International Journal of Algebra, Vol. 4, 2010, no. 2, 61 - 69

Atomic Modules

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Abstract

In this paper we introduce atomic module which is a dualization of coatomic modules studied by Zoschinger and study some of its properties. As a byproduct we get the result that every semiprime ring R which is cofinitely generated as a left R-module is semisimple Artinian. We also study rings over which every module is atomic. Also we give simplified proof of Lemma 3(3) of the paper "Coatomic Modules" by Gonca Gungoroglu.

Mathematics Subject Classifications: 16D99, 16D60, 16P99

Keywords: Supplemented modules, coatomic module, co-Noetherian ring

Introduction

In [7], Zoschinger defined and investigated coatomic module over commutative Noetherian rings. Also G. Gungoroglu[5] obtained more results on coatomic modules. A module M is called coatomic if every proper submodule of M is contained in a maximal submodule or equivalently, for a submodule N of M, whenever $\mathrm{Rad}(M/N) = M/N$ then $M = N$.

As dual to coatomic modules, we study in this paper, atomic modules. We define a left R-module M to be atomic if the socle of every nonzero submodule

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[2] The author is supported by Nilekani Scholarship.

of M is nonzero, that is, every nonzero submodule of M contains a simple submodule. Clearly "atomic" is a hereditary property. But a homomorphic image of an atomic left R-module is need not be atomic. However ring R is both left Noetherian and left co-Noetherian every homomorphic image of an atomic R-module is atomic. An extension of an atomic module by an atomic module is atomic. Any direct sum of atomic modules is atomic. We also study rings R over which every R-module is atomic. Over a left Artinian ring R every R-module is atomic. If R is a ring such that every R-module is atomic then $J(R)$ is T-nilpotent and the lower nil radical, the upper nil radical and the Jacobson radical of R all coincide. If M is a supplemented module, we prove that M is semisimple if and only if M is atomic and every simple submodule of M is a direct summand of M.

Throughout this paper, by a ring R we mean an associative ring with unity and by an R-module we mean unitary left R-module. Let R be a ring and M an R-module. For a subset A of R and a subset S of M we define following subsets of M and R respectively $\mathrm{Ann}_M(A) = \{m \in M \ / Am = 0\}$; $\mathrm{Ann}_R(S) = \{r \in R \ / rS = 0\}$. These are called respectively the annihilator of A in M and the annihilator of S in R. We write $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$, $E(M)$ for the radical , the socle and the injective hull of M respectively and $\mathrm{Nil}_*(R)$, $\mathrm{Nil}^*(R)$ and $J(R)$ for lower nil radical, upper nil radical and the Jacobson radical of R respectively.

A submodule N of M is said to be essential in M if N has nonzero intersection with every nonzero submodule of M. A submodule N of M is said to be small in M if $M = N + K$ for some $B \subseteq M$ implies $M = B$.

Let M be an R-module and N be a submodule of M. A submodule K of M is said to be complement of N in M if K is maximal with respect to the property $N \cap K = 0$.

Let M be an R-module and N and K submodules of M. N is called supplement of K in M if N is minimal with respect to the property $M = N + K$. N is called a supplement in M if it is supplement of some submodule K of M. It is clear that N is supplement of K in M if and only if $M = N + K$ and $N \cap K$ is small in N. M is called a supplemented module if every submodule of M is a supplement in M.

In section 2 of this paper, we define and study atomic module which is dualization of coatomic module and give some properties of atomic modules.

In section3, we prove Lemma 3(3) of G. Gungoroglu[5] over a more general ring. i.e. over a left co-Noetherian and left Noetherian ring.

2. Atomic modules

2.1. Definition: An R-module M is said to be atomic if socle of every nonzero submodule of M is nonzero. i.e. every nonzero submodule of M contains a

simple submodule. If a ring R is atomic as a left R-module we say the ring R is left atomic. Clearly semisimple module, Artinian module and finitely embedded modules of Vamos[9] (later called by J.P.Jans [8] as cofinitely generated) are atomic. Over a right perfect ring every nonzero module is atomic. This follows from the Theorem of Bass ([cf. [6, Theorem P]). We note that an R-module M is cofinitely generated if and only if M is atomic and socle of M is finitely generated [cf. 9, Lemma 1].

