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An Outline of Multiset Space Algebra

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Abstract

We briefly present the concept of a multiset and some basic operations between multisets used in this paper. We explicate the notion of a multiset space. We introduce two new operations on a multiset space and outline their properties, and show how they give rise to an algebraic structure. Finally, some directions for future work are indicated.

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1 The concept of a multiset

A multiset (mset, for short) is an unordered collection of objects in which, unlike a standard (Cantorian) set, duplicates or multiples of objects are admitted. Each individual occurrence of an object is called its *element*. All duplicates of an object in an mset are considered indistinguishable. The objects of an mset are the distinguishable or distinct elements of the mset. The distinction made between the terms *object* and *element* does enrich the multiset language. However, use of the term element alone may suffice if there does not arise any confusion. An mset containing one occurrence of a, two occurrences of b, and three occurrences of c is notationally written as $[[a, b, b, c, c, c]]$ or $[a, b, b, c, c, c]$ or $[a, b, c]_{1, 2, 3}$ or $[a, b, c]_{\ 1\ 2\ 3}$ or $[a\ b\ c]_{1, 2, 3}$ or $[a, b, c]_{\ 1\ 2\ 3}$ or $[a1, b2, c3]$, or $[1/a, 2/b, 3/c]$ or $[a^1, b^2, c^3]$ or $[a^1\ b^2\ c^3]$ depending on one's convenience.

The number of occurrences of an object x in an mset A, which is finite in most of the studies that involve msets, is called its *multiplicity* or *characteristic value*, usually denoted by $m_A(x)$ or $c_A(x)$ or simply by $A(x)$. A well formed formula $x \in^n y$ semantically reads that x belongs n times to y. The number of distinct elements in an mset A(which need not be finite) and their multiplicities jointly determine its cardinality usually denoted by $\#A$ or $C(A)$ or $|A|$. In other words, the cardinality of an mset is the sum of the multiplicities of all its objects.

The set of distinct elements of an mset is called its *root* or *support*. The cardinality of the root set of an mset is called its *dimension*.

An mset is called *regular* or *constant* if all its objects occur with the same multiplicity and the common multiplicity is called its *height*. For example, $[a, b]_{3,3}$is a regular mset of height 3.

An mset is called *simple* if all its elements are the same. For example,$[a]_3$ is a simple mset. It follows that the root set of a simple mset is a singleton.

A unique mset that does not contain any member is called the empty mset, denoted by ϕ.

Two mset A and B are called equal, denoted by $A = B$, if $m_A(x) = m_B(x)$ for all objects x.

An mset A is called a submultiset (submset, for short) or a multisubset (msubset, for short) of an mset B, denoted by $A \subseteq B$, if $m_A(x) \leq m_B(x)$ for all objects x. An mset is called the *parent* multiset or *overmultiset* in relation to its submsets. For example, $[a, b]_{1,2}$ is a submset of $[a, b, c]_{1,3,1}$ and the latter is a parent mset of the former. It follows that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. Also $A \subset B$ if $A \subseteq B$ and $A \neq B$.

A submset of a given mset is called *whole* if it contains all multiplicities of the common objects. For example, $[a, b]_{2,3}$ is a whole msubset of $[a, b, c]_{2,3,4}$.

A submset of a given mset is called *full* if it contains all objects of the parent mset. For example, $[a, b, c]_{1,2,3}$ is a full msubset of $[a, b, c]_{2,3,4}$.

The powermset of A, denoted by $\widetilde{\wp}(A)$, is the mset of all submsets of A (see [1] and [2] for various details).

An mset is called finite if its root set is finite and the multiplicity of its each object is finite; infinite otherwise. A cardinality-bounded mset space X^n can be defined as the set of all finite msets whose objects are drawn from a ground set X such that objects in any mset $A \in X^n$ occur at most n times.

1.1 Operations between multisets

Let $A, B, C, ...$ be msets over a given generic set D.

Definition 1 Union ($\cup$)

The union of A and B, denoted by $A \cup B$, is the mset defined by $m_{A\cup B}(x) = m_A(x) \cup m_B(x) =$ maximum $\{m_A(x), m_B(x)\}$, being the union of two numbers. That is, an object z occurring a times in A and b times in B, occurs maximum $\{a, b\}$ times in $A \cup B$, if such a maximum exists; otherwise the minimum of $\{a, b\}$ is taken which always exists. In other words, $A \cup B$ is the smallest mset C which contains both A and B i.e., $A \subseteq C$ and $B \subseteq C$.

