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Dihedral Homology of Polynomial Algebras

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Abstract

In this article we study the Polynomial Banach algebra and its dihedral homology group. We also interested in Dihedral homology of Laurent polynomial algebra and tensor algebra.

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1 Introduction

Let A be a Banach algebra, $[X]$ be class of polynomials, define:

$$\|[X]\| = \sup \|X(x_1, x_2, ..., x_n)\| : x_i \in A, \, \|x_i\| \leq 1 \tag{1}$$

Then $A[X]$ is Banach polynomial algebra with the multiplication of coefficient of polynomials. Firstly we recall some definitions and facts about dihedral homology and its properties (see [1]or [4]). Let A be unital Banach algebra with an involution over field K and $C_\bullet(A)$, $C_{\bullet\bullet}(A)$ are Hochschild complex and bicomplex,respectively. The homology group of these complexes give the Simplicial (Hochschild) and cyclic homology: $H_\bullet(A)$, $HC_{\bullet\bullet}(A)$ of algebra A. Consider the action of the group $Z/2$ on Connes -Tsygan bicomplex $C_{\bullet\bullet}(A)$ by means of operators

$$^{\varepsilon}r : a_0 \otimes a_1 \otimes ... \otimes a_n \rightarrow (-1)^{\frac{n(n+1)}{2}} \varepsilon . a_0^* \otimes a_n^*, \otimes ... \otimes a_1^*, \varepsilon = \pm 1 \tag{2}$$

The dihedral homology $HD_\bullet(A)$ of algebra A is defined by the hyper homology $H_\bullet(Z/2; C_{\bullet\bullet}(A))$ of group $Z/2$ with a coefficient of in bicomplex $C_{\bullet\bullet}(A)$ [6].

Consider chain maps:

$$\top : C_\bullet(A) \otimes C_\bullet(B) \to C_\bullet(A \otimes B) \tag{3}$$

$$\perp : C_\bullet((A \otimes B)) \to C_\bullet(A) \otimes C_\bullet(B) \tag{4}$$

where $C_\bullet\ (A), C_\bullet(B)$ are Hochschild complexes of algebra A and B respectively, such that

$$(a \otimes a_0 \otimes \cdots \otimes a_{n-1}) \top (b \otimes b_0 \otimes \cdots \otimes b_{n-1}) = \sum_\sigma sgn(\sigma).(a \otimes b) \otimes p_{\sigma^{-1}(0)} \otimes \cdots \otimes p_{\sigma^{-1}(n+m-1)}, \tag{5}$$

where the summation take over all permutation $\sigma \in \sum_{n+m}$ such that
$\sigma(0) < \ldots < \sigma(n-1), \sigma(n) < \ldots < \sigma(n+m-1)$, and
$(p_0, \ldots, P_{n+m-1}) = (a_0 \otimes 1, \ldots, a_{n-1} \otimes 1, 1 \otimes b_0, \ldots, 1 \otimes b_{m-1})$,

$$\begin{aligned} \perp \quad & ((a \otimes b) \otimes (a_0 \otimes b_0) \otimes \ldots \otimes (a_{n-1} \otimes b_{n-1})) = \\ = \quad & \sum_{i=-1}^{n-1} (aa_0 \cdot \ldots \cdot a_i) \otimes a_{i+1} \otimes \cdots \otimes a_{i-1} \otimes (b_{i+1} \cdot \ldots \cdot b_{n-1} b) \otimes b_0 \otimes \cdots \otimes b_{i.} \end{aligned} \tag{6}$$

It's known that:

- The chain maps $\perp$ and $\top$ induce isomorphism of Homology and the inverse is true[2] .
- Clearly if A and B involution k–algebrawe can define an the involution of tensor algebra $A \otimes_k B$ as follows: $(a \otimes b)^* = a^* \otimes b^*, a \in A, b \in B$.
- The action of group $Z/2$ on the Hochschild complex $C_\bullet(A)$ of algebra A by means of (1) induces action of $Z/2$ on the Hochschild homology $HH_\bullet(A)$ of algebra A.
- The spectral sequence of bicomplex $C_{\bullet\bullet}(A)$ is given by:

$$\hat{E}_{ij}^2(A) = \begin{cases} HH_j(A), & i \equiv 0 (\mathrm{mod}\, 2) \\ 0\ , & i\ \equiv 1 (\mathrm{mod}\ 2) \end{cases} \tag{7}$$

with the differentials :

$$\hat{d}_{ij}^2 = \begin{cases} B, & i \equiv 0 (\mathrm{mod}\, 2), i \succ 0 \\ 0, & i \equiv 1 (\mathrm{mod}\, 2); i = 0 \end{cases} \tag{8}$$

where B is Connes operator [7].

