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Band Congruences on E-Inversive E-Semigroups

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Abstract

In this paper, we investigate band congruences and rectangular band congruences on E-inversive E-semigroups. A semigroup S is a band if every element of S is an idempotent and a band S is rectangular band if $a = aba$ for all $a, b \in S$. A semigroup S is an E-inversive E-semigroup if for every $a \in S$, there exists $x \in S$ such that ax is an idempotent and the set of all idempotents of S forms a subsemigroup.

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1 Introduction

A congruence ρ on a semigroup S is a band congruence if S/ρ is a band and it is a rectangular band congruence if S/ρ is a rectangular band. In 1985, Janet E. Mills[3] studied a rectangular band congruence on an orthodox semigroup and Weipoltshammer [5] described some special congruences on E-inversive E-semigroup, such as the least group congruence, certain semilattice congruences, some regular congruences and a certain idempotent separating congruence. The strategy was to generalize known results for orthodox semigroups to E-inversive E-semigroup. Weipoltshammer showed that an E-inversive E-semigroup with a congruence ρ containing Green relation $\mathcal{D}$ has a semilattice congruence. These results generalize the corresponding results for regular semigroups [see [5], [1]].

For any a in a semigroup S is said to be *E-inversive* [5] if there exists an element x in S such that ax is an idempotent and semigroups consisting entirely of such elements are called *E-inversive semigroup*. E-inversive semigroups have received wide attention (see [4] and [5]). In this paper, $E(S)$ denote the set of all idempotents of a semigroup S. The set of all weak inverses of an E-inversive element a is denoted by $W(a)$, where $W(a) := \{x \in S \mid x = xax\}$. An element x is said to be an *idempotent* if $x^2 = x$, and semigroups consisting entirely of idempotent elements are called *bands*. A band S is called *rectangular band* [1] if $a = aba$ for all $a, b \in S$ and a commutative band is a *semilattice* and a band S is *normal* if $abca = acba$ for all $a, b, c \in S$. A semigroup whose idempotent elements form a subsemigroup are called an *E-semigroup*. Recall that a congruence τ defined on a semigroup S and σ is a congruence on H with $H \subseteq S$, and $\tau|_H = \sigma$, we say τ *restricted* to H [1].

The aim of this paper is to establish analogues to the results on congruences on an orthodox semigroups true for E-inversive E-semigroups. We describe the rectangular band congruence on an E-inversive E-semigroup S by mean of a congruence τ restricted to subset H of S. The Green relation $\mathcal{L}$ and $\mathcal{R}$ generalize the concept of band congruence on E-inversive E-semigroup, these results generalize the corresponding results in [3] for orthodox semigroup and in [5] for E-inversive E-semigroup.

2 Preliminary

The following lemmas and propositions give some properties of band congruence and rectangular band congruence on an E-inversive E-semigroup, which will be used frequently in the sequel.

Lemma 2.1. [5] *A semigroup S is an E-inversive if and only if $W(a) \neq \emptyset$ for all $a \in S$.*

We will be used the following properties of E-semigroup proved by Weipoltshammer [5];

Proposition 2.2. [5] *For any semigroup S, S is an E-semigroup if and only if $W(ab) = W(b)W(a)$ for all $a, b \in S$.*

Proposition 2.3. [5] *Let S be an E-semigroup. Then*

(i) *for all $a \in S, a' \in W(a), e, f \in E(S)$, $ea', a'f, fa'e \in W(a)$,*

(ii) *for all $a \in S, a' \in W(a), e \in E(S)$, $a'ea, aea' \in E(S)$,*

(iii) *for all $e \in E(S)$, $W(e) \subseteq E(S)$,*

(iv) *for all $e, f \in E(S)$, $W(ef) = W(fe)$.*

Proposition 2.4. [5] *For any E-inversive semigroup S. The following statements are equivalent.*

(i) *$E(S)$ is a rectangular band.*

(ii) *For all $a, b \in S, W(a) \cap W(b) \neq \emptyset$ implies $W(a) = W(b)$.*

Proposition 2.5. *Let S be an E-inversive semigroup. For all $a, b \in S, a' \in W(a), b' \in W(b)$, if $x \in W(a'abb')$ then $b'xa' \in W(ab)$.*

Proof. Let $a, b \in S$ and $a' \in W(a), b' \in W(b)$ and $x \in W(a'abb')$.

$$\text{Then} \quad \begin{aligned} x &= x(a'abb')x \\ b'xa' &= b'x(a'abb')xa' \\ &= (b'xa')\ ab\ (b'xa'). \end{aligned}$$

Therefore $b'xa' \in W(ab)$. □

3 Main Results

The purpose of this section is to characterize the condition of an E-inversive E-semigroup such that it is important to find band congruences and rectangular band congruences on an E-inversive E-semigroup.