It follows that for any given mset X there exists an mset Y which contains elements of elements of X, where the multiplicity of an element z in Y is the maximum multiplicity of z as an element of elements of X along with the above stipulation on the existence of such a maximum. We denote this fact by $Y = \cup X$.

Clearly, $Dom(\cup X) = \cup\{Dom(A)|A \in X\}$ and that the multipliticity of z in Y is the maximum of its multiplicities as an element of elements of X if it exists, otherwise the minimum is taken.

For example, if $A = [2\ 3\ 4\ 4]$, $B = [1\ 4\ 3\ 3]$ then $A \cup B = [1\ 2\ 3\ 3\ 4\ 4]$.

Also, it follows that for a finite mset X, the maximum multiplicity of elements of elements of X always exists. However, for certain infinite sets like $X = \{\{y\}, [y]_2, [y]_3 \ldots\}$, the maximum multiplicity of elements of elements of X does not exist, and hence $\cup X = \{y\}$. It is obvious by definition of the union that multiplicity of any $y \in X \neq \phi$ is irrelevant to $\cup X$, and hence $\cup X = \cup X^*$ (see [Bli 89], pp. 48–49, for details).

The above definition can be extended to an arbitrary number of msets as follows:

$\cup A_i = \{m_{\cup iAi}(x).x | m_{\cup iAi}(x) = max_{i\in I} m_{Ai}(x),\ for\ all\ x \in D\}$.

Definition 2 Intersection($\cap$)

The intersection of msets A and B, denoted by $A \cap B$, is the mset defined by $m_{A\cap B}(x) = m_A(x) \cap m_B(x) = minimum\{m_A(x), m_B(x)\}$, being the intersection of two numbers. In other words, $A \cap B$ is the largest mset C which is contained in both A and B i.e., $C \subseteq A, C \subseteq B$.

That is, an object x occurring a times in A and b times in B, occurs minimum $\{a, b\}$ times in $A \cap B$, which always exists. In general, for a given mset X, $Dom(\cap X) = \cap\{Dom(A) : A \in X\}$ and $z \in \cap X$ implies that the multiplicity of z is the minimum of its multiplicities as an element of elements of X.

For example, if $A = [3\ 3\ 3\ 4\ 4], B = [1\ 4\ 3\ 3]$ then$A \cap B = [3\ 3\ 4]$.

The above definition can be extended to an arbitrary number of msets as follows:

$\cap Ai = \{m_{\cap iAi}(x).x | m_{\cap iAi}(x) = min_{i\in I} m_{Ai}(x),\ for\ all\ x \in D\}$.

Note that for any mset X, we have $\cap X \subseteq \cup X$.

Definition 3 *Addition or Sum or Merge (+ or* $\uplus$ *or* $\oplus$*)*

The sum, $A+B$, of msets A and B is the mset defined by $m_{A+B}(x) = m_A(x)+m_B(x)$, for any $x \in D$, being the arithmetic addition of two numbers.

That is, an object x occurring a times in A and b times in B, occurs $a+b$ times in $A+B$. This, being the direct sum or arithmetic addition of two numbers, is sometimes called the counting law.

For example,
if $A = [1\ 1\ 2\ 2\ 4\ 4\ 4]$, $B = [1\ 2\ 3\ 3]$ then $A+B = [1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 4\ 4\ 4]$.

The above definition can be extended to an arbitrary number of msets as follows:

$$\sum A_i = \left\{ m_{\sum\limits_i A_i(x).x} : m_{\sum\limits_i A_i}(x) = \sum_{i \in I} m_{Ai}(x), \text{for all x in } D \right\}$$

Note that in a cardinality- bounded mset space X^n, the definition of arithmetic sum needs to be modified as follows:

$m_{A+B}(x) = minimum\ \{n,\ m_A(x) + m_B(x)\}$.

It follows that the insertion of an element x into an mset A gives rise to a new mset $A' = A + x$ such that $m_{A'}(x) = m_A(x) + 1$ and $m_{A'}(y) = m_A(y)$ for all $y \neq x$.