2.2. Remark: Hollow modules are coatomic modules[5]. But uniform modules need not be atomic.

2.3. Example: Z is uniform as Z-module but, it not atomic.

2.4. Remark: Clearly every submodule of an atomic module is atomic. But a homomorphic image of an atomic module need not be atomic.

2.5. Example: Osofsky [1] has constructed the following ring. Let $R = Z_{(p)} \oplus Z(p^\infty)$ be the group direct sum of the ring $Z_{(p)}$ of p-adic integers and the p-Prufer group $Z(p^\infty)$ and define multiplication in R as follows. If $(\lambda, x)(\mu, y) \in R$ we define $(\lambda, x)(\mu, y) = (\lambda\mu, \lambda y + \mu x)$. She has proved that R is a commutative self-injective ring. If I is any proper ideal of R then either I is contained in $Z(p^\infty)$ or contains $Z(p^\infty)$. The socle of R is $Z(p)$, the cyclic subgroup of $Z(p^\infty)$ of order p. Hence R is cofinitely generated. Hence R is atomic as an R-module. Clearly $Z(p^\infty)$ is an ideal of R and the factor ring $R/Z(p^\infty)$ is isomorphic to the ring Z_p of p-adic integers. Since, Z_p is a commutative integral domain, its socle is zero and hence the socle of the factor rings $R/Z(p^\infty)$ is also zero. Hence the socle of $R/Z(p^\infty)$ as R-module is also zero. So $R/Z(p^\infty)$ cannot be atomic as an R-module.

We give below a sufficient condition for a factor module of an atomic module is atomic.

2.6. Proposition: Let R be a left co-Noetherian and a left Noetherian ring and M be an R-module. Let N be a submodule of M. If M is atomic then M/N is also atomic.

Proof: Let K/N be a nonzero submodule of M/N. Let $f : R \to K/N$ be any nonzero map. Then, there exists a homomorphism $g : R \to M$ such that $\eta g = f$, where, $\eta : M \to M/N$ is a canonical epimorphism. Since, $kerg \subseteq kerf$ we get an epimorphism $\phi : R/kerg \to R/kerf$ and a one one map from $R/kerg$ to M. Since, by hypothesis, M is atomic, $R/kerg$ is atomic. Also $R/kerg$ is finitely generated. Hence $R/kerg$ is cofinitely generated. Since, by hypothesis R is left co-noetherian, $R/kerf$ is cofinitely generated. Therefore $\mathrm{Soc}(R/kerf)$ is essential in $R/kerf$ and hence $\mathrm{Soc}(R/kerf) \neq 0$. But we have a one one map from $R/kerf$ to K/N. So $\mathrm{Soc}(K/N) \neq 0$. Hence, M/N is atomic.

2.7. Proposition: Let M be an R-module and N be a submodule of M. If N and M/N are atomic then M is atomic.

Proof: Let N be a nonzero submodule of M such that N and M/N are atomic. Let K be a nonzero submodule of M. If $K \subseteq N$, then $\mathrm{Soc}(K) \neq 0$, since, by

hypothesis, N is atomic and hence we are done. So suppose K does not contained in N then $(K+N)/N \neq 0$. Since, by hypothesis, M/N is atomic, $\mathrm{Soc}((K+N)/N) \neq 0$. Since $(K+N)/N \cong K/(N \cap K)$, $\mathrm{Soc}(K/(N \cap K)) \neq 0$. If $N \cap K = 0$ then, $\mathrm{Soc}(K) \neq 0$. If $N \cap K \neq 0$ then, $N \cap K \subseteq N$ and since, by hypothesis, N is atomic, $\mathrm{Soc}(N \cap K) \neq 0$, and hence, $\mathrm{Soc}(K) \neq 0$. Therefore M is atomic.

2.8. Corollary: Consider the short exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules. If both M' and M'' are atomic then so is M.

2.9. Proposition: If M_1 and M_2 are R-modules then $M = M_1 \oplus M_2$ is atomic if and only if both M_1 and M_2 are atomic.

Proof: Only if: Let M be atomic then clearly M_1 and M_2 are atomic.

If: Consider the canonical short exact sequence $0 \to M_1 \xrightarrow{i} M_1 \oplus M_2 \xrightarrow{\pi} M_2 \to 0$. Now our required assertion follows from the above Corollary 2.8.

