

# A Cohomological Approach for Classifying Nilpotent Leibniz Algebras

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## Abstract

In this paper a cohomological approach (the Skjelbred-Sund method) to classify nilpotent Leibniz algebras in low dimensional cases will be presented. We distinguish six isomorphism classes (one parametric family and five concrete) of three dimensional nilpotent Leibniz algebras and show that they exhaust all possible cases.

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## 1 Introduction

The concept of Leibniz algebra was first introduced by Loday [6] in the study of the so-called Leibniz (co)homology of Lie algebras as a “noncommutative” analogue of Lie algebra (co)homology initially found by Cuvier [3] and [7]. By definition, a Lie algebra is Leibniz.

There is a cohomology theory for these algebraic objects whose properties are similar to those of the classical Chevalley-Eilenberg cohomology theory for Lie algebras. The central extensions of Lie algebras play a central role in the classification theory of Lie algebras. In 1978 Skjelbred and Sund reduced the classification of nilpotent Lie algebras in a given dimension to the study of orbits under the action of a group on the space of second degree cohomology of a smaller Lie algebra with coefficients in a trivial module.

Like in the Lie algebras case, the Leibniz central extensions play an crucial role in the structural theory of Leibniz algebras. Loday-Pirashvili [8] proved that the Virasoro algebra  $Vir$  is a universal central extension of  $Der(\mathbf{C}[t, t^{-1}])$

both in the Lie and in the Leibniz framework. In 1978, Skjelbred and Sund [10] used the central extension in the classification problem of nilpotent Lie algebras. The aim of this paper is to apply the Skjelbred-Sund classification method to the classification problem of complex nilpotent Leibniz algebras. The content of the paper is as follows.

In section 1, we prove analogues of some classical results of Lie algebras for Leibniz ones. Among them, Theorem 1.1 is quite essential in proving the classification Theorem 3.3.

Section 2 describes the classification procedure which is a modification of the similar process for Lie algebras [4]. And then we treat nilpotent Leibniz algebras applying the classification procedure to low dimensional nilpotent Leibniz algebras. We distinguish six non-isomorphic classes (one parametric family and five concrete) of three dimensional Leibniz algebras and show that they exhaust all possible cases (table in subsection 2.1.3).

**Definition 1.1.** A Leibniz algebra  $L$  is a vector space over a field  $F$  equipped with a bilinear map

$$[\cdot, \cdot] : L \times L \longrightarrow L$$

satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \quad \text{for all } x, y, z \in L.$$

The first pure algebraic motivation of Loday to introduce this class of algebras was the search of an “obstruction” to the periodicity in algebraic  $K$ -theory. Besides, this purely algebraic motivation some relationships with classical geometry have recently been discovered, which could lead to investigate the (co)homological theory of Leibniz algebras in view of concrete applications in non-commutative geometry and its physical interpretations. Obviously, a Lie algebra is Leibniz. However, a Leibniz algebra is Lie if and only if

$$[x, x] = 0, \quad \text{for all } x \in L.$$

Let  $L$  be a Leibniz algebra and  $V$  be a vector space over  $F$ . Then, the bilinear maps  $\theta : L \times L \longrightarrow V$  with

$$\theta(x, [y, z]) = \theta([x, y], z) - \theta([x, z], y), \quad \text{for all } x, y, z \in L$$

are called Leibniz cocycles. The set of all Leibniz cocycles is denoted by  $ZL^2(L, V)$ . Let  $\theta \in ZL^2(L, V)$ . Then, we set  $L_\theta = L \oplus V$  and define a bracket  $[\cdot, \cdot]$  on  $L_\theta$  by

$$[x + v, y + w] = [x, y]_L + \theta(x, y),$$

where  $[\cdot, \cdot]_L$  is the bracket on  $L$ .

The proof of the following lemma can be found by a simple computation.