Note also that if X be an infinite mset, then the multiplicity of some mset $A \in \uplus X$ may not be finite. In that case, the multiplicity of A in $\cup X$ is used.

For example, if $X = \{\{z\}, [z]_2, [z]_3\ ...\}$ then $+X = \cup X = \{z\}$ (see ([Bli 89], p. 51, for details).

The following are some properties of mset operations: $\cup$, $\cap$ and $+$ are commutative and associative. Identity law holds for $\cup, \cap$ and $+$. Idempotency holds for $\cup$ and $\cap$ but not for $+$. Distributivity holds for $\cup$ and $\cap$. However, $+$ is distributive over $\cup$ and $\cap$ but the converse is false (see [1] and [2] for details).

2 Multiset space and some related notions

Definition 4 *Multiset Space : Let X be a finite set with cardinality m, usually called an $m-set$. Let $X^n(m)$ denote the set of all multisets each having m objects occurring with multiplicities at most n times, including 0. We call $X^n(m)$, a cardinality bounded multiset space for X.*

Let $X = \{a_1, a_2, a_3, ..., a_m\}$ be an ordered $m-$set and let $X^n(m)$ be a cardinality bounded space. Let an arbitrary element of $X^n(m)$ be denoted by $X_{<pi>}$ where $< p_i >$ is an ordered $m-tuple$, p_i is the multiplicity of i^{th} object in X, $1 \leq i \leq m$ and $0 \leq p_i \leq n$. For convenience $X_{<pi>}$ will be denoted by X_{pi}. Following the aforesaid notation, the term X_p of $X^n(m)$ would mean that all its objects have the same multiplicity p, that is X_p is a regular mset. Also, X_0 will refer to the empty multiset or the origin of $X^n(m)$.

It is easy to see that $X^n(m)$ will have $(n+1)^m$ elements where n and m are positive integers.

2.1 Construction of $X^n(m)$ and their patterns

We wish to bring our point here by way of considering some concrete cases as below:

Let $X = \{a, b\}$, that is, $m = 2$. Then,

$X^0(2) = \{[a, b]_{0\,0}\}$

$X^1(2) = \{[a, b]_{0\,0}, [a, b]_{0\,1}, [a, b]_{1\,0}, [a, b]_{1\,1}\}$

$X^2(2) = \{[a, b]_{0\,0}, [a, b]_{0\,1}, [a, b]_{1\,0}, [a, b]_{1\,1}, [a, b]_{2\,0}, [a, b]_{0\,2}, [a, b]_{2\,1}, [a, b]_{1\,2}, [a, b]_{2\,2}\}$.

$X^3(2) = \{[a, b]_{0\,0}, [a, b]_{0\,1}, [a, b]_{1\,0}, [a, b]_{1\,1}, [a, b]_{1\,2}, [a, b]_{2\,1}, [a, b]_{0\,2}, [a, b]_{2\,0}, [a, b]_{2\,2},$
$[a, b]_{3\,0}, [a, b]_{0\,3}, [a, b]_{1\,3}, [a, b]_{3\,1}, [a, b]_{3\,2}, [a, b]_{2\,3}, [a, b]_{3\,3}\}$

$X^4(2) = \{[a, b]_{0\,0}, [a, b]_{0\,1}, [a, b]_{1\,0}, [a, b]_{1\,1}, [a, b]_{1\,2}, [a, b]_{2\,1}, [a, b]_{0\,2}, [a, b]_{2\,0}, [a, b]_{2\,2},$
$[a, b]_{3\,0}, [a, b]_{0\,3}, [a, b]_{1\,3}, [a, b]_{3\,1}, [a, b]_{3\,2}, [a, b]_{2\,3}, [a, b]_{3\,3}, [a, b]_{4\,0}, [a, b]_{0\,4},$
$[a, b]_{0\,4}, [a, b]_{1\,4}, [a, b]_{2\,4}, [a, b]_{4\,2}, [a, b]_{4\,3}, [a, b]_{3\,4}, [a, b]_{4\,4}\}$.