2.10. Corollary: Every finite direct sum of atomic R-modules is atomic

Proof: Follows by the Proposition 2.9 and induction.

The more general is also true. That is, any direct sum of atomic modules is atomic.

2.11. Proposition: If $M = \oplus_{i \in I} M_i$, where each M_i is atomic then M is also atomic.

Proof: Since, by hypothesis, each M_i is atomic, $\mathrm{Soc}(M_i) \subseteq' M_i$. Hence $\oplus_{i \in I} \mathrm{Soc} M_i \subseteq' \oplus_{i \in I} M_i$. By [11, Exercise 6.12(6)], $\mathrm{Soc}(\oplus_{i \in I} M_i) \subseteq' \oplus_{i \in I} M_i$. i.e. $\mathrm{Soc}(M) \subseteq' M$. Hence M is atomic.

2.12. Remark: We have seen above an atomic module is a generalization of a semisimple module. We know that if R is a semisimple R-module then every R-module is semisimple. But the same thing is not true in the case of atomic modules. In the Example given above for Remark 2.4, R is atomic as an R-module but the factor module $R/Z(p^\infty)$ is not atomic. However we prove the following.

We are not able to characterize the ring R for which every R-module is atomic. However we have the following result.

2.13. Proposition: If R is a left Artinian ring then every R-module is atomic.

Proof: Let M be an R-module and N be any nonzero submodule of M. Let x be a nonzero element of N .Then Rx being a (nonzero) homomorphic image of R as an R-module, it is Artinian and hence it is atomic. So, $\mathrm{Soc}(Rx) \neq 0$ and hence $\mathrm{Soc}(N) \neq 0$. So M is atomic.

2.14. Remark: We know that every left Artinian ring is right perfect [6, Theorem P]. So the above proposition follows from the Bass's theorem stated in the beginning of this section. But the Bass proof is very complicated whereas our proof is quite simple.

2.15. Proposition: A ring R is left Artinian if and only if it is left Noetherian and every R-module is atomic.

Proof: By the above proposition we need only to prove the 'if' part. It is

enough to prove that every cyclic R-module is cofinitely generated. Let I be any left ideal of R. Then, by hypothesis, R/I is atomic and finitely generated and hence cofinitely generated since, by hypothesis, R is left Noetherian. So R is a left Artinian ring.

2.16. Proposition: If every R-module is atomic, then $J(R)$ is right T-nilpotent.

Proof: Follows from Remark 28.5(2) of [4].

2.17. Corollary: If every R-module is atomic and $R/J(R)$ is semisimple then, R is a right perfect ring.

Proof: Follows from Proposition 2.16.

We also have the following result.

2.18. Proposition: If a ring R is left atomic and semiprime then $J(R) = 0$.

Proof: Let for convenience $J = J(R)$. Suppose $J \neq 0$. Since, by hypothesis, R is left atomic, J contains a minimal left ideal, say, S. Then $JS = 0$ and hence $S^2 = 0$, which contradicts our hypothesis that R is semiprime. So $J = 0$.

2.19. Corollary: If R is semiprime ring and cofinitely generated as an R-module then R is semisimple Artinian.

Proof: We know that $J(R) = \cap M_\alpha$, where each M_α is a maximal left ideal of R. From above Proposition 2.18, it follows that $J(R) = 0$. i.e.$M_\alpha = 0$. Since, by hypothesis, R is cofinitely generated, there exists $\alpha_1, ..., \alpha_n$ such that $\cap_n^{i=1} M_{\alpha_i} = 0$. Then the map $\phi : R \to \oplus_n^{i=1} R/M_{\alpha_i}$ defined by $\phi(r) = (r + M_{\alpha_1}, ...r + M_{\alpha_n})$ for all $r \in R$ is a one-one map. Since $\oplus_n^{i=1} R/M_{\alpha_i}$ is semisimple so is R. Hence R is semisimple Artinian.

We now generalize the Proposition 2.18. Prior to this we prove the following proposition.

2.20. Proposition: Let I be an ideal of ring R and let M be an R/I-module. If M is atomic as an R-module then M is atomic as an R/I-module also.

Proof: Let N be a nonzero R/I-submodule of M. Clearly N is a nonzero R-submodule of M. Since, by hypothesis, M is atomic as an R-module, N contains a simple R-submodule, say, S. Then S becomes a simple R/I-submodule of N. Hence $\mathrm{Soc}_{R/I}(N) \neq 0$. Thus M is atomic as an R/I-module.

