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A Normal Form for Cubic Surfaces

Zane Kun Li

1969 Frist Center
Princeton, NJ 08544, USA
zkli@princeton.edu

Abstract

We consider cubic surfaces with rational coefficients that contain a rational point and satisfy a certain condition regarding their coefficients. Each such cubic surface is shown to be birationally equivalent to a surface of the form $z^2 = f(x, y)$, where $f(x, y)$ is a polynomial of degree at most 4. Our method is similar to that which Tate and Silverman used in [2] to put cubic curves into Weierstrass form.

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1 Introduction

In $\mathbb{CP}^3$, the general homogeneous equation for a cubic surface with rational coefficients is given by

$$\begin{aligned} a_1X^3 + a_2Y^3 + a_3Z^3 + a_4W^3 + 3a_5X^2Y + 3a_6XY^2 + 3a_7Y^2Z + 3a_8YZ^2 \\ + 3a_9W^2X + 3a_{10}WX^2 + 3a_{11}W^2Y + 3a_{12}WY^2 + 3a_{13}W^2Z \\ + 3a_{14}WZ^2 + 3a_{15}X^2Z + 3a_{16}XZ^2 + 6a_{17}WXY + 6a_{18}WXZ \\ + 6a_{19}WYZ + 6a_{20}XYZ = 0 \end{aligned} \tag{1}$$

where $a_1, a_2, a_3, \ldots, a_{20} \in \mathbb{Q}$. We can obtain its affine part, and hence the general equation for a cubic surface with rational coefficients in affine space, by dehomogenizing by W.

This note is organized as follows: we first prove the following theorem in Section 2.

Theorem 1.1. *Suppose we have an affine space cubic surface with rational coefficients and a rational point. Furthermore, suppose that its coefficients satisfy one of the following conditions:*

a) $a_8 = 0$ *and* $a_{16} \neq 0$*,*

b) $a_8 \neq 0$ *and* $a_{16} = 0$*, or*

c) $a_8 \neq 0$ *and* $a_{16} \neq 0$*.*

Then the cubic surface is birationally equivalent to a surface of the form $z^2 = f(x, y)$ *where* $f(x, y)$ *is a polynomial of the degree at most 4.*

In Section 3, we obtain a proposition in which if $a_6a_{16} \neq a_{20}^2$ and $a_{10}a_{16} \neq 0$ for the cubic surface equation, we can remove one more term from the equation $z^2 = f(x, y)$ and find that the degree of $f(x, y)$ is 4. Finally, in Section 4, we end with a few questions that are some continuations of this research.

From the theorem and proposition proven in this note we know that under certain conditions, a cubic surface is birationally equivalent to a quartic surface of the form

$$c_1x^2y^2 + c_2xy^3 + x^3 + c_3xy^2 + c_4x^2 + c_5y^2 + c_6xy + c_7x + c_8y + c_9 - z^2 = 0$$

where c_i is defined in the proof of Proposition 3.1. This allows us to analyze some cases of the intersection in space between a quadric surface and a cubic surface as a special case of an intersection between a quadric surface and a quartic surface.

The author originally analyzed the intersection between a quadric surface and a cubic surface in [1] by adapting the methods of parameterization demonstrated by Wang et al. in [3]. This however was limited because the author assumed that the intersection had a 3-fold point. By analyzing the intersection space curve as an intersection of a quadric surface and a special type of quartic surface, we may be able to allow a more general case to be analyzed.

This normal form may make analyzing the intersection between a quadric surface and a cubic surface computationally more efficient since most of the terms of the normal form are in either x or y and the only term not in x or y is a square. This normal form may also help in trying to find an explicit normal form for the intersection of a quadric surface with a cubic surface; see [4] for the case of the intersection of two quadric surfaces.

2 Proof of Theorem 1.1

Proof. Given a cubic surface with rational coefficients and a rational point O on the cubic surface, we consider the cubic surface S in $\mathbb{CP}^3$ obtained by homogenizing the equation for the affine space cubic surface. It is of a form as displayed in (1).

