(x) \wedge d(y)) = (d(x) \wedge f(y)) \vee D(x,x,y) \vee D(x,y,y) \vee (f(x) \wedge d(y))$.
Thus $d(x\wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)) \vee D(x,x,y) \vee D(x,y,y)$

**Proposition 3.12** *Let $(L,\wedge,\vee)$ be a distributive lattice and $f : L \to L$ a mapping. Let $D$ be a permuting $f$-triderivation of $L$ with trace $d$. Then the following hold:*

*(i)* $D(x,x,y) \leq d(x \wedge y)$ *and* $D(x,y,y) \leq d(x \wedge y)$*,*
*(ii)* $f(x) \wedge d(y) \leq d(x \wedge y)$*,*
*(iii)* $d(x) \wedge f(y) \leq d(x \wedge y)$*,*
*(iv)* $d(x) \wedge d(y) \leq d(x \wedge y)$*.*

*Proof.* Using Theorem 3.11, we get
$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)) \vee D(x,x,y) \vee D(x,y,y)$ for all $x, y \in L$, which implies $D(x,x,y) \leq d(x \wedge y)$, $D(x,y,y) \leq d(x \wedge y)$, $f(x) \wedge d(y) \leq d(x \wedge y)$ and $d(x) \wedge f(y) \leq d(x \wedge y)$. By proposition 3.6, we have $d(x) \leq f(x)$ for all $x \in L$. Thus $d(x) \wedge d(y) \leq f(x) \wedge d(y) \leq d(x \wedge y)$, this implies
$d(x) \wedge d(y) \leq d(x \wedge y)$ for all $x, y \in L$.

**Remark 3.13** *Since a distributive lattice is a modular lattice, therefore Theorem 3.11 and Proposition 3.12 are also true for modular lattices.*

**Corollary 3.14** *Let* $(L, \wedge, \vee)$ *be a distributive lattice and* $f : L \to L$ *a mapping. Let* $D$ *be a permuting* $f$*-triderivation of* $L$ *with trace* $d$*. If* $L$ *has a greatest element 1 then* $D(x,x,1) \leq d(x)$ *and* $f(x) \wedge d(1) \leq d(x)$*.*

*Proof.* Replacing $y$ by 1 in Proposition 3.12(i), we get $D(x,x,1) \leq d(x \wedge 1) = d(x)$, which implies $D(x,x,1) \leq d(x)$ and replacing $y$ by 1 in Proposition 3.12(ii), we get $f(x) \wedge d(1) \leq d(x \wedge 1) = d(x)$, from which we get $f(x) \wedge d(1) \leq d(x)$ for all $x \in L$.

**Corollary 3.15** *Let* $(L, \wedge, \vee)$ *be a modular lattice and* $f : L \to L$ *a mapping. Let* $D$ *be a permuting* $f$*-triderivation of* $L$ *with trace* $d$*. If* $L$ *has a greatest element 1 then* $D(x,x,1) \leq d(x)$ *and* $f(x) \wedge d(1) \leq d(x)$*.*

**Proposition 3.16** *Let* $(L, \wedge, \vee)$ *be a distributive lattice with least element 0 and greatest element 1 and* $f : L \to L$ *be a mapping. Let* $D$ *be a permuting* $f$*-triderivation of* $L$ *with trace* $d$*. Then the following hold:*
*(i) If* $d(1) \leq f(x)$*, then* $d(1) \leq d(x)$*,*
*(ii) If* $f(x) \leq d(1)$*, then* $d(x) = f(x)$*.*

*Proof.* (i) Since $d(1) \leq f(x)$, therefore using Proposition 3.12(ii), we get
$d(1) = f(x) \wedge d(1) \leq d(x \wedge 1) = d(x)$. Thus $d(1) \leq d(x)$.
(ii) Since $f(x) \leq d(1)$, therefore using Proposition 3.12(ii), we get
$f(x) = f(x) \wedge d(1) \leq d(x \wedge 1) = d(x)$. Thus $f(x) \leq d(x)$.
From proposition 3.6, we have $d(x) \leq f(x)$, this alongwith $f(x) \leq d(x)$ implies $d(x) = f(x)$.