$X^5(2) = \{[a, b]_{0\,0}, [a, b]_{0\,1}, [a, b]_{0\,2}, [a, b]_{0\,3}, [a, b]_{0\,4}, [a, b]_{1\,0}, [a, b]_{1\,1}, [a, b]_{1\,2}, [a, b]_{1\,3},$
$[a, b]_{1\,4}, [a, b]_{2\,0}, [a, b]_{2\,1}, [a, b]_{2\,2}, [a, b]_{2\,3}, [a, b]_{2\,4}, [a, b]_{3\,0}, [a, b]_{3\,1}, [a, b]_{3\,2},$
$[a, b]_{3\,3}, [a, b]_{3\,4}, [a, b]_{4\,0}, [a, b]_{4\,1}, [a, b]_{4\,2}, [a, b]_{4\,3}, [a, b]_{4\,4}, [a, b]_{5\,0}, [a, b]_{0\,5},$
$[a, b]_{5\,1}, [a, b]_{1\,5}, [a, b]_{2\,5}, [a, b]_{5\,2}, [a, b]_{3\,5}, [a, b]_{5\,3}, [a, b]_{5\,4}, [a, b]_{4\,5}, [a, b]_{5\,5}\}$

Similarly, $X^n(2)$ are obtained for $n = 6, 7, 8, ...$

We take into account the cardinality of the elements of $X^n(m)$ to generate a pattern with reference to the frequency of their occurrences. Tables 2.1 and 2.2 below provide a schematic representation of the said patterns for $X^n(2)$ and $X^n(3)$ respectively, for some finite values of n.

Table 2.1

Frequency values corresponding to the cardinality of the elements of $X^n(2)$:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$X^0(2)$	1																
$X^1(2)$	1	2	1														
$X^2(2)$	1	2	3	2	1												
$X^3(2)$	1	2	3	4	3	2	1										
$X^4(2)$	1	2	3	4	5	4	3	2	1								
$X^5(2)$	1	2	3	4	5	6	5	4	3	2	1						
$X^6(2)$	1	2	3	4	5	6	7	6	5	4	3	2	1				
$X^7(2)$	1	2	3	4	5	6	7	8	7	6	5	4	3	2	1		
$X^8(2)$	1	2	3	4	5	6	7	8	9	8	7	6	5	4	3	2	1

In the table 2.1, the first row 0, 1, 2, 3, ..., 16 represents the cardinality of the elements in $X^n(2)$ while the remaining rows represent the corresponding frequency of their occurrence. The set $X^0(2)$ contains only one element with cardinality zero, the set $X^1(2)$ contains four elements; one element with cardinality zero, two elements with cardinality one and one element with cardinality two and so on.

It can be easily seen from the above that the number of elements in $X^n(m)$ is the same as the sum of the frequencies of the objects. The highest cardinality of an element $X_{pi} \in X^n(m)$ is mn which is unique while the smallest is zero. The highest frequency of an element in $X^n(m)$ is $(n+1)$ and the corresponding cardinality is n. A graphical representation of $X^n(2)$ can be found in figure 1 on page 1524.

The graph of $X^n(2)$ is symmetric about n.
Now, let $m = 3$. We compute $X^n(m)$ for $n = 0, 1, 2, ..., 6$.
Let $X = \{a, b, c\}$
$X^0(3) = [a, b, c]_{0\,0\,0}$
$X^1(3) = \{[a, b, c]_{0\,0\,0}, [a, b, c]_{0\,0\,1}, [a, b, c]_{0\,1\,0}, [a, b, c]_{0\,1\,1}, [a, b, c]_{1\,0\,0}, [a, b, c]_{1\,0\,1}, [a, b, c]_{1\,1\,0}, [a, b, c]_{1\,1\,1}$
$X^2(3) = \{[a, b, c]_{0\,0\,0}, [a, b, c]_{0\,0\,1}, [a, b, c]_{0\,1\,0}, [a, b, c]_{0\,1\,1}, [a, b, c]_{1\,0\,0}, [a, b, c]_{1\,0\,1}, [a, b, c]_{1\,1\,0}, [a, b, c]_{1\,1\,1}, [a, b, c]_{0\,0\,2}, [a, b, c]_{0\,1\,2}, [a, b, c]_{0\,2\,0}, [a, b, c]_{0\,2\,1}, [a, b, c]_{0\,2\,2}, [a, b, c]_{1\,2\,0}, [a, b, c]_{1\,0\,2}, [a, b, c]_{1\,2\,1}, [a, b, c]_{1\,1\,2}, [a, b, c]_{1\,2\,2}, [a, b, c]_{2\,0\,0}, [a, b, c]_{2\,1\,0}, [a, b, c]_{2\,0\,1}, [a, b, c]_{2\,1\,1}, [a, b, c]_{2\,0\,2}, [a, b, c]_{2\,2\,0}, [a, b, c]_{2\,1\,2}, [a, b, c]_{2\,2\,1}, [a, b, c]_{2\,2\,2}\}$
Similarly, $X^n(3)$ are obtained for $n = 3, 4, ...$
Table 2.2: Frequency Values corresponding to the cardinality of the elements of $X^n(3)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$X^0(3)$	1																		
$X^1(3)$	1	3	3	1															
$X^2(3)$	1	3	6	7	6	3	1												
$X^3(3)$	1	3	6	7	10	12	12	6	3	1									
$X^4(3)$	1	3	6	10	15	18	19	18	15	10	6	3	1						
$X^5(3)$	1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1			
$X^6(3)$	1	3	6	10	15	21	28	33	36	37	36	33	28	21	15	10	6	3	1