- The action of group $Z/2$ on the spectral sequence $\hat{E}_{\bullet\bullet}^2(A)$ generates to the action of $Z/2$ on every term of $\hat{E}_{\bullet\bullet}^s(A)$, $2 \leq S \leq \infty$.

2 Polynomial algebra and its dihedral homology group

In this part we discuss the action of $Z/2$ of Hochschild homology by considering the chain operators $\top$ and $\perp$ by the following lemma to prove theorem 2.

Lemma 1 *Let $\alpha \in H_n(A)$ and $\beta \in H_m(\beta)$. Then ${}^{\varepsilon}r_n(\alpha)\top({}^{\delta}r_n(\beta)) ={}^{(\varepsilon\delta)} r_{n+m}(\alpha\top\beta)$.*

proof: *Let $\tilde{C}_\bullet(A)$ be the normalized Hochschild complex of polynomial Banach algebra $A[x]$. Suppose the representations $a\otimes a_0\otimes ...\otimes a_{n-1} \in \alpha$ and $b\otimes b_0\otimes\cdots\otimes b_{m-1} \in \beta$, then*

$$\begin{aligned}(a \otimes a_0 \otimes ... \otimes a_{n-1}) \quad \top \quad & (b \otimes b_0 \otimes \cdots \otimes b_{m-1}) = \\ = \quad & \sum sgn(\sigma)(a \otimes b) \otimes p_{\sigma^{-1}(0)} \otimes \cdots \otimes p_{\sigma^{-1}(n+m-1)} \underrightarrow{(\in \sigma)} \\ & \underrightarrow{(\in \sigma)} (-1)^{\frac{(n+m)(n+m+1)}{2}} (\varepsilon\sigma) \sum sgn(\sigma)(a \otimes b)^* \otimes \\ & \otimes p^*_{\sigma^{-1}(m+n-1)} \otimes \cdots \otimes p^*_{\sigma^{-1}(0)}\end{aligned}$$

Acting on $\alpha\top\beta$ by means of operator $\perp$, all elements in summation $(\alpha\top\beta)$ tend to zero except the elements :

$$(a\otimes b)\otimes(a_0\otimes 1)\otimes a_1\otimes 1)\otimes\cdots\otimes(a_{n-1}\otimes 1)\otimes(1\otimes b_0)\otimes(1\otimes b_1)\otimes\cdots\otimes(1\otimes b_{m-1}). \quad (9)$$

such that

$$^{(\varepsilon\delta)}r((a \otimes b) \otimes (a_0 \otimes 1) \otimes \cdots \otimes (a_{n-1} \otimes 1) \otimes (1 \otimes b_0) \otimes \cdots \otimes (1 \otimes b_{m-1})) =$$

$$= (-1)^{\frac{(m+n)(m+n+1)}{2}}(\varepsilon\delta)(a \otimes b)^* \otimes (1 \otimes b_{m-1})^* \otimes \cdots \otimes (1 \otimes b_0)^* \otimes (a_{n-1} \otimes 1)^*\otimes$$

$$\otimes\cdots\otimes(a_0\otimes 1)^{**} \underrightarrow{\perp} (-1)^{\frac{(m+n)(m+n+1)}{2}}(\varepsilon\delta).(a^*\otimes a^*_{n-1}\otimes\cdots\otimes a^*_0)\otimes(b^*\otimes b^*_{m-1}\otimes\cdots\otimes b^*_0) =$$

$$={}^{\varepsilon} 2(a \otimes a_0 \otimes \cdots \otimes a_{n-1}) \otimes^{\delta} 2(b \otimes b_0 \otimes \cdots \otimes b_{m-1}).$$

Thus ${}^{\varepsilon}r(\alpha) \otimes^{\delta} r(\beta) = \perp({}^{(\varepsilon\delta)}r(\alpha\top\beta)$ and hence:${}^{\varepsilon}r(\alpha)\top({}^{\delta}r)(\beta)) ={}^{(\varepsilon\delta)} r(\alpha\top\beta)$.