**Proposition 3.17** *Let* $(L, \wedge, \vee)$ *be a distributive lattice and* $f : L \to L$ *a mapping. Let* $D$ *be a permuting* $f$*-triderivation of* $L$ *with trace* $d$*. If* $x \leq y$*,* $f(x) \leq f(y)$ *and* $d(y) = f(y)$*, then* $d(x) = f(x)$*.*

*Proof.* Since $x \leq y$, $x \wedge y = x$ and $d(y) = f(y)$. By Proposition 3.7 $D(x, x, y) \leq f(x)$ and $D(x, y, y) \leq f(x)$, therefore by proposition 3.6 and hypothesis $f(x) \leq f(y)$ we get $d(x) \leq f(x) \leq f(y)$.
By Theorem 3.11, we get
$d(x) = d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)) \vee D(x, x, y) \vee D(x, y, y) = f(x)$.
Hence $d(x) = f(x)$ for all $x \in L$.

**Remark 3.18** *Since a distributive lattice is a modular lattice, therefore Proposition 3.16 and Proposition 3.17 are also true for modular lattices.*

**Definition 3.19** [3] Let $(L, \wedge, \vee)$ be a lattice. A mapping $f : L \to L$ is called a lattice homomorphism if
(1) $f(x \wedge y) = f(x) \wedge f(y)$,
(2) $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L$.

**Definition 3.20** *Let $(L, \wedge, \vee)$ be a lattice and $f : L \to L$ a mapping. Let $D$ be a permuting $f$-triderivation of $L$ with trace $d$. If $x \leq y$ implies $d(x) \leq d(y)$, then $d$ is called an isotone mapping.*

**Theorem 3.21** *Let $(L, \wedge, \vee)$ be a lattice with greatest element 1. Let $f : L \to L$ be a homomorphism and $d$ a trace of a permuting $f$-triderivation $D$ of $L$. Then the following conditions are equivalent:*
*(i) $d$ is an isotone mapping,*
*(ii) $d(x) = f(x) \wedge d(1)$*
*(iii) $d(x \wedge y) = d(x) \wedge d(y)$,*
*(iv) $d(x) \vee d(y) \leq d(x \vee y)$.*

*Proof.* (i)$\Rightarrow$ (ii) Since $d$ is isotone and $x \leq 1$, therefore $d(x) \leq d(1)$. By Proposition 3.6 $d(x) \leq f(x)$, we get $d(x) \leq f(x) \wedge d(1)$. By Proposition 3.12(ii), we have $f(x) \wedge d(1) \leq d(x)$, this alongwith $d(x) \leq f(x) \wedge d(1)$, we get $d(x) = f(x) \wedge d(1)$ for all $x \in L$.
(ii)$\Rightarrow$ (iii) Since $d(x) = f(x) \wedge d(1)$ for all $x \in L$. Then $d(x \wedge y) = f(x \wedge y) \wedge d(1) = (f(x) \wedge f(y)) \wedge (d(1) \wedge d(1)) = (f(x) \wedge d(1)) \wedge (f(y) \wedge d(1)) = d(x) \wedge d(y)$ for all $x, y \in L$.
(iii)$\Rightarrow$ (i) Since $d(x \wedge y) = d(x) \wedge d(y)$. Let $x \leq y$, then $d(x) = d(x \wedge y) = d(x) \wedge d(y)$. Hence $d(x) \leq d(y)$. Hence $d$ is an isotone.
(i)$\Rightarrow$ (iv) Let $d$ be isotone. Since $x \leq x \vee y$ and $y \leq x \vee y$, therefore $d(x) \leq d(x \vee y)$ and $d(y) \leq d(x \vee y)$. Hence $d(x) \vee d(y) \leq d(x \vee y)$.
(iv)$\Rightarrow$ (i) Let $x \leq y$. Since $d(x) \leq d(x \vee y) = d(y)$, this implies $d(x) \leq d(y)$. Hence $d$ is an isotone.