A graphical representation of $X^n(3)$ can be found in figure 2 on page 1525.

Likewise, $X^n(m)$ can be obtained for various values of n and m. It is observed that for $m \geq 3$ the graph of $X^n(m)$ is normal.

2.2 Formula for computing the frequency number of a multiset space

Let us call the number of elements of $X^n(m)$ with the same cardinality as the frequency number of that class of elements and denote it by $FX^n(m)$. This number can be generated using the recurrence formula as follows:

For example, the frequency number of $X^n(2)$, denoted by $FX^n(2)$, is given by

$$FX^n(2) = \begin{cases} \{k+1\}_n & when \;\; 0 \leq k \leq n \\ 2n - k + 1, & when \;\; n+1 \leq k \leq 2n \end{cases}$$

Likewise, the frequency number of $X^n(3)$, denoted by $FX^n(3)$, is given by

$$FX^n(3) = \begin{cases} \sum_{i=1}^{k}(i+1) & \text{when } 0 \le k \le n \\ \sum_{i=1}^{n}(i+1) + \sum_{i=n+1}^{k}(3n-2i+1), & \text{when } n+1 \le k \le 2n \\ \sum_{i=1}^{n}(i+1) + \sum_{i=n+1}^{2n}(3n-2i) + \sum_{i=2n+1}^{k}(i-3n-2), & \text{when } 2n+1 \le k \le 3n \end{cases}$$

Looking at the aforesaid formulation for $X^n(3)$ it is observed that for $n < 2$ the first two summations would suffice to generate the required frequency numbers; however, for $n \ge 3$, all the three summations are necessary. No general formulation seems to be forthcoming in this regard.

3 Multiset space algebra

3.1 Operations on multiset space

Definition 5 *Union modulo* $\left(\bar{\cup}\right)$ *and Product modulo* $\left(\mathring{\cup}\right)$

Let $X^n(m)$ be an mset space. Consider the set $X = \{a_1, a_2, a_3, \ldots, a_m\}$. Let $X_{pi} = [a_1, a_2, a_3, \ldots, a_m]_{p1\ p2\ p3\ \ldots\ pm}$ and $X_{qi} = [a_1, a_2, a_3, \ldots, a_m]_{q1\ q2\ q3\ \ldots\ qm}$ be two elements of $X^n(m)$, $i = 1, 2, 3, \ldots, m$.

The union modulo $\bar{\cup}$ on $X^n(m)$ is defined as $X_{pi} \mathbin{\bar{\cup}} X_{qi} = X_{(pi+qi)\bmod(n+1)}$, for $i = 1, 2, 3, \ldots, m$. For example let $X = \{a, b, c\}$ and let $X_{3,2,4}$ and $X_{4,1,1} \in X^5(3)$ then $X_{3,2,4} \mathbin{\bar{\cup}} X_{4,1,1} = X_{1,3,5}$.