The main theorem of this part is the following assertion.

Theorem 2 *Let A be commutative involutive unital Banach algebra over K. then*

${}^{\varepsilon}HD_n(A\,[x]) ={}^{\varepsilon} HD_n(A) \oplus^{\varepsilon} HR^{\infty}_n(A), \varepsilon = \pm 1.$where

$$^{\varepsilon}HR^{\infty}_n(A) ={}^{\varepsilon} H_n(Z/2, C_\bullet(A)) \otimes^{\varepsilon} H_n(Z/2, C_\bullet(A)) \otimes \cdots. \quad (10)$$

Proof: It's known from [3] that:

- If the algebra A has involution *, then the algebra A[x] also has involution:

$$(\sum_i a_i x^i)^* = \sum_i a_i^* x^i \qquad a_i \in A$$

- $A[x] = A \otimes_k k\,[x] : ax^i \quad \leftrightarrow \quad a \otimes x^i$ is involutive isomorphism..
- Hochshid Homology $H_\bullet(k\,[x])$ isomorphic to exterior algebraic form in $K\,[x]$ and Conne‘s operator B in Hochschild Homology corresponds to exterior differential form d.
- A given $A[x] = A \otimes k[x]$ tends to an isomrphism $H_\bullet(A[x]) \simeq H_\bullet(A) \otimes H_\bullet(k[x])$ and the Conne's operator B acts on $H_\bullet(A) \otimes H_\bullet(k[x])$ by means of $B(\alpha \otimes \beta) = B(\alpha) \otimes \beta + (-1)^{\deg \alpha} \alpha \otimes B(\beta)$,where $B(\alpha \top \beta) = B(\alpha) \top \beta + (-1)^{\deg \alpha} \alpha \top B(\beta)$[2, 7].

  Since

$$H_i(k[x]) = \begin{cases} \Omega^0, \; i = 0 \\ \Omega^i, \; i = 1 \\ 0, \; i > 1 \end{cases} \tag{11}$$

where Ω^i- exterior algebraic form in $K\,[x]$, then the second term in spectral sequence $\left\{\hat{E}^s_{\bullet\bullet} A\,[x]\right\}$takes the form

$$\hat{E}^2_{ij}(A\,[x]) = \begin{cases} Hj(A) \otimes \Omega^0 \otimes Hj_{-1}(A) \otimes \Omega^1, & i \equiv 0(\mathrm{mod}\,2). \\ 0, & i \equiv 1(\mathrm{mod}\,2), \end{cases} \tag{12}$$

$$\hat{d}^2_{ij}(\alpha \otimes \beta) = \begin{cases} B(\alpha) \otimes \beta \otimes (-1)^{\deg \alpha} \alpha \otimes d(\beta), \; if \; i \equiv 0(\mathrm{mod}\,2) \text{ and } i > 0 \\ 0, \qquad if \; i \not\cong 0(\mathrm{mod}\,2) \end{cases} \tag{13}$$

In other wise, the cohomology of complex $H_0(A\,[x]) \quad \underrightarrow{\hat{d}^2} \ldots \quad \underrightarrow{\hat{d}^2} \quad H_n(A\,[x]) \; \underrightarrow{\hat{d}^2} \ldots$ isomorphic to the cohomology of complex $H_0(A) \; \underrightarrow{B} \; H_1(A) \; \underrightarrow{B} \ldots \; \underrightarrow{B} \; H_n(A) \; \underrightarrow{B} \ldots$ Using Kenneth formula[2]we get: $H^\bullet(H_\bullet(A) \otimes \Omega^\cdot_\bullet = H^\bullet(H_{\bullet\cdot}(A) \otimes H^\bullet(\Omega^\cdot) = H^\bullet(H_\bullet(A))$.

Since

$$H^\bullet(\Omega^\bullet) = \begin{cases} k, & \bullet = 0 \\ 0, & \bullet > 0. \end{cases} \tag{14}$$

, then the term $\hat{E}^3_{\bullet\bullet}(A\,[x])$ has the form

$$\hat{E}^3_{ij}(A\,[x]) = \begin{cases} (H_j(A) \otimes \Omega^0 \otimes H_{j-1}(A) \otimes \Omega^1) \;\mathrm{Im}\; \hat{d}^2, & i = 0 \\ \ker B \,\mathrm{Im}\; B, \qquad i \equiv 0(\mathrm{mod}\,2) & i > 0, \\ 0, & i \equiv 1(\mathrm{mod}\,2). \end{cases} \tag{15}$$

Note that the groups $\hat{E}^3_{ij}(A\,[x])$, $i > 0$ isomorphic to groups $\hat{E}^3_{ij}(A), i > 0$.