Let X_1 be the rational point on S, that is, the point equivalent to O on the affine space cubic surface. We choose $W = 0$ to be the tangent plane tangent to S at X_1. Call this plane T_1. Let $C_1 = S \cap T_1$. Since X_1 is on both S and T_1, X_1 is on C_1. Now draw a line that is tangent to C_1 at X_1 which intersects C_1 at another point. Denote it X_2. Draw another line tangent to C_1 at X_2. This line will intersect C_1 at another point. Denote it X_3. We note that X_2 and X_3 are on C_1.

We now take $X = 0$ to be the tangent plane tangent to S at X_2. Call this plane T_2. Let $C_2 = S \cap T_2$. Since X_2 is on both S and T_2, X_2 is on C_2. We also note that X_3 is on C_2 since the tangent plane at a point on a surface contains all lines that are tangent to the surface at that point and X_2X_3 is tangent to the surface at X_2. Draw the line that is tangent to C_2 at X_3 and denote the line's intersection with C_2 by X_4.

We choose coordinates (X, Y, Z, W) in $\mathbb{CP}^3$ for the cubic surface S such that $X_1 = (1, 0, 0, 0)$, $X_2 = (0, 1, 0, 0)$, $X_3 = (0, 0, 1, 0)$, and $X_4 = (0, 0, 0, 1)$. We have the following lines

$$\begin{array}{ll} X_1X_2: & Z = W = 0, \\ X_1X_3: & Y = W = 0, \\ X_2X_3: & X = W = 0, \\ X_2X_4: & X = Z = 0, \text{ and} \\ X_3X_4: & X = Y = 0. \end{array}$$

The line X_1X_2 is tangent to C_1 at X_1. The line X_1X_3 is tangent to S at X_1. The line X_2X_3 is tangent to C_1 at X_2. The line X_2X_4 is tangent to S at X_2. The line X_3X_4 is tangent to C_2 at X_3. The lines X_1X_3 and X_2X_4 are tangent to S because they are in the planes of T_1 and T_2, respectively. With this construction, we have the following lemma.

Lemma 2.1. *By the construction described above, the terms a_1, a_2, a_3, a_4, a_5, a_7, a_{12}, a_{14}, and a_{15} in* (1) *are equal to 0.*

Proof. First, we consider the case when S intersects the line X_1X_2. Since both $Z = 0$ and $W = 0$, (1) becomes

$$a_1X^3 + a_2Y^3 + 3a_5X^2Y + 3a_6XY^2 = 0. \tag{2}$$

Since X_1X_2 contains X_1, from (2) we can see that

$$a_1 + a_2Y^3 + 3a_5Y + 3a_6Y^2 = 0. \tag{3}$$

We note that X_1X_2 is tangent to S at X_1 and at X_1, $Y = 0$. So (3) has $Y = 0$ as a double root. Hence $a_1 = 0$ and $a_5 = 0$.

Second, we consider the case when S intersects the line X_1X_3. As both $Y = 0$ and $W = 0$, (1) becomes

$$a_1X^3 + a_3Z^3 + 3a_{15}X^2Z + 3a_{16}XZ^2 = 0. \tag{4}$$

Since X_1X_3 contains X_1, by (4),

$$a_1 + a_3Z^3 + 3a_{15}Z + 3a_{16}Z^2 = 0. \tag{5}$$

We note that X_1X_3 is tangent to S at X_1 and at X_1, $Z = 0$. So (5) has $Z = 0$ as a double root. Hence $a_1 = 0$ and $a_{15} = 0$.

Third, we consider the case when S intersects the line X_2X_3. Since for this line both $X = 0$ and $W = 0$, (1) becomes

$$a_2Y^3 + a_3Z^3 + 3a_7Y^2Z + 3a_8YZ^2 = 0. \tag{6}$$

Since X_2X_3 contains X_2, from (6) we find that

$$a_2 + a_3Z^3 + 3a_7Z + 3a_8Z^2 = 0. \tag{7}$$

We note that X_2X_3 is tangent to S at X_2 and at X_2, $Z = 0$. So (7) has $Z = 0$ as a double root. Hence $a_2 = 0$ and $a_7 = 0$.