**Lemma 1.1.**  $L_\theta$  is a Leibniz algebra if and only if  $\theta$  is a Leibniz cocycle.

The Leibniz algebra  $L_\theta$  is called a **central extension** of  $L$  by  $V$ . Let  $\nu : L \rightarrow V$  be a linear map, and define  $\eta(x, y) = \nu([x, y])$ . Then it is easy to see that  $\eta$  is a Leibniz cocycle called **coboundary**. The set of all coboundaries is denoted by  $BL^2(L, V)$ . Clearly,  $BL^2(L, V)$  is a subgroup of  $ZL^2(L, V)$ . We call the factor space, denoted by  $HL^2(L, V) = ZL^2(L, V) / BL^2(L, V)$ , the **second cohomology group** of  $L$  by  $V$ .

The following lemma shows that the central extension of a given Leibniz algebra  $L$  is limited to the coboundary level.

**Lemma 1.2.** Let  $L$  be a Leibniz algebra and  $\eta$  be a coboundary, then the central extensions  $L_\theta$  and  $L_{\theta+\eta}$  are isomorphic.

*Proof.* We define  $\sigma : L_\theta \rightarrow L_{\theta+\eta}$  by  $\sigma(x + v) = (x + \nu(x) + v)$ . Then, an easy computation shows that  $\sigma$  is an invertible linear transformation. Moreover,

$$\begin{aligned} \sigma([x + v, y + w]_{L_\theta}) &= \sigma([x, y]_L + \theta(x, y)) \\ &= ([x, y]_L + \nu([x, y]_L) + \theta(x, y)) = ([x, y]_L + \eta(x, y) + \theta(x, y)) \\ &= ([x, y]_L + (\theta + \eta)(x, y)) = [x + \nu(x) + v, y + \nu(y) + w]_{L_{\theta+\eta}} \\ &= [\sigma(x + v), \sigma(y + w)]_{L_{\theta+\eta}}, \text{ for all } x + v, y + w \in L_\theta. \end{aligned}$$

□

When constructing a Leibniz algebra as  $L_\theta = L \oplus V$ , we want to restrict  $\theta$  to such an extent the center of  $L_\theta$  equals  $V$ . In this way, we discard constructing the same Leibniz algebra as central extension of different Leibniz algebras.

The center of a Leibniz algebra  $L$  is defined as follows:

$$C(L) = \{x \in L \mid [x, L] = [L, x] = 0\}.$$

For  $\theta \in ZL^2(L, V)$  set

$$\theta^\perp = \{x \in L \mid \theta(x, L) = \theta(L, x) = 0\},$$

which is called the radical of  $\theta$  ( $\text{Rad}(\theta) = \theta^\perp$ ). We conclude that any Leibniz algebra with a nontrivial center can be obtained as a central extension of a Leibniz algebra of smaller dimension.

The proof of the following lemma is straightforward.

**Lemma 1.3.** If  $\theta \in ZL^2(L, V)$  then  $C(L_\theta) = (\theta^\perp \cap C(L)) + V$ .

As a consequence of this lemma we get the following criterion.

**Corollary 1.1.**  $\theta^\perp \cap C(L) = \{0\}$  if and only if  $C(L_\theta) = V$ .

Let  $e_1, \dots, e_k$  be a basis of  $V$  and  $\theta \in ZL^2(L, V)$ . Then

$$\theta(x, y) = \sum_{i=1}^k \theta_i(x, y) e_i,$$

where  $\theta_i \in ZL^2(L, F)$ . Furthermore,  $\theta$  is a coboundary if and only if all  $\theta_i$  are.

The automorphism group  $\text{Aut}(L)$  acts on  $ZL^2(L, V)$  by  $\phi\theta(x, y) = \theta(\phi(x), \phi(y))$  and  $\eta \in BL^2(L, V)$  if and only if  $\phi\eta \in BL^2(L, V)$ . This induces an action of  $\text{Aut}(L)$  on  $HL^2(L, V)$ . The proof of the following theorem can be carried out for Leibniz algebras by a minor modification of the corresponding result in [5].