Clearly the groups $\hat{E}^3_{0j}$ isomorphic to group $H_j(A)\;/B(H_{j-1}(A) \otimes H_j(A) \otimes \Omega^0/k$ where Ω^0/k is space of algebraic 0-form factored by k. Hence:

$$\hat{E}^3_{ij}(A\,[x]) = \begin{cases} \hat{E}^3_{ij}(A) \otimes H_j(A) \otimes \Omega^0 k, i = 0 \\ \hat{E}^3_{ij}(A), \qquad i \equiv 0(\mathrm{mod}\,2), \quad i > 0 \\ 0, \qquad\qquad i \equiv 1(\mathrm{mod}\,2). \end{cases} \tag{16}$$

The spectral sequence $E^3_{0j}(A\,[x])$ without term $H^\infty_j(A) = H_j(A) \otimes \Omega^0 k$ tends to the cyclic homology group $HC_\bullet(A)$ and hence :

$$HC_\bullet(A)\,[x]) = HC_\bullet(A) \otimes H^\infty_{\bullet}(A) \tag{17}$$

The group $Z/2$ acts on groups $E^3_{0j}(A\,[x]) = H_j(A)B(H_{j-1}(A)) \oplus (H_j(A) \otimes \Omega^0 K)$ by means of

$${}^\varepsilon r(\alpha \otimes \beta \otimes w) =^\varepsilon r(\alpha) \otimes^\varepsilon r(\beta) \otimes {}^1 r(w), \tag{18}$$

where $\alpha \in H_j(A)/B(H_{j-1}(A)), \beta \in H_j(A), w \in \Omega^0/K$.

For a Banach algebra $A$the spectral sequence $\{{}^\varepsilon E^s_{\bullet\bullet}(A[x])\}$ of $f[A]$ is given by ${}^\varepsilon E^2_{pr}(A\,[x]) =^\varepsilon H_p(Z/2 \cdot HC_q(A))$. By factoring the group $HC_\bullet(A) \otimes H^\infty_\bullet(A)$ by the operator $(1-{}^\varepsilon r_.)$, we get required dihedral homolology ${}^\varepsilon HD_n(A\,[x])$. □

Corollary 3 *Let A be commutativeinvolution, unital $K-$Banach algebra, then:*

$${}^\varepsilon HD_i(A[x_1, ..., x_n]) =^\varepsilon HD_i(A) \otimes^\varepsilon HR^\infty_i(A) \otimes (\otimes^{n-1\varepsilon}_{j=1} HR^\infty_i(A[x_1, ..., x_j]). \tag{19}$$

3 Dihedral homolog of Laurent polynomial algebra.

In this part we get the dihedral homology of Laurent polynomial algebra as follows:

Theorem 4 *Let A be involution Commutative unital Banach k-algebra, then:* ${}^\varepsilon HD_n(A[x, x^{-1}]) =^\varepsilon HD_n(A) \otimes^\varepsilon HD_{n-1}(A) \otimes HR^\infty_n(A)$

proof: Clearly :

- There is an involution isomorphism k-algebra

$$A[x, x^{-1}] = A \otimes_k k[x, x^{-1}] : ay \longleftrightarrow a \otimes y,\; a \in A,\; y \in k[x, x^{-1}]. \tag{20}$$

- The following involution is true:

$$\left(\sum_i a_i x^i + \sum_j b_j x^{-j}\right)^* = \sum_i a_i^* x^i + \sum_j b_j^* x^{-j}, \quad a_i, b_j \in A. \tag{21}$$

Following [8], the Hochshild homology $H_\bullet(k[x,x^{-1}])$ isomorphic to exterior algebraic

form in $k[x,x^{-1}]$ and is given by :

$$H_\bullet(k[x,x^{-1}]) = \begin{cases} \Omega^0 & ,i=0 \\ \Omega^1 & ,i=1 \\ 0 & ,i>1 \end{cases}$$

Clearly the differential d of exterior form $\Omega^i, i \geq 0$ on $k[x,x^{-1}]$ corresponds to connes's operator B on Hochshild Homology $H_\bullet(k[x,x^{-1}])$. Since

$A[x,x^{-1}] = A \otimes_k k[x,x^{-1}]$,then $H_\bullet(A[x,x^{-1}]) = H_\bullet(A) \otimes H_\bullet(k[x,x^{-1}])$.