Fourth, we consider the case when S intersects the line X_2X_4. As both $X = 0$ and $Z = 0$, (1) becomes

$$a_2Y^3 + a_4W^3 + 3a_{11}W^2Y + 3a_{12}WY^2 = 0. \tag{8}$$

Since X_2X_4 contains X_2, from (8) we have

$$a_2 + a_4W^3 + 3a_{11}W^2 + 3a_{12}W = 0. \tag{9}$$

We note that X_2X_4 is tangent to S at X_2 and at X_2, $W = 0$. So (9) has $W = 0$ as a double root. Hence $a_2 = 0$ and $a_{12} = 0$.

Finally, we consider the case when S intersects the line X_3X_4. Since both $X = 0$ and $Y = 0$, (1) becomes

$$a_3Z^3 + a_4W^3 + 3a_{13}W^2Z + 3a_{14}WZ^2 = 0. \tag{10}$$

Since X_3X_4 contains X_3, from (10) we can see that

$$a_3 + a_4W^3 + 3a_{13}W^2 + 3a_{14}W = 0. \tag{11}$$

We note that X_3X_4 is tangent to S at X_3 and at X_3, $W = 0$. So (11) has $W = 0$ as a double root. Hence $a_3 = 0$ and $a_{14} = 0$.

We now remove one more term from (1). Since X_4 is on the surface S, $X = Y = Z = 0$ and $W = 1$. Therefore all terms in (1) vanish except for a_4W^3. We now have $a_4W^3 = 0$, and hence $a_4 = 0$.

This completes the proof of the lemma. $\square$

By the lemma above, the terms X^3, Y^3, Z^3, W^3, X^2Y, Y^2Z, WY^2, WZ^2, and X^2Z are eliminated from (1) since their coefficients are 0. The terms XY^2, YZ^2, W^2X, WX^2, W^2Y, W^2Z, XZ^2, WXY, WXZ, WYZ, and XYZ remain. We dehomogenize the remaining terms in (1) with respect to W and obtain that

$$a_6xy^2+(a_8y+a_{16}x)z^2+a_9x+a_{10}x^2+a_{11}y+a_{13}z+2a_{17}xy+2a_{18}xz+2a_{19}yz+2a_{20}xyz=0.$$

Since either $a_8=0, a_{16}\neq 0$; $a_8\neq 0, a_{16}=0$; or $a_8\neq 0, a_{16}\neq 0$, we know that $(a_8y+a_{16}x)$ is 0 for only finitely many x and y. By applying the birational transformation $z\to\dfrac{z}{a_8y+a_{16}x}$, we have

$$a_6xy^2+\frac{z^2}{a_8y+a_{16}x}+a_9x+a_{10}x^2+a_{11}y+\frac{a_{13}z}{a_8y+a_{16}x} \\ +2a_{17}xy+\frac{2a_{18}xz}{a_8y+a_{16}x}+\frac{2a_{19}yz}{a_8y+a_{16}x}+\frac{2a_{20}xyz}{a_8y+a_{16}x}=0.$$

We then multiply both sides by $(a_8y+a_{16}x)$ and get

$$a_6xy^2(a_8y+a_{16}x)+z^2+a_9x(a_8y+a_{16}x)+a_{10}x^2(a_8y+a_{16}x)+a_{11}y(a_8y+a_{16}x) \\ +a_{13}z+2a_{17}xy(a_8y+a_{16}x)+2a_{18}xz+2a_{19}yz+2a_{20}xyz=0.$$

For simplicity, define $g(x,y)=(a_6xy^2+a_9x+a_{10}x^2+a_{11}y+2a_{17}xy)(a_8y+a_{16}x)$. Then the previous equation can be written as

$$z^2+(a_{13}+2a_{18}x+2a_{19}y+2a_{20}xy)z+g(x,y)=0$$

which can be rewritten as

$$\left(z+\frac{a_{13}+2a_{18}x+2a_{19}y+2a_{20}xy}{2}\right)^2+g(x,y)-\frac{(a_{13}+2a_{18}x+2a_{19}y+2a_{20}xy)^2}{4}=0.$$