**Theorem 1.1.** Let  $\theta(x, y) = \sum_{i=1}^k \theta_i(x, y) e_i$  and  $\eta(x, y) = \sum_{i=1}^k \eta_i(x, y) e_i$  be two elements of  $HL^2(L, V)$ . Suppose that  $\theta^\perp \cap C(L) = \eta^\perp \cap C(L) = \{0\}$ . Then  $L_\theta \cong L_\eta$  if and only if there is a  $\varphi \in \text{Aut}(L)$  such that  $\varphi\eta_i$  span the same subspace of  $HL^2(L, V)$  as  $\theta_i$ .

Let  $L = I_1 \oplus I_2$ , where  $I_1$  and  $I_2$ , are ideals of  $L$ . Suppose that  $I_2$  is contained in the center of  $L$ . Then  $I_2$  is called a central component of  $L$ . In order to keep away from the Leibniz algebras with central components we use the following criterion.

**Lemma 1.4.** Let  $\theta(x, y) = \sum_{i=1}^k \theta_i(x, y) e_i \in HL^2(L, V)$  be such that  $\theta^\perp \cap C(L) = \{0\}$ . Then  $L_\theta$  has no central components if and only if  $\theta_1, \dots, \theta_k$  are linearly independent in  $HL^2(L, F)$ .

Let  $G_k(HL^2(L, F))$  be the Grassmanian of subspaces of dimension  $k$  in  $HL^2(L, F)$ .

One makes  $\text{Aut}(L)$  act on  $G_k(HL^2(L, F))$  as follows:  $W = \langle \vartheta_1, \vartheta_2, \dots, \vartheta_k \rangle \in G_k(HL^2(L, F))$ ,

$$\phi W = \langle \phi\vartheta_1, \phi\vartheta_2, \dots, \phi\vartheta_k \rangle.$$

This definition is legitimate because if  $\{\vartheta_1, \vartheta_2, \dots, \vartheta_k\}$  is linear independent so is  $\{\phi\vartheta_1, \phi\vartheta_2, \dots, \phi\vartheta_k\}$ .

Define

$$U_k(L) = \{W = \langle \vartheta_1, \vartheta_2, \dots, \vartheta_k \rangle \in G_k(HL^2(L, F)) : \vartheta_i^\perp \cap Z(L) = \{0\}, i = 1, 2, \dots, k\}.$$

**Lemma 1.5.** The set  $U_k(L)$  is stable under the action of  $\text{Aut}(L)$ .

*Proof.* Let  $\phi \in \text{Aut}(L)$ , and  $W = \langle \vartheta_1, \vartheta_2, \dots, \vartheta_k \rangle \in U_k(L)$ . Obviously, one has  $(\phi\vartheta_i)^\perp = \phi^{-1}\vartheta_i^\perp$  and  $\phi^{-1}(C(L)) = C(L)$ . Therefore  $(\phi\vartheta_i)^\perp \cap C(L) = \phi^{-1}(\vartheta_i^\perp \cap C(L))$  which prove that  $W \in U_k(L)$  if and only if  $\phi W \in U_k(L)$ .  $\square$

The set of orbits under the action  $Aut(L)$  on  $U_k(L)$  will be denoted by  $U_k(L)/Aut(L)$ .

Here is an analogue of the Skejlbred-Sund theorem [10] for Leibniz algebras.

**Theorem 1.2.** *There exists a canonical one-to-one map from  $U_k(L)/Aut(L)$  onto the set of isomorphism classes of Leibniz algebras without direct abelian factor which are central extensions of  $L$  by  $F^k$  and have  $k$ -dimensional center.*

The following theorem will be used in the computation of the automorphism groups [9].