The Connes's operators B acts on $H_\bullet(A) \otimes H_\bullet(k[x,x^{-1}])$ by means of: $B(\alpha\otimes\beta) = \beta(\alpha) \otimes \beta + (-1)^{\deg\alpha} \alpha \otimes B(\beta)$ and give the following first term of spectral sequence $\left\{\hat{E}^s_{\bullet\bullet}(A[x,x^{-1}])\right\}$:

$$\hat{E}^2_{ij}(A[x,x^{-1}]) = \begin{cases} H_j(A) \otimes \Omega^0 \otimes H_{j-1}(A) \otimes \Omega^1, & i \equiv 0 (\mathrm{mod}\, 2), \\ 0 \quad , & i \equiv 1 (\mathrm{mod}\, 2), \end{cases} \tag{22}$$

$$\hat{d}^2_{ij}(\alpha\otimes\beta) = \begin{cases} B(\alpha) \otimes \beta \otimes (-1)^{\deg\alpha} \alpha \otimes d(\beta) \, if \;\; i \equiv 0 (\mathrm{mod}\, 2) \text{ and } i>0 \\ 0, & if \;\; i \not\equiv 0 (\mathrm{mod}\, 2), \end{cases}$$

where $\alpha \in H_\bullet(A), \beta \in \Omega^\bullet$.

The direct calculation gives:

$$\begin{aligned} & H_0(A[x,x^{-1}]) \xrightarrow{\hat{d}^2} \dots \xrightarrow{\hat{d}^2} H_n(A[x,x^{-1}]) \xrightarrow{\hat{d}^2} \dots H^\bullet(H_\bullet(A) \otimes \Omega^\bullet) \qquad (23) \\ = \;& H^n(H_\bullet(A)) \otimes H^\bullet(\Omega^\bullet))_n = H^0(\Omega^\bullet) \otimes H^n(H_\bullet(A)) \otimes H^1(\Omega^\bullet) \otimes H^{n-1}(H_\bullet(A)) \\ = \;& H^n(H_\bullet(A)) \otimes H^{n-1}(H_\bullet(A)), \end{aligned}$$

where $H^i(\Omega^\bullet) = \begin{cases} k & ,i=0 \\ k & ,i=1 \\ 0 & ,i>1 \end{cases}$

Consequently the second term of spectral sequence $\hat{E}^3_{\bullet\bullet}(A[x,x^{-1}])$ has the form:

$$\hat{E}^3_{ij}(A[x,x^{-1}]) = \begin{cases} H_j(A) \otimes \Omega^0 \otimes H_{j-1}(A) \otimes \dots \otimes \Omega^1 / \operatorname{Im} \hat{d}^2, & i=0 \\ \hat{E}^3_{ij}(A) \oplus \hat{E}^3_{i,j-1}(A), & i \equiv 0 (\mathrm{mod}\, 2), \; i>0 \\ 0 \, , & i \equiv 1 (\mathrm{mod}\, 2) \end{cases}$$

Clearly the group $\hat{E}^3_{ij}(A[x,x^{-1}])$ isomorphic to groups:

$$(H_j(A)/B(H_{j-1}(A))) \otimes (H_j(A) \otimes \Omega^0/k) \otimes (H_{j-1}(A)/B(H_{j-2}(A))) \tag{24}$$

In Spectral sequence $\{\hat{E}^s_{\bullet\bullet}(A[x,x^{-1}])\}$,the group $\hat{E}^3_{0j}(A[x,x^{-1}])$ without the term $H^\infty_j(A) = H_j(A) \otimes \Omega^0/k$ converges to group $HC_\bullet(A) \otimes HC_{\bullet-1}(A)$, then we have the cyclic homology of Laurent polynomial algebra:

$$HC_n(A[x,x^{-1}]) = HC_n(A) \otimes HC_{n-1}(A) \otimes HH^\infty_n(A). \tag{25}$$