2.21. Corollary: If a ring R is such that every R-module is atomic and if I is an ideal of R then every R/I-module is atomic

2.22. Proposition: If a ring R is such that every R-module is atomic then $\mathrm{Nil}_*(R) = \mathrm{Nil}^*(R) = J(R)$.

Proof: By Proposition 10.27 of [10], we have $\mathrm{Nil}_*(R) \subseteq \mathrm{Nil}^*(R) \subseteq J(R)$. We now prove that $J(R) = \mathrm{Nil}_*(R)$. Clearly $R/\mathrm{Nil}_*(R)$ is a semiprime ring and it is left atomic, by the above corollary. Hence by the Proposition 2.18, $J(R/\mathrm{Nil}_*(R)) = (\bar{0})$. Since $\mathrm{Nil}_*(R) \subseteq J(R)$, $J(R/\mathrm{Nil}_*(R) = J(R)/\mathrm{Nil}_*(R)$. Hence $J(R)/\mathrm{Nil}_*(R) = (\bar{0})$ which implies $J(R) = \mathrm{Nil}_*(R)$.

We note that the above proposition generalizes the well known property of left Artinian rings that all the three radicals $\mathrm{Nil}_*(R)$, $\mathrm{Nil}^*(R)$ and $J(R)$

coincide [cf. 10, Proposition 10.27].

By the Remark 28.5 and the Exercise 28.5 of [4] there exists a ring R such that every R-module is atomic but R is not right perfect.

2.23. Proposition: Let M be a supplemented module. Then M is semisimple if and only if M is atomic and every simple submodule of M is a direct summand of M.

Proof: We need only to prove 'if' part.

If: Let N be any submodule of M. Since, by hypothesis, M is supplemented, N has a supplement, say K. Then $N + K = M$ and $N \cap K$ is small in K. If $N \cap K = 0$ then we are done. Suppose $N \cap K \neq 0$. Since, by hypothesis, M is atomic, $\mathrm{Soc}(N \cap K) \neq 0$. Therefore there exists a simple submodule S in $N \cap K$. By hypothesis, S is a direct summand of M and hence of K. Also S is small in K. So $S = 0$, which is a contradiction. Hence $N \cap K = 0$. So N is a direct summand of M. Hence M is semisimple.

2.24. Proposition: Let R be commutative integral domain and M be an R-module. If M is atomic module then there are no nonzero torsion-free submodules.

Proof: Suppose the contrary. Then there exists a nonzero submodule N of M which is torsion-free. Since, by hypothesis, M is atomic, $\mathrm{Soc}(N) \neq 0$. So $\mathrm{Soc}(N)$ contains a simple submodule S, which is torsion. Being a submodule of N, $\mathrm{Soc}(N)$ is torsion-free. This is a contradiction. Hence M contains no nonzero torsion-free submodule.

2.25. Proposition: Let M be an R-module and N be a submodule of M which is essential in M. If N is atomic then M is also atomic.

Proof: Let K be a nonzero submodule of M. Since, by hypothesis, N is essential in M, $N \cap K \neq 0$. Since, by hypothesis, N is atomic, $\mathrm{Soc}(N \cap K) \neq 0$, and hence, $\mathrm{Soc}(K) \neq 0$. Hence M is atomic.

2.26. Proposition: Let M be an R-module. U and V are submodules of M such that U is a complement of V. Then V is atomic if and only if M/U is atomic.

Proof: Only if: Let V be atomic. Since, $(U + V)/U \cong V/(U \cap V) \cong V$, $(U + V)/U$ is atomic. Since, by hypothesis, U is complement of V, $(U + V)/U$ is essential in M/U. Since, by hypothesis, V is atomic, it follows, by Proposition 2.25, that M/U is atomic.

If: Let M/U be atomic. Let $\eta : M \to M/U$ be the canonical epimorphism. Let $f = \eta|_V : V \to M/U$. Since, by hypothesis, U is complement of V, $U \cap V = 0$. So $\ker f = U \cap V = 0$. Therefore f is one-one map. Hence V is atomic.

2.27. Proposition: Let M be an R-module and U be a submodule of M such that $\mathrm{Soc}(M) \subseteq U$. If M/U is atomic then U is essential in M.