**Theorem 1.3.** *Let  $\theta$  be a Leibniz cocycle on the Leibniz algebra  $L$  with values in  $V$ , and assume  $\theta^\perp \cap C(L) = \{0\}$ . Then the automorphism group  $Aut(L_\theta)$  of the extension algebra  $L_\theta$  consists of all linear operators of the matrix form  $\alpha = \left( \begin{array}{c|c} \alpha_0 & 0 \\ \hline \varphi & \psi \end{array} \right)$ , where  $\alpha_0 \in Aut(L)$ ,  $\psi = \alpha|_{C(L_\theta)} \in GL(k)$ , and  $\varphi \in Hom(L, V)$ , and*

$$\theta(\alpha_0x, \alpha_0y) = \varphi[x, y] + \psi\theta(x, y), \quad \text{all } x, y \in L.$$

## 2 The classification procedure

This section deals with the procedure to classify nilpotent Leibniz algebras. Here all the theoretical observations of the previous sections will be applied.

**Definition 2.1.** If  $L$  is a Leibniz algebra, We may define

$$L^1 = L, L^k = [L, L^{k-1}] \quad (k > 1).$$

The series

$$L^1 \supseteq L^2 \supseteq L^3 \supseteq \dots$$

is called the descending central series of  $L$ . If the series terminates for some positive integer  $s$ , then the Leibniz algebra  $L$  is said to be nilpotent.

In this section a procedure will be described through which a nilpotent Leibniz algebra  $L$  of dimension  $n - k$  is considered as input. It outputs all nilpotent Leibniz algebras  $K$  of dimension  $n$  such that  $K/C(K) \cong L$ , and  $K$  has no central components. It runs as follows (for Lie case see[4]).

1. For a given algebra of smaller dimension, we list at first its center (or the generators of its center), to help us identify the 2-cocycles satisfying  $\theta^\perp \cap C(L) = 0$ .

2. We also list its derived algebra (or the generators of the derived algebra), which is needed in computing the coboundaries  $BL^2(L, F)$ .
3. Then we compute all the 2-cocycles  $ZL^2(L, F)$  and  $BL^2(L, F)$  and compute the set  $HL^2(L, F)$  of cosets of  $BL^2(L, F)$  in  $ZL^2(L, F)$ . For each fixed algebra  $L$  with given base  $\{e_1, e_2, \dots, e_n\}$ , we may represent a 2-cocycles  $\theta$  by a matrix  $\theta = \sum_{i,j=1}^n c_{ij}\Delta_{ij}$ , where  $\Delta_{ij}$  is the  $n \times n$  matrix with  $(i, j)$  element being 1 and all the others 0. When computing the 2-cocycles, we will just list all the constraints on the elements  $c_{ij}$  of the matrix  $\theta$ .
4. We have  $ZL^2(L, F) = BL^2(L, F) \oplus W$ , where  $W$  is a subspace of  $ZL^2(L, F)$ , complementary to  $BL^2(L, F)$ , and

$$BL^2(L, F) = \{df \mid f \in C^1(L, F) = L^*\}$$

( $d$  is the coboundary operator). One easy way to obtain  $W$  is as follows. When a nilpotent Leibniz algebra  $L$  of dimension  $n = r + k$  has a basis in the form  $\{e_1, \dots, e_r, e_{r+1}, \dots, e_{r+k}\}$ , where  $\{e_1, \dots, e_r\}$  are the generators, and  $\{e_{r+1}, \dots, e_{r+k}\}$  forms a basis for the derived algebra  $[L, L]$ , with  $e_{r+t} = [e_{i_t}, e_{j_t}]$ , where  $1 \leq i_t, j_t < r + t$  and  $1 \leq t \leq k$ . Consider  $C^1(L, F) = L^*$  generated by the dual basis

$$\langle f_1, \dots, f_r, g_1, \dots, g_k \rangle$$

of

$$\langle e_1, \dots, e_r, e_{r+1}, \dots, e_{r+k} \rangle.$$

Then

$$BL^2(L, F) = \{dh \mid h \in L^*\} = \langle df_1, \dots, df_r, dg_1, \dots, dg_k \rangle.$$

Since  $df_i(x, y) = -f_i([x, y]) = 0$ , we have  $BL^2(L, F) = \langle dg_1, \dots, dg_k \rangle$ . Now we have

$$ZL^2(L, F) = \langle dg_1, \dots, dg_k \rangle \oplus W.$$

For  $\theta \in W$ , we may assume that  $\theta(e_{i_t}, e_{j_t}) = 0$ ,  $t = 1, \dots, k$ , otherwise, if  $\theta(e_{i_t}, e_{j_t}) = u_{i_t j_t} \neq 0$ , we choose  $\theta + u_{i_t j_t} dg_t$  instead. When we carry out the group action on  $W$ , we do it as if it were done in  $HL^2(L, F)$ , and may identify  $HL^2(L, F)$  with  $W$ , by calling all the nonzero elements in  $W$  the normalized 2-cocycles.