Further more, the group $Z/2$ acts on groups $\hat{E}^3_{ij}(A[x,x^{-1}]), i > 0$,by means of operators $({}^{\in}r_j \otimes^{\in} r_{j-1})$ and on $\hat{E}^3_{0j}(A[x,x^{-1}])$ by:

$$^{\varepsilon}r(\alpha \otimes (\beta \otimes w) \otimes \delta) =^{\varepsilon} r(\alpha) \otimes^{\varepsilon} r(\beta) \otimes^{1} r(w)) \otimes^{\varepsilon} r(\gamma) \tag{26}$$

where $\alpha \in H_j(A)/B(H_{j-1}(A)), \beta \in H_j(A)_j(A), w \in \Omega^0/k, \gamma \in H_{j-1}(A)/B(H_{j-2}(A))$.
Then the spectral sequence $\{^{\varepsilon}E^s_{\bullet\bullet}(A[x,x^{-1}]\}$ isomorphic to:

$$^{\varepsilon}E^3_{pq}(A[x,x^{-1}]) =^{\varepsilon} H_p(Z/2; HC_q(A[x,x^{-1}]) =^{\varepsilon} HD_n(A[x,x^{-1}]) \tag{27}$$

Therefore factoring the group $HC_{\bullet}(A) \otimes HC_{\cdot-1}(A) \otimes HH^{\infty}_{\cdot}(A)$ by the operator $(1 -^{\varepsilon} r)$ we get $^{\varepsilon}HD_n(A) \otimes^{\varepsilon} HD_{n-1}(A) \otimes HR^{\infty}_n(A)$ which isomorrphic to $^{\varepsilon}HD_n(A[x,x^{-1}])$.

In [8] Masuda has calculated the cyclic homology of algebra $A = k[x]/(x^n)$, $n \geq 2$ with $char(k) = 0$. we use this results for Banach algebra with an involution and study its dihedral homology.

Theorem 5 *For Banach k-algebra A (where k is a field of real or complex numbers), the following dihedral homology holds for $n \geq 2$.*

$$^{1}HD_i(k[x]/(x^n)) = \begin{cases} k^n \ , i \equiv 0 (\mathrm{mod}\, 4) \\ k \ , i \equiv 2 (\mathrm{mod}\, 4) \\ 0 \ , i \equiv 1,3 (\mathrm{mod}\, 4) \end{cases} \tag{28}$$

$$^{-1}HD_i(k[x]/(x^n)) = \begin{cases} k^{n-1} \ , i \equiv 2 (\mathrm{mod}\, 4) \\ 0 \ , i \neq 2 (\mathrm{mod}\, 4) \end{cases} \tag{29}$$

proof: Masuda [8]considered two groups of Hochchinld homology $H_{2m-1}(A) = k^{n-1}$with generators $1, x, ..., x^{n-2}$ and $H_{2m}(A) = k^{n-1}$with generators $x, ...x^{n-1}$and spectral sequence of bicomplex $C_{\bullet\bullet}(A)$

$$\hat{E}^2_{ij}(A) = \begin{cases} k^n \ , \quad i \equiv 0 (\mathrm{mod}\, 2), \ j = 0 \\ k^{n-1} \ , i \equiv 2 (\mathrm{mod}\, 2), \ i > 0 \\ 0 \ , \qquad i \equiv 1 (\mathrm{mod}\, 2) \end{cases} \tag{30}$$

The Connes's operator B In [8] is given as follows : $B : H_{2m-1}(A) \longrightarrow H_{2m}(A)$ is zero homomorphism, $B : H_{2m}(A) \longrightarrow H_{2m+1}(A)$ is an isomrphism and $B : H_0(A) \longrightarrow H_1(A)$ is epimorphism. Under these consideration we get the spectral sequence $\hat{E}^3_{ij}(A)$:

$$\hat{E}^3_{ij}(A) = \begin{cases} k^n \ , \qquad i = j = 0 \\ k^{n-1} \ , \ i = 0, \ j \equiv 0 (\mathrm{mod}\, 2) \\ k \ , \quad i = 0, i \equiv 0 (\mathrm{mod}\, 2) \end{cases}$$