The product modulo $\mathring{\cup}$ on $X^n(m)$ is defined as $X_{pi} \mathbin{\mathring{\cup}} X_{qi} = X_{(piqi)\bmod n}$ for $i = 1, 2, 3, \ldots, m$. For example let $X = \{a, b, c\}$ and let $X_{3,2,4}$ and $X_{4,1,1} \in X^5(3)$ then $X_{3,2,4} \mathbin{\mathring{\cup}} X_{4,1,1} = X_{0,2,4}$.

3.2 Properties of Union Modulo and Product Modulo

Let $X_{pi} = X_{p1\ p2\ \ldots\ pm}$; $X_{qi} = X_{q1\ q2\ \ldots\ qm}$ and $X_{ri} = X_{r1\ r2\ \ldots\ rm}$ be any three elements in $X^n(m)$.Then we have the following:

i) *Closure:* $X^n(m)$ is closed under $\bar{\cup}$ and $\mathring{\cup}$

ii) *Commutativity:* $\bar{\cup}$ and $\mathring{\cup}$ are both commutative

iii) *Associativity:* $\bar{\cup}$ and $\mathring{\cup}$are both associative

iv) *Identity law:* X_0 and X_1 are identity elements of $X^n(m)$ with respect to $\bar{\cup}$ and $\mathring{\cup}$ respectively

v) Existence of inverse: each element of $X^n(m)$ has union modulo inverse and product modulo inverse.

vi) Cardinality: $\left|X_{pi} \bar{\cup} X_{qi}\right| \leq |X_{pi}| + |X_{qi}|$ and $\left|X_{pi} \mathring{\cup} X_{qi}\right| \leq |X_{pi}|\,|X_{qi}|$

vii) $X^n(m)$ is not zerosumfree with respect to $\bar{\cup}$ since there is existence of inverse for each element in $X^n(m)$, i.e., $X_p \bar{\cup} X_q = X_0$ does not necessaraly mean $p = q = 0$.

viii) $X^n(m)$ is not cancellative under $\bar{\cup}$ .For example $X_{1\,1\,0}$, $X_{1\,0\,1}$, $X_{1\,1\,1}$ be elements of $X^2(3)$. Then $X_{1\,0\,1}\mathring{\cup}\, X_{1\,1\,1} = X_{1\,0\,1}$ and $X_{1\,0\,1}\mathring{\cup}\, X_{1\,0\,1} = X_{1\,0\,1}$, But $X_{1\,1\,1} \neq X_{1\,0\,1}$

ix) X_0 is absorbing under $\mathring{\cup}$. For example $X_0 \mathring{\cup} X_p = X_0$.

x) Distributive properties: X_{pi}, X_{qi}, $X_{ri} \in X^n(m)$,the following hold:

a) $X_{pi}\mathring{\cup}\left(X_{qi} \bar{\cup} X_{ri}\right) = \left(X_{pi} \mathring{\cup} X_{qi}\right) \bar{\cup} \left(X_{pi} \mathring{\cup} X_{ri}\right)$

b) $X_{pi}\bar{\cup}\left(X_{qi} \mathring{\cup} X_{ri}\right) \neq \left(X_{pi} \bar{\cup} X_{qi}\right) \mathring{\cup} \left(X_{pi} \bar{\cup} X_{ri}\right)$

c) $X_{pi} \mathring{\cup} \left(X_{qi} \cup X_{ri}\right) \neq \left(X_{pi} \mathring{\cup} X_{qi}\right) \cup \left(X_{pi} \mathring{\cup} X_{ri}\right)$

d) $X_{pi} \cup \left(X_{qi} \mathring{\cup} X_{ri}\right) \neq \left(X_{pi} \cup X_{qi}\right) \mathring{\cup} \left(X_{pi} \cup X_{ri}\right)$

e) $X_{pi} \mathring{\cup} \left(X_{qi} \cap X_{ri}\right) \neq \left(X_{pi} \mathring{\cup} X_{qi}\right) \cap \left(X_{pi} \mathring{\cup} X_{ri}\right)$

f) $X_{pi} \cap \left(X_{qi} \mathring{\cup} X_{ri}\right) \neq \left(X_{pi} \cap X_{qi}\right) \mathring{\cup} \left(X_{pi} \cap X_{ri}\right)$