Proof: Let K be a submodule of M such that $U \cap K = 0$. If suppose $K \neq 0$. Then, $(K + U)/U \neq 0$. Since, by hypothesis, M/U is atomic, $\mathrm{Soc}((K + U)/U) \neq 0$. Since, $(K + U)/U \cong K$, $\mathrm{Soc}(K) \neq 0$ and $\mathrm{Soc}(K) \subseteq$

$\mathrm{Soc}(M) \subseteq U$. Therefore $0 \neq \mathrm{Soc}(K) \subseteq U \cap K = 0$, a contradiction. Hence $K = 0$. Therefore, U is essential in M.

2.28. Corollary: Let M be an R-module such that $M/\mathrm{Soc}(M)$ is atomic module then, M is also atomic.

Proof: Since, by hypothesis, $M/\mathrm{Soc}(M)$ is atomic, by Proposition 2.27, $\mathrm{Soc}(M)$ is essential in M. Hence, M is atomic.

Remark: The above corollary also follows from Proposition 2.27 since $\mathrm{Soc}(M)$, being semisimple, is atomic.

2.29. Remark: In [2], a module M is defined as semi perfect if M is projective and every homomorphic image of M has a projective cover. If M is semiperfect then M is coatomic[5, Proposition 6]. But dual of a semiperfect module is injective module and injective modules need not be atomic.

2.30. Example: Q is injective as Z-module. Since Z is not atomic as Z-module, it follows that Q is also not atomic.

2.31. Definition: An R-module M is said to be coreduced if, for every submodule N of M, $\mathrm{Soc}(M/N) = 0$ implies $M = N$.

2.32. Proposition: If M is coreduced then M is atomic module.

Proof: Let N be a nonzero submodule of M. Suppose $\mathrm{Soc}(N) = 0$. Let K be a complement of N in M. Let $\eta : M \to M/K$ be the canonical epimorphism. Let $f = \eta|_N : N \to M/K$. Since, $N \cap K = 0$, f is one one map and since, K is complement of N, $f(N) = (N + K)/K$ is essential in M/K. Therefore $f(\mathrm{Soc}(N)) = \mathrm{Soc}(M/K)$, by Corollary 9.1.5(a)[3]. Hence $\mathrm{Soc}(M/K) = f(\mathrm{Soc}N) = f(0) = 0$. Since, by hypothesis, M is coreduced, it follows that $M = K$ which implies that $N = 0$ which is a contradiction. Hence $\mathrm{Soc}(N) \neq 0$ which proves that M is atomic.

2.33. Proposition: Let R be a left co-Noetherian and a left Noetherian ring. An R-module M is atomic if and only if every submodule of M is coreduced.

Proof: Only if: Let M be atomic and let N be any nonzero submodule of M. Let K be a submodule of N such that $\mathrm{Soc}(N/K) = 0$. Since, N is atomic by hypothesis, and since, by hypothesis R is left co-Noetherian and left Noetherian, it follows, by Proposition 2.6, that N/K is atomic, and hence $N/K = 0$. Therefore $N = K$. So N is coreduced.

If: Follows from the above proposition.

3. Simplified proof of G. Gungoroglu:

We conclude our paper by a simplified and a more generalized proof of a result of Gungoroglu that over a discrete valuation ring R, every submodule of a coatomic R-module is coatomic.

Gonca Gungoroglu[5], in her paper on "Coatomic Modules" in Lemma 3(3), has proved that over a discrete valuation ring R, every submodule of a coatomic R-module is coatomic. We prove the same result under far weaker condition

"commutative Noetherian ring". Prior to this, we prove a lemma.

Recall that Vamos [9] defined an R-module M to be finitely embedded (later called by Jans [8] as cofinitely generated) if M has finitely generated essential socle.

We consider the dual question. An R-module M is finitely generated if and only if its radical, $\mathrm{Rad}(M)$, is small in M and $M/\mathrm{Rad}(M)$ is cofinitely generated.

Here 'only if' part is not true.

3.1. Example: Consider the ring Z of integers. Z, as a Z-module, is finitely generated. Clearly, $\mathrm{Rad}(Z)=0$, and hence, small in Z. And Z is not cofinitely generated, since $\mathrm{Soc}(Z)= 0$ is not essential in Z.