5. Consider  $\theta \in HL^2(L, V)$  with  $\theta(x, y) = \sum_{i=1}^n \theta_i(x, y)e_i$  where  $\theta_i \in HL^2(L, F)$  are linearly independent, and  $\theta^\perp \cap C(L) = 0$ .
6. Find a (maybe redundant) list of representatives of the orbits of  $\text{Aut}(L)$  acting on the  $\theta$  from 5.
7. For each  $\theta$  found, construct  $L_\theta$ . Discard the isomorphic ones.

## 2.1 Nilpotent Leibniz algebras of dimension 3

In this section we use the Skjelbred-Sund method to classify nilpotent Leibniz algebras of dimension 3. Here and onward all algebras are supposed to be over the field of complex numbers  $\mathbb{C}$  and the omitted products of basis vectors are supposed to be zero. We have two subsections: in each subsection the central extensions of one particular Leibniz algebra are considered. We denote the  $j$ -th algebra of dimension  $i$  by  $L_{i,j}$ . To classify three dimensional case we need the classification of two dimensional nilpotent Leibniz algebras. For the following lemma we refer to [6].

**Lemma 2.1.** *In dimension two there are two non-isomorphic nilpotent Leibniz algebras. One of them is abelian denoted  $L_{2,1}$ , another one is  $L_{2,2}$ , can be given by the following table  $[e_1, e_1] = e_2$ .*

### 2.1.1 Central extensions of $L_{2,1}$

We can get a basis for  $HL^2(L_{2,1}, F)$ , and write “Basis:  $\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22}$ ”. In this case, we are considering the 1-dimensional central extension of  $L_{2,1}$ . We need to find a set of representatives of the orbits of 1-dimensional subspaces of  $HL^2(L_{2,1}, F)$  under the action of the automorphism group  $\text{Aut}(L_{2,1})$ . Here  $\text{Aut}(L_{2,1})$  consists of

$$\phi = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ where } a_{11}a_{22} - a_{12}a_{21} \neq 0.$$

With the chosen basis, we may denote an arbitrary element in  $HL^2(L_{2,1}, F)$  by

$$\theta := [a, b, c, d] = a\Delta_{11} + b\Delta_{12} + c\Delta_{21} + d\Delta_{22}.$$

When a generic element  $\phi$  in  $\text{Aut}(L_{2,1})$  acts on  $\theta$ , we get

$$\phi\theta = a'\Delta_{11} + b'\Delta_{12} + c'\Delta_{21} + d'\Delta_{22},$$

we will simply write  $a \longrightarrow a'$ ,  $b \longrightarrow b'$ ,  $c \longrightarrow c'$ ,  $d \longrightarrow d'$ . Now we have

$$\begin{cases} a \longmapsto a_{11}(aa_{11} + ba_{21}) + a_{21}(ca_{11} + da_{21}) \\ b \longmapsto a_{11}(aa_{12} + ba_{22}) + a_{21}(ca_{12} + da_{22}) \\ c \longmapsto a_{12}(aa_{11} + ba_{21}) + a_{22}(ca_{11} + da_{21}) \\ d \longmapsto a_{12}(aa_{12} + ba_{22}) + a_{22}(ca_{12} + da_{22}) \end{cases}$$