If A is Banach $k-$algebra ($char(k)=0$),then the spectral sequence $\hat{E}^3_{ij}(A)$ give the cyclic homology $HC_\bullet(A) = \underset{\bullet=p+q}{\oplus} \hat{E}^3_{ij}(A) \overset{\wedge 3}{E_{pq}} (A)$. Factoring the group $HC_\bullet(A)$ by $Im(1-^\varepsilon r_\bullet)$,we get:

$$\hat{E}^3_{ij}(A)/(1-^1 r_j) = \begin{cases} k^n, & i=j=0 \\ k, & j=0,\ i\equiv 0(\mathrm{mod}\,2),\ i>0 \\ k^{n-1}, & i=0,\ j\equiv 0(\mathrm{mod}\,4),\ j>0, \end{cases} \tag{31}$$

$$E^{\wedge 3}_{ij}(A)/(1-^{(-1)} r_j) = \begin{cases} k^{n-1}, i=0,\ j\equiv 2(\mathrm{mod}\,4),\ j>0, \\ 0, \quad i\succ 0,\ j\neq 2(\mathrm{mod}\,4). \end{cases} \tag{32}$$

That is

$$^1HD_i(K\,[x]\,/(x^n)) = \begin{cases} k^n,\ i\equiv 0(\mathrm{mod}\,4) \\ k,\ i\equiv 2(\mathrm{mod}\,4) \\ 0,\ i\equiv 1{\cdot}3(\mathrm{mod}\,4), \end{cases} \tag{33}$$

$$^{-1}HD_i(K\,[x]\,/(x^n)) = \begin{cases} k^{n-1}, i\equiv 2(\mathrm{mod}\,4) \\ 0, \quad i\neq 2(\mathrm{mod}\,4) \end{cases} \tag{34}$$

4 Dihedral Homology of tensor algebra

Let A be an involution k-Banach algebrawhere field k (k is real or complex number), M is A-bimodule. Consider the tensor algebra $T_A(M)$. Define an involution on M(involutoin is an automorphism $*: M \longrightarrow M$ of order two such that $(a\ m\ b)^* = b^* m^* a^*$,a,b$\in A$). Suppose the complex $S^{(k)}(A, P_\bullet) = A \otimes_{A\otimes A^{op}} P_\bullet^{\otimes(k+1)}$, where P$_\bullet$ is a chain complex of A-bimodules. Acting on $S^{(k)}(A, P_\bullet)$ by two automorphisms :

$$\begin{aligned} t_k(p_0\otimes ...\otimes p_k) &= (-1)^s p_k \otimes p_0 \otimes ... \otimes p_{k-1}, \\ {}^\varepsilon\theta_k(p_0\otimes ...\otimes p_k) &= (-1)^\upsilon \varepsilon p_i^* \otimes ... \otimes P_1^* \otimes p_0^*, \end{aligned} \tag{35}$$

where

$$\begin{aligned} s &= (\deg\ p_k)(\sum_{i=0}^{k-1} \deg\ p_i), \\ \upsilon &= (k + \sum_{i=0}^{k-1} \deg\ p_i)(k + \sum_{i=0}^{k} \deg\ p_i - 1)/2 \end{aligned} \tag{36}$$

Clearly the automorphisms t_k and ${}^\varepsilon\theta_k$ represent to dihedral group D_{k+1} of order $2(k+1)$. If $P_\bullet$ is free resolution of involutive A-bimodule M , then the complex $S^{(k)}(A,\, P_\bullet)$ can be written as $S^{(k)}(A,\, M)$.

Theorem 6 *Let M be an involutive A-bimodule, and $Tor_i^A(M,M) = 0$, $i > 0$. Then*

$HD_i(T_A(M)) =^{\varepsilon} HD_i(A) \otimes (\otimes_{k=0}^{\infty}(D_{k+1}S^{(k)}(A,M)))$.

proof: *Following [5] the long exact sequence of relative dihedral homology for the short exact sequence $0 \to A \xrightarrow{i} T_A(M) \longrightarrow 0$, is given by*

$$\begin{aligned} ... \quad \longrightarrow \quad & {}^{\varepsilon}HD_i(A \xrightarrow{i} T_A(M)) \xrightarrow{0}{}^{\varepsilon} HD_i(A \longrightarrow 0) \longrightarrow \\ \to \quad & {}^{\varepsilon}HD_i(T_A(M) \longrightarrow 0) \longrightarrow^{\varepsilon} HD_{i-1}(A \xrightarrow{i} T_A(M)) \xrightarrow{0} ... \end{aligned} \tag{37}$$