If a is an element of a semigroup S without identity then Sa need not contain a [1]. The following notations will be standard :

$$\begin{aligned} &S^1a := Sa \cup \{a\}, \\ &aS^1 := aS \cup \{a\}, \\ &S^1aS^1 := SaS \cup Sa \cup aS \cup \{a\}. \end{aligned}$$

Equivalence relations $\mathcal{L}$ and $\mathcal{R}$ on a semigroup S are called *Green's relations* [1] and are defined by

$$\begin{aligned} &a\mathcal{L}b \text{ if and only if } S^1a = S^1b, \\ &a\mathcal{R}b \text{ if and only if } aS^1 = bS^1. \end{aligned}$$

Remark 3.1. [1] *Let a, b be elements of a semigroup S. Then*
$a\mathcal{L}b$ if and only if there exist $x, y \in S^1$ such that $xa = b, yb = a$,
$a\mathcal{R}b$ if and only if there exist $u, v \in S^1$ such that $au = b, bv = a$.

The notation $\mathcal{L}^\sharp$ is the smallest congruence on S containing $\mathcal{L}$. For the notation $\mathcal{R}^\sharp$ it is defined similarly.

We obtain the following band congruences on E-inversive E-semigroup as follows :

Theorem 3.2. *Let S be an E-inversive E-semigroup and ρ a congruence on S containing $\mathcal{L}[\mathcal{R}]$. Then a relation*

$$\begin{aligned}\alpha \ := \ &\{(a,b) \in S \times S \mid \ \text{for all } a' \in W(a) \text{ there exists } b' \in W(b) \\ &\text{such that } a'\rho b' \text{ and for all } b' \in W(b) \text{ there exists } a' \in W(a) \\ &\text{such that } a'\rho b'\}\end{aligned}$$

is a band congruence on S. Moreover, $\rho|_{Reg(S)} \subseteq \alpha|_{Reg(S)}$ where $Reg(S)$ denote the set of all regular elements in S.

Proof. Clearly, α is an equivalence relation on S. Let $a, b, c \in S$ be such that $a\alpha b$. Let $x \in W(ac), a' \in W(a)$ and $c' \in W(c)$ such that $x = c'a'$. Since $a\alpha b$, there exists $b' \in W(b)$ such that $a'\rho b'$. Since ρ a congruence on S, we get $c'a'\rho c'b'$. Note that $c'b' \in W(bc)$ by Proposition 2.2. Similarly, we can show that if $x \in W(bc)$ then there exists $y \in W(ac)$ such that $x\rho y$. Hence $ac\alpha bc$, so α is a right compatible. By the similar argument, we can show that α is a left compatible. Hence α is a congruence on S.

Next, to show that α is a band congruence on S, let $a \in S$ and $a' \in W(a)$. Then $a' = a'aa'$ and so $a'\mathcal{L}aa'$ by Remark. Since ρ contains $\mathcal{L}, a'\rho aa'$ and $a'a'\rho a'aa' = a'$. By Proposition 2.2, $a'a' \in W(a^2)$.

Let $x \in W(a^2)$. Then $x = xa^2x$. Let $y = ax$. We get $yay = (ax)a(ax) = a(xa^2x) = ax = y$. Thus $y \in W(a)$. Since $y = ax$ and $xay = xa(ax) = xa^2x = x$, it follows from Remark that $x\mathcal{L}y$, and so $x\rho y$. Therefore $a\alpha a^2$ and so α is a band congruence on S.