**Theorem 3.22** *Let $(L, \wedge, \vee)$ be a modular lattice and $f : L \to L$ a mapping and $d$ a trace of a permuting $f$-triderivation $D$ of $L$. Then the following*

*conditions are equivalent:*
*(i) $d$ is an isotone,*
*(ii) $d(x \wedge y) = d(x) \wedge d(y)$.*

*Proof.* (i)$\Rightarrow$ $(ii)$ Let $d$ be isotone. Since $x \wedge y \leq x$ and $x \wedge y \leq y$, therefore $d(x \wedge y) \leq d(x)$ and $d(x \wedge y) \leq d(y)$. Thus $d(x \wedge y) \leq d(x) \wedge d(y)$. Since $L$ is a modular lattice, therefore
$d(x \wedge y) = (d(x) \wedge f(y)) \vee D(x, x, y) \vee (f(x) \wedge d(y)) \vee D(x, y, y) = [(d(x) \vee D(x, x, y)) \wedge f(y)] \vee [(d(y) \vee D(x, y, y)) \wedge f(x)]$, which implies $(d(x) \wedge f(y)) \wedge (d(y) \wedge f(x)) \leq d(x \wedge y)$. So $(d(x) \wedge f(x)) \wedge (d(y) \wedge f(y)) \leq d(x \wedge y)$. Using Proposition 3.6, we get $d(x) \wedge d(y) \leq (d(x) \wedge f(x)) \wedge (d(y) \wedge f(y)) \leq d(x \wedge y)$, which implies $d(x) \wedge d(y) \leq d(x \wedge y)$. This alongwith $d(x \wedge y) \leq d(x) \wedge d(y)$ gives $d(x \wedge y) = d(x) \wedge d(y)$ for all $x, y \in L$.
(ii)$\Rightarrow$ $(i)$ Let $x \leq y$ and $d(x \wedge y) = d(x) \wedge d(y)$. Since $x \wedge y = x$, we get $d(x) = d(x \wedge y) = d(x) \wedge d(y) \leq d(y)$. Hence $d(x) \leq d(y)$ for all $x, y \in L$.

# 4 Two sided $f$-generalized $d$-derivations

In this section we describe the concept of a two sided $f$-generalized $d$-derivation on lattices and prove our results regarding this notion.

**Definition 4.1** Let $(L, \wedge, \vee)$ be a lattice and $f : L \to L$ a mapping. Let $d$ be a trace of a permuting $f$-triderivation $D$. Let $G : L \to L$ be a mapping, then $G$ is called a two sided $f$-generalized $d$-derivation on $L$ if it satisfies the following condition
$G(x \wedge y) = (G(x) \wedge f(y)) \vee (f(x) \wedge d(y))$ for all $x, y \in L$.

**Proposition 4.2** *Let $(L, \wedge, \vee)$ be a lattice and $f : L \to L$ a mapping. Let $D$ be a permuting $f$-triderivation of $L$ with trace $d$. If $G : L \to L$ is a two sided $f$-generalized $d$-derivation on $L$, then*
*(i) $d(x) \leq G(x)$,*
*(ii) $G(x \wedge y) \leq G(x) \vee G(y)$.*

*Proof.* (i) Since $x \wedge x = x$ for all $x \in L$, we get
$G(x) = G(x \wedge x) = (G(x) \wedge f(x)) \vee (f(x) \wedge d(x))$. By Propostion 3.6, we get $G(x) = (G(x) \wedge f(x)) \vee d(x)$.
Hence $d(x) \leq G(x)$ for all $x \in L$.
(ii) Since $G(x) \wedge f(y) \leq G(x)$ and $f(x) \wedge d(y) \leq d(y)$, which alongwith $d(y) \leq G(y)$ gives
$G(x \wedge y) = (G(x) \wedge f(y)) \vee (f(x) \wedge d(y)) \leq G(x) \vee G(y)$.
Hence $G(x \wedge y) \leq G(x) \vee G(y)$ for all $x, y \in L$.