Assume  $a \neq 0$ . By taking  $a_{21} = 0$  (and ensuring at the same time that the matrix of  $L_{2,1}$  is nonsingular). We set  $a_{11} = \frac{1}{\sqrt{a}}$ , then  $a \longmapsto 1$ . We get

$$\begin{cases} a \longmapsto 1 \\ b \longmapsto a_{12} + ba_{22} \\ c \longmapsto a_{12} + ca_{22} \\ d \longmapsto a_{12}^2 + ba_{12}a_{22} + ca_{12}a_{22} + da_{22}^2 \end{cases}$$

By taking  $a_{12} = -ca_{22}$ , we get  $c \longmapsto 0$ , and to fix  $c$ , we require that  $a_{12} = 0$ . In this case we get

$$\begin{cases} a \longmapsto 1 \\ b \longmapsto ba_{22} \\ c \longmapsto 0 \\ d \longmapsto da_{22}^2 \end{cases}$$

Case 1:  $b = 0$ . Depending  $d = 0$  or not, we would have two representatives (1)  $[1, 0, 0, 0]$  and (2)  $[1, 0, 0, 1]$ . But (1) is split algebra and (2)  $\cong [1, 1, 1, 0]$  by  $e_1 \longmapsto e'_1, e_2 \longmapsto ie'_1 - ie'_2, e_3 \longmapsto e'_3$  (such that  $i^2 = -1$ ). The corresponding Leibniz algebra with  $[1, 1, 1, 0]$  is

$$L_{3,5} : [e_1, e_1] = e_3, [e_1, e_2] = e_3, [e_2, e_1] = e_3$$

Case2:  $b \neq 0$ . By taking  $a_{22} = \frac{1}{\sqrt{b}}$ , we get  $b \longmapsto 1$ . Therefore, we get the representative  $[1, 1, 0, \alpha]$ . The corresponding Leibniz algebra is given by

$$L_{3,4} : [e_1, e_1] = e_3, [e_1, e_2] = e_3, [e_2, e_2] = \alpha e_3, \alpha \in \mathbb{C}$$

If  $a = 0$ , then we have

$$\begin{cases} a \longmapsto 0 \\ b \longmapsto ba_{11}a_{22} \\ c \longmapsto ca_{11}a_{22} \\ d \longmapsto ba_{12}a_{22} + ca_{12}a_{22} + da_{22}^2 \end{cases}$$

At least one of  $b, c$  and  $d$  is nonzero. Suppose that  $b \neq 0$ . We choose  $a_{22} = 1$  and  $a_{11} = \frac{1}{b}$  and get  $b \longmapsto 1$ . We get

$$\begin{cases} a \longmapsto 0 \\ b \longmapsto 1 \\ c \longmapsto c \\ d \longmapsto a_{12}(1 + c) + d \end{cases}$$

If  $c = -1$ , depending on  $d = 0$  or not, we have two representatives (3)[0, 1, -1, 0] and (4) [0, 1, -1,  $\alpha$ ]. The corresponding Leibniz algebra with (3) is

$$L_{3,3} : [e_1, e_2] = e_3, [e_2, e_1] = -e_3.$$

But in (4) if  $\alpha \neq 0$  then it maps to 1, and then (4)  $\cong L_{3,4}$  ( $\alpha = \frac{1}{4}$ ) by

$$e_1 \mapsto e'_1 + \frac{1}{\alpha}e'_2, e_2 \mapsto \frac{1}{2\alpha}e'_2, e_3 \mapsto \frac{1}{\alpha}e'_3$$

If  $\alpha = 0$ , then we get  $L_{3,3}$  again. If  $c \neq -1$ , we choose  $a_{12} = \frac{-d}{c+1}$ , we get the representative [0, 1,  $\alpha$ , 0] ( $\alpha \neq -1$ ), which we denote by  $\lambda$ . But  $\lambda(\alpha = 1) \cong L_{3,5}$  by

$$e_1 \mapsto e'_1 - \frac{1}{2}e'_2, e_2 \mapsto \frac{1}{2}e'_2, e_3 \mapsto \frac{1}{2}e'_3$$