Finally, we shall show that $\rho|_{Reg(S)} \subseteq \alpha|_{Reg(S)}$.
Let $a, b \in Reg(S)$ with $a\rho b$. Let $a' \in W(a), b^* \in V(b) \subseteq W(b)$. Then $b = bb^*b$, $a' = a'aa'\rho a'ba' = a'(bb^*b)a'$. Since $ab^*a\rho bb^*b, a'ab^*aa'\rho a'bb^*ba'$. It follows that $a'\rho a'ab^*aa'$. Note that $a'ab^*aa' \in W(b)$ by Proposition 2.3(i).

Similarly, we can show that for every $x \in W(b)$, there exists $y \in W(a)$ such that $x\rho y$. Then $a\alpha b$ and so $\rho|_{Reg(S)} \subseteq \alpha|_{Reg(S)}$. □

Corollary 3.3. *Let S be an E-inversive E-semigroup. Then a relation*

$$\begin{aligned}\alpha' \ := \ &\{(a,b) \in S \times S \mid \ \text{for all } a' \in W(a) \text{ there exists} \\ &b' \in W(b) \text{ such that } a'\mathcal{L}^\sharp b' \text{ and for all } b' \in W(b) \\ &\text{there exists } a' \in W(a) \text{ such that } a'\mathcal{L}^\sharp b'\}\end{aligned}$$

is a band congruence on S with $\mathcal{L}^\sharp|_{Reg(S)} \subseteq \alpha'|_{Reg(S)}$.

Proof. It follows from Theorem 3.2 where $\mathcal{L}^\sharp$ is the smallest congruence on S containing the Green's relation $\mathcal{L}$. □

Proposition 3.4. *On any semigroup $S, \mathcal{L}^\sharp$ is a band congruence on $Reg(S)$.*

Proof. Let $a \in Reg(S)$. Then $a = aa'a$ for some $a' \in V(a)$. We get that $a\mathcal{L}a'a$ and $a^2\mathcal{L}aa'a = a$. Thus $a^2\mathcal{L}^\sharp a$, hence $\mathcal{L}^\sharp$ is a band congruence on $Reg(S)$. $\square$

Secondly, we characterize a rectangular band congruence on an E-inversive E-semigroup. Let R denote the least rectangular band congruence on S.

We have the following results :

Proposition 3.5. *If σ is a rectangular band congruences on an E-inversive semigroup S, then every σ-class contains an idempotent of S.*

Proof. Let $a \in S$. Then $(a, a^2) \in \sigma$. Let $x \in W(a^2)$. Thus $x = xa^2x$ and $(axa)(axa) = a(xa^2x)a = axa$ and so $axa \in E(S)$. Therefore $(axa)\sigma = a\sigma x\sigma a\sigma = a\sigma$ and so $axa \in a\sigma$. $\square$

Proposition 3.6. *If σ is a rectangular band congruence on E-inversive semigroup S, then*

(i) *for all $a \in S, x \in W(a^2)$ implies $axa \in E(S)$ and $(axa)\sigma a$,*

(ii) *for all $a, b, c \in S, a\sigma a^2, (abc)\sigma ac$.*

Proof. By the proof of Proposition 3.5 and definition of rectangular band. $\square$

Lemma 3.7. *Let σ and τ be rectangular band congruences on E-inversive semigroup S such that $\sigma|_{E(S)} = \tau|_{E(S)}$. Then $\sigma = \tau$.*

Proof. Let $a, b \in S$ be such that $a\sigma b$. Let $x \in W(a^2)$ and $y \in W(b^2)$. By Proposition 3.5, $axa\sigma a, byb\sigma b$ and $axa, byb \in E(S)$. Then $axa\sigma byb$. By hypothesis, we have $axa\tau byb$. By Proposition 3.5 again, $axa\tau a, byb\tau b$. It implies that $a\tau b$, so $\sigma \subseteq \tau$. Similarly, we can show that $\tau \subseteq \sigma$. Hence $\sigma = \tau$. $\square$

Corollary 3.8. *Let S be an E-inversive E-semigroup. If σ is a rectangular band congruence on S such that $\sigma|_{E(S)}$ is the least rectangular band congruence on $E(S)$, then σ is the least rectangular band congruence on S.*

Proof. Let R be the least rectangular band congruence on S. Then $R \subseteq \sigma$. That is $R|_{E(S)} \subseteq \sigma|_{E(S)}$. Since $\sigma|_{E(S)}$ is the least rectangular band congruence on $E(S)$, we have $R|_{E(S)} = \sigma|_{E(S)}$. By Lemma 3.7, hence $R = \sigma$.