**Corollary 4.3** *Let $(L, \wedge, \vee)$ be a lattice, $f : L \to L$ a mapping and $D$ a permuting $f$-triderivation of $L$ with trace $d$. Let $G : L \to L$ be a two sided $f$-generalized $d$-derivation on $L$. If 1 is the greatest element of $L$ and $d(1) = 1$, $f(1) = 1$, then $G(1) = 1$.*

*Proof.* Let $d(1) = 1$ and $f(1) = 1$. Then $G(1) = G(1 \wedge 1) = (G(1) \wedge f(1)) \vee (f(1) \wedge d(1)) = 1$.

**Proposition 4.4** *Let $(L, \wedge, \vee)$ be a lattice, $f : L \to L$ a mapping and $D$ a permuting $f$-triderivation of $L$ with trace $d$. Let $G : L \to L$ be a two sided $f$-generalized $d$-derivation on $L$. If 1 is the greatest element of $L$, then*
*(i) $G(1) \leq d(x) \Rightarrow d(x) = G(x)$,*
*(ii) $d(x) \leq G(1) \Rightarrow G(x) \leq G(1)$.*

*Proof.* (i) Since $G(1) \leq d(x)$ and $x \wedge 1 = x$ for all $x \in L$, we have $G(x) = G(1 \wedge x) = (G(1) \wedge f(x)) \vee (f(1) \wedge G(x)) \leq (d(x) \wedge f(x)) \vee d(x) = d(x)$. Which implies $G(x) \leq d(x)$ for all $x \in L$. By Proposition 4.2, we have $d(x) \leq G(x)$ for all $x \in L$, this alongwith $G(x) \leq d(x)$ gives $G(x) = d(x)$ for all $x \in L$.
(ii) Let $d(x) \leq G(1)$ for all $x \in L$. Since $x \wedge 1 = x$ for all $x \in L$, therefore we have $G(x) = G(1 \wedge x) = (G(1) \wedge f(x)) \vee (f(1) \wedge d(x)) \leq (G(1) \wedge f(x)) \vee G(1) = G(1)$. Thus $G(x) \leq G(1)$ for all $x \in L$.

**Definition 4.5** *Let $(L, \wedge, \vee)$ be a lattice, $f : L \to L$ a mapping and $D$ a permuting $f$-triderivation of $L$ with trace $d$. Let $G : L \to L$ be a two sided $f$-generalized $d$-derivation on $L$. We define the set by : $F = \{x \in L : G(x) = d(x)\}$.*

**Theorem 4.6** *Let $(L, \wedge, \vee)$ be a lattice, $f : L \to L$ a mapping and $D$ a permuting $f$-triderivation of $L$ with trace $d$. Let $G : L \to L$ be a two sided $f$-generalized $d$-derivation on $L$. If $d$ is decreasing function on $L$, then $y \leq x$ and $x \in F$ imply $y \in F$.*

*Proof.* Let $y \leq x$, $x \in F$ and $d$ is decreasing function. Then $d(x) \leq d(y)$. Thus $G(y) = G(x \wedge y) = (G(x) \wedge f(y)) \vee (f(x) \wedge d(y)) \leq G(x) \vee d(y) = d(y)$. Hence $G(y) \leq d(y)$ for all $y \in L$. By Proposition 4.2, $d(y) \leq G(y)$ for all $y \in L$, which along with $G(y) \leq d(y)$ implies $G(y) = d(y)$ for all $y \in L$. Hence $y \in F$.