g) $X_{pi} \bar{\cup} \left(X_{qi} \cup X_{ri}\right) \neq \left(X_{pi} \bar{\cup} X_{qi}\right) \cup \left(X_{pi} \bar{\cup} X_{ri}\right)$

h) $X_{pi} \cup \left(X_{qi} \bar{\cup} X_{ri}\right) \neq \left(X_{pi} \cup X_{qi}\right) \bar{\cup} \left(X_{pi} \cup X_{ri}\right)$

i) $X_{pi} \bar{\cup} \left(X_{qi} \cap X_{ri}\right) \neq \left(X_{pi} \bar{\cup} X_{qi}\right) \cap \left(X_{pi} \bar{\cup} X_{ri}\right)$

j) $X_{pi} \cap \left(X_{qi} \bar{\cup} X_{ri}\right) \neq \left(X_{pi} \cap X_{qi}\right) \bar{\cup} \left(X_{pi} \cap X_{ri}\right)$

A nonempty multiset space $X^n(m)$,equipped with the operations of union modulo $\bar{\cup}$ and product modulo $\mathring{\cup}$ resembles with some well known algebraic structures such as

a) $\left(X^n(m), \bar{\cup}\right)$ is a monoid, a semigroup and an abelian group,

b) $\left(X^n(m), \mathring{\cup}\right)$ is a monoid, a semigroup and an abelian group,

c) $\left(X^n(m), \bar{\cup}, \mathring{\cup}\right)$ is a semiring and a commutative ring,

d) $\left(X^n(m), \bar{\cup}, \mathring{\cup}\right)$ is a semidioids, and

e) $\left(X^n(m), \bar{\cup}, \mathring{\cup}\right)$ is an integral domain, to mention a few.

3.3 Future perspective

A variety of results related to the aforesaid structures can be investigated. The notion of open and closed sets, and hence that of topologies can be defined in a multiset space. Also, the notions of topological group and topological ring of a multiset space can be explicated.

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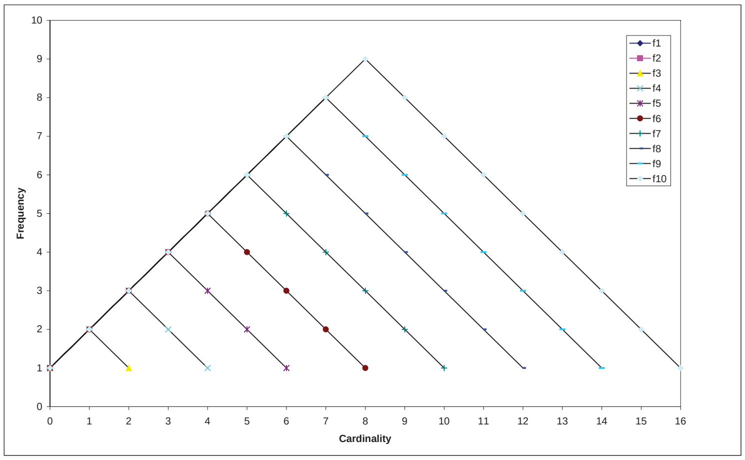

Figure 1: Symmetry

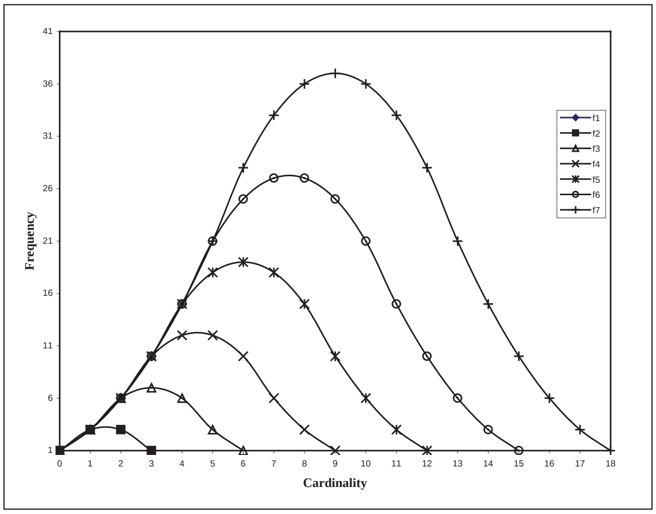

Figure 2: Normal distribution