Therefore σ is the least rectangular band congruence on S. $\square$

Theorem 3.9. *Let S be an E-inversive E-semigroup with σ a congruence on S. Then σ is a rectangular band congruence on S if and only if $\sigma|_{E(S)}$ is a rectangular band congruence on $E(S)$ and every class of σ contains an idempotent.*

Proof. By Proposition 3.5 and properties of rectangular band congruence, the results holds.

Conversely, let $a, b \in S$. By hypothesis, there exist $e, f \in E(S)$ such that $e\sigma a$ and $f\sigma b$. Then $efe\sigma aba$. Since $efe \in E(S)$ and $\sigma|_{E(S)}$ is a rectangular band congruence on $E(S)$, we have $efe\sigma|_{E(S)}e$ and $efe\sigma e$. Thus $aba\sigma efe\sigma e\sigma a$, hence σ is a rectangular band congruence on S. □

Next, we shall extend a rectangular band congruence on $E(S)$ to a rectangular band congruence on an E-inversive E-semigroup S.

Theorem 3.10. *Let S be an E-inversive E-semigroup. If R is the least rectangular band congruence on S such that $R|_{E(S)}$ is the least rectangular band congruence on $E(S)$, then every rectangular band congruence on $E(S)$ can be extended to a rectangular band congruence on S.*

Proof. Let γ be a rectangular band congruence on $E(S)$. A relation σ on S is defined by

$$\sigma := \{(a, b) \in S \times S \mid aRe, e\gamma f, fRb \text{ for some } e, f \in E(S)\}.$$

We shall show that σ is a congruence on S, let $a \in S$. By Proposition 3.5, aRe and eRa for some $e \in E(S)$. Since γ is a congruence on $E(S)$, $e\gamma e$. Hence $a\sigma a$. Clearly, σ is symmetric. Let $a, b, c \in S$ be such that $a\sigma b$ and $b\sigma c$. Then there exist $e, f, g, h \in E(S)$ such that $aRe, e\gamma f, fRb$ and $bRg, g\gamma h, hRc$. Then fRg and $fR|_{E(S)}g$. Since $R|_{E(S)}$ is the least rectangular band congruence on $E(S)$, we have $f\gamma g$. Since $e\gamma f, f\gamma g, g\gamma h$, we have $e\gamma h$. Hence $aRe, e\gamma h, hRc$ and so $a\sigma c$. Let $a, b, c \in S$ be such that $a\sigma b$. Then $aRe, e\gamma f$ and fRb for some $e, f \in E(S)$.

Since R is a rectangular band congruence on S, there exists $g \in E(S)$ such that gRc. Hence $acReg, egRfg, fgRbc$. Then $ac\sigma bc$.

Similarly, we can show that $ca\sigma cb$. Hence σ is a congruence on S.

Let $a, b \in S$ such that aRb. By Proposition 3.5, there exists $e \in E(S)$ such that $aRe, e\gamma e, eRb$. Then $a\sigma b$ and so $R \subseteq \sigma$. Hence σ contains an idempotent.

Let $e, f \in E(S)$ be such that $e\gamma f$. Note that eRe, fRf, then $eRe, e\gamma f$, fRf which implies that $e\sigma f$ and so $e\sigma|_{E(S)}f$. Hence $\gamma \subseteq \sigma|_{E(S)}$.

Let $e, f \in E(S)$ be such that $e\sigma f$. Then there exist $g, h \in E(S)$ such that $eRg, g\gamma h, hRf$. Since $R|_{E(S)}$ is the least rectangular band congruence on $E(S)$, we have $R|_{E(S)} \subseteq \gamma$ and so $e\gamma g, g\gamma h, h\gamma f$. Hence $e\gamma f$ and so $\sigma|_{E(S)} \subseteq \gamma$ and so $\sigma|_{E(S)} = \gamma$ is a rectangular band congruence on $E(S)$. By Theorem 3.9, σ is a rectangular band congruence on S. □

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