We prove below the 'if' part if the ring R is left co-Noetherian.

3.2. Lemma: If M is an R-module such that $\mathrm{Rad}(M)$ is small in M and $M/\mathrm{Rad}(M)$ is cofinitely generated, then M is finitely generated.

Proof: Let M is an R-module such that $\mathrm{Rad}(M)$ is small in M and $M/\mathrm{Rad}(M)$ is cofinitely generated. We know that $\mathrm{Rad}(M/\mathrm{Rad}(M))= 0$. Since, by hypothesis, $M/\mathrm{Rad}(M)$ is cofinitely generated, it follows, by Exercise 19.3 of [11]. (Also see Proposition 19.1 and Definition 19.2 of [11]) that $M/\mathrm{Rad}(M)$ is finitely generated Since, by hypothesis, $\mathrm{Rad}(M)$ is small in M, it follows that M is finitely generated.

We now give a simplified proof of Lemma 3(3) of Gonca Gungoroglu. For this we recall the definition of a coatomic module. A module M is called coatomic if every proper submodule of M is contained in a maximal submodule or equivalently, for a submodule N of M, whenever $\mathrm{Rad}(M/N)= M/N$ then $M = N$.

3.3. Lemma: Over a left co-Noetherian and left Noetherian ring, every submodule of a coatomic module is coatomic.

Proof: Let R be a left co-Noetherian and left Noetherian ring and let M be coatomic module. Let L be any submodule of M. Let U be a submodule of L. Assume $L \neq U$. Since, by [12, Proposition 2], $\{E(S)/S$ is simple R-module$\}$ is a family of cogenerators for R-mod, there exists a simple R-module S and a nonzero homomorphism $f : L/U \to E(S)$. Let π denote the canonical projection from L onto L/U. Then there exists $g : M \to E(S)$ extending $f\pi : M \to E(S)$. Since, by hypothesis, M is coatomic, $g(M)$ is coatomic, by Lemma 3(1) of [5]. Since $g(M) \subseteq E(S)$, $g(M)$ is cofinitely generated. Let, for convenience, $N = g(M)$. Since, N is coatomic, $\mathrm{Rad}(N)$ is small in N. Since, by hypothesis, R is left co-Noetherian, $N/\mathrm{Rad}(N)$ is cofinitely generated. By the Lemma 3.2 above, N is finitely generated. Hence $g(L)$ is finitely generated since, by hypothesis, R is left Noetherian ring. Hence $g(L)$ has a maximal submodule. Since $g(L) = f\pi(L) = f(L/U)$, it follows that L/U has also a maximal submodule. Hence $L/U \neq \mathrm{Rad}(L/U)$. This proves that L is coatomic.

3.4. Corollary: Over a commutative Noetherian ring, every submodule of a coatomic module is coatomic.
Proof: The proof follows from the above and the from the fact that every commutative Noetherian ring is co-Noetherian [9, Theorem 2].

References

1)B. L. Osofsky, A generalizations of quasi-Frobenius rings, J. Algebra 4, (1966) 373-387.
2)E. A. Mares, Semiperfect modules, Math. Z. 82 (1963), 347-360.
3)F. Kasch, Modules and Rings, Academic Press Inc.(London) Ltd. (1982).
4)F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Springer-Verlag New York. Heidelberg. Berlin, (1991).
5)G. Gungoroglu, Coatomic Modules, Far East J. Math. Soc. Special Vol. (1998), Part-II, 153-162.
6)H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, Trans. Amer. Math. Soc., 95(1960), 466-488.
7)H. Zoschinger, Komplementierte moduln ber dedekindringen, J. Algebra 29 (1974), 42-56.
8)J. P. Jans, On co-Noetherian rings, J.London Math. Soc.(2)1 (1969), 588-590.
9)P. Vamos, On the dual of the notion of finitely generated, J.London Math.Soc., 43 (1968), 643-646.
10)T. Y. Lam, A First Course in Noncommutative Rings, Springer publications (1990).
11)T. Y. Lam, Lectures on Modules and Rings, Springer-Verlag Newyork, Inc. (1999).
12)V. A. Hiremath, Cofinitely generated and cofinitely related modules, Acta Math. Acad. Sci.Hungar. 39(1-3)(1982), 1-9.

Received: June, 2009