Since A is the direct sum in $T_A(M)$, then the long exact sequence (37) splits and hence

$$^{\varepsilon}HD_i(T_A(M) =^{\varepsilon} HD_i(A) \otimes^{\varepsilon} HD_i(A \xrightarrow{i} T_A(M)). \tag{38}$$

we show that

$$^{\varepsilon}HD_i(A \longrightarrow T_A(M)) = \otimes_{k=0}^{\infty} {}^{\varepsilon}DH_i(D_{k+1}, S^{(k)}(A,M)) \tag{39}$$

Really

$$\begin{aligned} & T_A(P_\bullet)/(A + [T_A(P_\bullet), T_A(P_\bullet)] + \mathrm{Im}(1 -^{\varepsilon} r)) \\ = \quad & \otimes_{k=0}^{\infty} P_\bullet^{\otimes(k+1)}/(A + [T_A(P_\bullet), T_A(P_\bullet)] + \mathrm{Im}(1 -^{\varepsilon} r)), \end{aligned} \tag{40}$$

since

$$^{\varepsilon}r(p_0 \otimes ... \otimes p_k) = (-1)^{\acute{\upsilon}}\varepsilon(p_0 \otimes ... \otimes p_k)^* = (-1)^{\acute{\upsilon}}\varepsilon p_k^* \otimes ... \otimes p_1^* \otimes p_0^* \tag{41}$$

where

$$\acute{\upsilon} = (k + \deg(p_0 \otimes p_1 \otimes ... \otimes p_k))(k + \deg(p_0 \otimes ... \otimes p_k) - 1)/2 = \upsilon \tag{42}$$

then, we have an isomorphism

$$\begin{aligned} & \otimes_{k=0}^{\infty} P_\bullet^{\otimes(k+1)}/([T_A(P_\bullet), T_A(P_\bullet)] + \mathrm{Im}(1 -^{\varepsilon} r)) \\ = \quad & \otimes_{k=0}^{\infty} A \otimes_{A \otimes A^{op}} P_\bullet^{\otimes(k+1)}/(\mathrm{Im}(1 - t_k) + \mathrm{Im}(1 -^{\varepsilon} \theta_k)). \end{aligned} \tag{43}$$

The homology of chain complex $\otimes_{k=0}^{\infty} A \otimes_{A \otimes A^{op}} P_\bullet^{\otimes(k+1)}/ \mathrm{Im}(1 - t_k) + \mathrm{Im}(1 -^{\varepsilon} \theta_k)$ coincide with $\otimes_{k=0}^{\infty} {}^{\varepsilon}H_\bullet(CD_{k+1}, S^{(k)}(A,M))$.From relations (38), (39) the theorem is proved □

Note that if A is augmented k-algebra with involution, then its dihedral homology is denoted by $\overline{^{\varepsilon}HD_\bullet}(A)$.

Corollary 7 . *Let V be Banach space with an involution, then*

$$\overline{^{\varepsilon}HD_\bullet}(T_k(V) = \otimes_{k=0}^{\infty} {}^{\varepsilon}H_\bullet(D_{k+1}, V^{\otimes(k+1)}), \tag{44}$$

proof: Since

$$ {}^{\varepsilon}HD_\bullet(T_k(V)) ={}^{\varepsilon} HD_\bullet(k) \otimes (\otimes_{i=0}^{\infty}\ {}^{\varepsilon}H_\bullet(D_{i+1}, S^{(i)}(k, v))). \tag{45} $$

we have an isomorphism

$$ \otimes_{k=0}^{\infty\varepsilon} H_\bullet(D_{k+1}, S^{(k)}(k, v)) \simeq \oplus_{i=0}^{\infty\varepsilon} H_\bullet(D_{k+1}, D_{k+1}, V^{\otimes(k+1)}) \tag{46} $$

Really $S_\bullet^{(i)}(k, v) = R_\bullet^{\otimes(i+1)}$, where $R_\bullet$ is free k- module resolution of k-module $V_\bullet$.We can get an isomorphism (46) by considering the the sequence $v \longleftarrow 0 \longleftarrow 0 \longleftarrow ...$ from the resolution $R_\bullet$. The corollary is proved.

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