**Theorem 4.7** *Let $(L, \wedge, \vee)$ be a lattice, $f : L \to L$ a homomorphism and $D$ a permuting $f$-triderivation of $L$ with trace $d$. Let $G : L \to L$ be a two sided $f$-generalized $d$-derivation on $L$. If $G$ is a decreasing function on $L$, then $x \vee y \in F$ for all $x, y \in F$.*

*Proof.* Since $x \leq x \vee y$ and $y \leq x \vee y$ for all $x, y \in F$ and $G$ is a decreasing function on $L$, therefore $G(x \vee y) \leq G(x)$ and $G(x \vee y) \leq G(y)$ for all $x, y \in F$. So $G(x \vee y) \leq G(x) \vee G(y) = d(x) \vee d(y)$. By Theorem 3.21, we get $d(x) \vee d(y) \leq d(x \vee y)$. Thus $G(x \vee y) \leq d(x) \vee d(y) \leq d(x \vee y)$. So $G(x \vee y) \leq d(x \vee y)$ for all $x, y \in F$. By Proposition 4.2, we have $d(x \vee y) \leq G(x \vee y)$, which alongwith $G(x \vee y) \leq d(x \vee y)$ implies $G(x \vee y) = d(x \vee y)$ for all $x, y \in F$. Hence $x \vee y \in F$ for all $x, y \in F$.

**Theorem 4.8** *Let $(L, \wedge, \vee)$ be a lattice, $f : L \to L$ a homomorphism and $D$ a permuting $f$-triderivation of $L$ with trace $d$. Let $G : L \to L$ be a two sided $f$-generalized $d$-derivation on $L$. If $G$ and $d$ are decreasing functions on $L$, then the set $F = \{x \in L : G(x) = d(x)\}$ is an ideal of $L$.*

*Proof.* From Theorem 4.6 and Theorem 4.7, Proof follows.

**ACKNOWLEDGEMENT.** The authors are thankful to the Bahauddin Zakariya University, Multan for the support and facilities provided during this research.

# References

[1] R. Balbes and P. Dwinger, Distributive lattices,*University of Missouri Press, Columbia, United States,* 1974.

[2] A. J. Bell, The co-information lattice, in: *4th International Symposium on Independent Component Analysis and Blind Signal Separation(ICA2003), Japan,* 921-926.

[3] G. Birkhoff, Lattice Theory, *American Mathematical Society, New York,* 1940.

[4] G. Durfee, Cryptanalysis of RSA using algebraic and lattice methods, *A dissertation submitted to the Department of Computer Science and the committee on graduate studies of Stanford University,* (2002), 1-114.

[5] M. A. Ozturk, Permuting tri-derivations in prime and semiprime rings, *East Asian Math. J.,* **15** (2)(1999), 177-190.

[6] D. Ozden and M. A. Ozturk, Permuting tri-derivations in prime and semiprime gamma rings, *Kyungpook Math. J.,* **46** (2006), 153-167.

[7] M. A. Ozturk, H. Yazarli and K. H. Kim, Permuting tri-derivations in lattices, *Quaestions Mathematicae,* **32** (3)(2009), 415-425.

[8] X. L. Xin, T. Y. Li and J. H. Lu, On derivations of Lattices, *Information sciences,* **178** (2008), 307-316.

[9] H. Yazarli, M. A. Ozturk and Y. B. Jun, Tri-additive maps and permuting tri-derivations, *Commun. Fac. Univ. Ank. Series A1,* **54** (1)(2005), 1-8.

[10] C. Yilmaz and M. A. Ozturk, On the traces of a permuting tri-additive mapping in left s-unital rings, *International Journal of Pure and Applied Mathematics,* **23** (4)(2005), 465-474.

[11] C. Yilmaz and M. A. Ozturk, On $f$-derivations of Lattices, *Bull. Korean. Math. Soc.,* **4** (45)(2008), 701-707.

[12] J. Zhan and Y. L. Liu, On $f$-derivations of BCI-algebras, *Int. J. Math. Sci.,* **11**(2005), 1675-1684.

**Received: September, 2010**