$\lambda(\alpha \neq 1) \cong L_{3,4}(\frac{-\alpha}{(\alpha-1)^2})$  by

$$e_1 \mapsto \frac{1}{\alpha+1}e'_1 + \frac{\alpha-1}{\alpha+1}e'_2, e_2 \mapsto \frac{\alpha}{\alpha^2-1}e'_1 - \frac{1}{\alpha+1}e'_2, e_3 \mapsto \frac{1}{\alpha^2-1}e'_3.$$

### 2.1.2 Central extensions of $L_{2,2}$

$\text{Aut}(L_{2,2})$  consists of all operators

$$\phi = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{11}^2 \end{bmatrix}, \text{ where } a_{11} \neq 0$$

$HL^2(L_{2,2}, F)$  consists of  $\theta = a\Delta_{21}$ . Moreover,  $\theta^\perp \cap C(L) = 0$  if and only if  $a \neq 0$ . The automorphism group acts as follows  $a \longrightarrow aa_{11}^3$ . (Here we write  $a \longrightarrow a'$  to show that the coefficient of  $\Delta_{21}$  in  $\phi\theta$  is  $a'$ .) Then we choose  $a_{11} = \frac{1}{\sqrt[3]{a}}$  such that  $a$  is mapped to 1. This yields

$$L_{3,6} : [e_1, e_1] = e_2, [e_2, e_1] = e_3$$

*Remark 1.* There is no one-dimensional Leibniz algebra except for abelian.

**Definition 2.2.** An  $n$ -dimensional Leibniz algebra  $L$  is said to be nulfiliform if  $\dim L^i = n - i + 1$ , where  $2 \leq i \leq n + 1$ .

*Remark 2.* Up to isomorphism there is only one nulfiliform Leibniz algebra in each dimension [2].

The  $L_{2,2}$  is particular case of the following more general result that can be spelled out in the next theorem. The proof can be found in [9].

**Theorem 2.1.** A nulfiliform Leibniz algebra in dimension  $(n+1)$  is the central extension of nulfiliform Leibniz algebra in dimension  $n$ .

### 2.1.3 Table of three-dimensional nilpotent Leibniz algebras

According to lemma 3.2 in dimension 2 there are two nilpotent Leibniz algebras, we have denoted them as  $L_{2,1}$ , and  $L_{2,2}$ . And combining them we get the algebras  $L_{3,1} = L_{2,1} \oplus I$ ,  $L_{3,2} = L_{2,2} \oplus I$  (where  $I$  is a 1-dimensional Abelian ideal) in dimension 3. The others are presented in the following table:

$L$	Basis for the set of cocycles on $L$ ( $HL^2(L)$ )	$\text{Aut}(L)$	Representative for defining orbit in $(H^2(L))$	Product in $L_\theta$ only non-zero bracket are given	$L_\theta$
$L_{2,1}$	$\Delta_{11}, \Delta_{12}$ $\Delta_{21}, \Delta_{22}$	$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $a_{11}a_{22} - a_{12}a_{21} \neq 0$	$[0, 1, -1, 0]$	$[e_1, e_2] = e_3$ $[e_2, e_1] = -e_3$	$L_{3,3}$
			$[1, 1, 0, \alpha]$ $\alpha \in \mathbb{C}$	$[e_1, e_1] = e_3$ $[e_1, e_2] = e_3$ $[e_2, e_2] = \alpha e_3$	$L_{3,4}$
			$[1, 1, 1, 0]$	$[e_1, e_1] = e_3$ $[e_1, e_2] = e_3$ $[e_2, e_1] = e_3$	$L_{3,5}$
$L_{2,2}$	$\Delta_{21}$	$\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{11}^2 \end{bmatrix}$ $a_{11} \neq 0$	$[1]$	$[e_1, e_1] = e_2$ $[e_2, e_1] = e_3$	$L_{3,6}$

This result agrees with the classification of three dimensional nilpotent Leibniz algebras from [1]. However, the classification method that has been used in [1] and [2] depended on result of classification of associative algebras that can not be applied in higher dimensions.

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