We then apply the transformation $z\to z-\dfrac{a_{13}+2a_{18}x+2a_{19}y+2a_{20}xy}{2}$ and get that

$$z^2=\frac{(a_{13}+2a_{18}x+2a_{19}y+2a_{20}xy)^2}{4}-g(x,y). \tag{12}$$

We note that the right hand side of (12) is of at most degree 4 since the coefficients of the x^2y^2 and xy^3 terms could both be 0. Hence a cubic surface with rational coefficients satisfying the initially stated conditions on the coefficients and containing a rational point is birationally equivalent to a surface of the form $z^2=f(x,y)$ where the degree of $f(x,y)$ is at most 4.

This completes the proof of the theorem. □

3 Eliminating the Term x^2y

By expanding (12), we obtain the following proposition.

Proposition 3.1. *If $a_6a_{16} \neq a_{20}^2$ and $a_{10}a_{16} \neq 0$, the term x^2y can be eliminated from* (12).

Proof. We note that if we expand (12), then

$$\begin{aligned} z^2 = & - a_{10}a_{16}x^3 - a_6a_{16}x^2y^2 + a_{20}^2x^2y^2 - 2a_{16}a_{17}x^2y - a_8a_{10}x^2y \\ & + 2a_{18}a_{20}x^2y - a_9a_{16}x^2 + a_{18}^2x^2 - a_6a_8xy^3 - 2a_8a_{17}xy^2 \\ & + 2a_{19}a_{20}xy^2 - a_{11}a_{16}xy - a_8a_9xy + a_{13}a_{20}xy + 2a_{18}a_{19}xy \\ & + a_{13}a_{18}x - a_8a_{11}y^2 + a_{19}^2y^2 + a_{13}a_{19}y + \frac{1}{4}a_{13}^2. \end{aligned}$$

If $a_{10}a_{16} \neq 0$, the above equation can be rewritten as

$$z^2 = x^3 + x^2(b_1y^2 + b_2y + b_3) + xy(b_4y^2 + b_5y + b_6) + b_7x + (b_8y^2 + b_9y + b_{10})$$

where each b_i is a combination of a_i. If $a_6a_{16} \neq a_{20}^2$, by completing the square the above equation can be rewritten as

$$z^2 = x^3+b_1x^2\left(\left(y + \frac{b_2}{2b_1}\right)^2 + \frac{b_3}{b_1} - \frac{b_2^2}{4b_1^2}\right)+xy(b_4y^2+b_5y+b_6)+b_7x+(b_8y^2+b_9y+b_{10}).$$

We apply the transformation $y \to y-\dfrac{b_2}{2b_1}$, which transforms the above equation to the form

$$z^2 = x^3 + c_1x^2y^2 + c_2xy^3 + c_3xy^2 + c_4x^2 + c_5y^2 + c_6xy + c_7x + c_8y + c_9 \quad (13)$$

where each c_i is a combination of b_i. Hence we see that we have eliminated the term x^2y from equation (12).

This completes the proof of the proposition. □

We note that if we add the conditions $a_6a_{16} \neq a_{20}^2$ and $a_{10}a_{16} \neq 0$ to Theorem 1.1, then the cubic surface is birationally equivalent to a surface of the form $z^2 = f(x, y)$ where the degree of $f(x, y)$ is 4.

4 Questions

1. Can we eliminate more terms from (13)?

2. Does there exist a relation between the coefficients (or roots) of $f(x, y)$ and the singularity of cubic surfaces?

3. We note the pattern that a cubic curve is birationally equivalent to $y^2 = f(x)$ where the degree of $f(x)$ is 3, and that a cubic surface under certain conditions is birationally equivalent to $z^2 = f(x, y)$ where the degree of $f(x, y)$ is 4. Is it true that a cubic hypersurface under certain conditions is birationally equivalent to $x_n^2 = f(x_1, x_2, \dots, x_{n-1})$ with the degree of $f(x_1, x_2, \dots, x_{n-1})$ being $n + 1$?

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