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Calculation of Determinant of a Polynomial Matrix in Complex Basis

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Abstract

In this work a fast and numerical computational method for the calculation of determinant of a polynomial matrix is proposed. The method modifies the Evaluation-Interpolation technique for the calculation of determinant and reduces the number of fixed required points to half with the use of complex basis. The performance of the proposed numerical computational method is evaluated through a comparative analysis with the simple computational method and built-in function of Matlab in software Matlab.

Keywords: Determinant, Polynomial Matrix, Evaluation-Interpolation, Complex basis

1 Introduction

The purpose of this paper is the development of numerical computational methods for solving problems of Control Theory [1], [2], [4]. The computational method *evaluation-interpolation* enables us to solve several Control Theory problems[5], [6]. The advantage of this method is the use of numerical analysis against analytical solutions of the corresponding problems.
In many Control Theory problems the solution is one or more one-variable polynomials (e.g. calculation of the determinant, calculation of the inverse o a polynomial matrix, etc.) With the method *evaluation-interpolation* we can compute the specific polynomials rather than just an approximate solution.
In this paper, using polynomial interpolation with Newton's method, we present

a method for calculating the determinant of one-variable polynomial matrices and we propose a new method using complex basis.
More specifically, in the first section the basic concepts of a polynomial matrix and the polynomial interpolation are introduced. In the second section, we present the calculation of the determinant of polynomial matrix using the method evaluation-interpolation with Newton's interpolation. In the next section, we describe the same method with the use of complex basis. In the fourth section, execution times of the methods, with the use of Matlab Software, are presented and finally the comparison of computational methods and their execution times are presented.

1.1 Definitions for the degree of a Polynomial Matrix

Definition 1.1 (Corresponding Degree Matrix)**.** Let the polynomial matrix $A \in \mathbb{R}[s]^{n\times n}$

$$A(s) = \begin{bmatrix} a_{11}(s) & \dots & a_{1n}(s) \\ \vdots & \ddots & \vdots \\ a_{n1}(s) & \dots & a_{nn}(s) \end{bmatrix}$$

Corresponding Degree Matrix of Polynomial Matrix A is a numerical matrix $D \in \mathbb{N}^{n\times n}$ which defined by $D = [d_{ij}]$ where $d_{ij} = deg(a_{ij}(s))$ is the degree of each element (polynomial) of polynomial matrix A.

Definition 1.2 (Degree of a Polynomial Matrix)**.** Let the polynomial matrix $A \in \mathbb{R}[s]^{n\times n}$. Degree of a one-variable polynomial matrix is $d = min\{d_c, d_r\}$ where d_c and d_r are the sum of the maximum degrees per column and per row of the Corresponding Degree Matrix D, respectively.

For example the polynomial matrix

$$A(s) = \begin{bmatrix} s^2 & 2s & s+3 \\ s-1 & s^3 & 9 \\ -1 & s & s+8 \end{bmatrix}$$

has Corresponding Degree Matrix

$$D = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The maximum degrees per column and per row of the Corresponding Degree Matrix D are

$$\left.\begin{matrix} d_{r_1} = 2 \\ d_{r_2} = 3 \\ d_{r_3} = 1 \end{matrix}\right\} \Rightarrow d_r = 2+3+1 = 6, \text{ and } \left.\begin{matrix} d_{c_1} = 2 \\ d_{c_2} = 3 \\ d_{c_3} = 1 \end{matrix}\right\} \Rightarrow d_c = 2+3+1 = 6$$

Thus, we have the Degree of a Polynomial Matrix $d = min\{d_r, d_c\} = 6$.

1.2 Evaluation-Interpolation technique

The Evaluation-Interpolation technique is used in control theory problems to minimize the computational complexity of the calculation-solution specific problems such as: a) calculation of greatest common divisor of two polynomials of one, two or more variables, b) calculation of the determinant of a polynomial matrix of one, two or more variables [6], [7], [8], [9] and c) calculation of the inverse matrix of a polynomial matrix of one, two or more variables [3], [5]. The Evaluation-Interpolation technique has 2 parts:

1. the computation of fixed points (evaluation part) and
2. the interpolation at these points (interpolation part).

The advantages of this technique are: a) the calculation of the determinant using arithmetic operations and not symbolic and b) the finding of the unique polynomial that verifies the initial values derived from a polynomial which is the unique solution.

2 Calculation of Determinant with Evaluation-Interpolation technique

A Control Theory problem in which can applied the Evaluation-Interpolation technique is the computation of the determinant of a one variable polynomial matrix. The steps of the computational method are the follows:

1. We calculate the degree $d = \min\{d_r, d_c\}$ of the polynomial matrix. The number of required points is $n = d + 1$.
2. We determine n random fixed points.
3. We evaluate the constant matrices for each point. For each constant matrix the corresponding determinant is calculated.
4. The set of the interpolation points is the values of fixed points and corresponding determinants
5. We apply the interpolation in the above set to calculate the Newton polynomial which is the determinant of the polynomial matrix

The above computational method is entitled DCEI (Determinant Calculation with Evaluation-Interpolation technique)

Example 1. *Let the matrix* $A \in \mathbb{R}^{3\times 3}[s]$

$$A(s) = \begin{bmatrix} s & 2s & 1 \\ 3 & s+1 & s \\ 6s & 4 & s+1 \end{bmatrix}$$

Step 1: The Corresponding Degree matrix is

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

We compute

$$\left.\begin{array}{l} d_{c_1} = 1 \\ d_{c_2} = 1 \\ d_{c_3} = 1 \end{array}\right\} \Rightarrow d_c = 1+1+1 = 3 \quad and \quad \left.\begin{array}{l} d_{r_1} = 1 \\ d_{r_2} = 1 \\ d_{r_3} = 1 \end{array}\right\} \Rightarrow d_r = 1+1+1 = 3$$

Then, $d = min\{d_c, d_r\} = 3$. Therefore, the number of required fixed points is $n = d + 1 = 4$.
Step 2: We determine 4 random fixed points, $x_i = -1, 0, 1, 2$.
Step 3: For each point we have

$$A(-1) = \begin{bmatrix} -1 & -2 & 1 \\ 3 & 0 & -1 \\ -6 & 4 & 0 \end{bmatrix}, \; A(0) = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$
$$A(1) = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 6 & 4 & 2 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 3 & 2 \\ 12 & 4 & 3 \end{bmatrix}$$

and $det(A(-1)) = -4$, $det(A(0)) = 12$, $det(A(1)) = 0$ and $det(A(2)) = 38$, respectively.
Step 4: The interpolation set is

x_i	-1	0	1	2
$\det(A(x_i))$	-4	12	0	38

Step 5: We calculate the Newton interpolation polynomial in above set

$$P(s) = 13s^3 - 14s^2 - 11s + 12$$

which is the determinant of polynomial matrix A.

3 Calculation of Determinant with Evaluation-Interpolation technique in Complex basis

In this case we follow the same steps with the above technique (DCEI). The only difference is that due to the conjugate of the complex points it is not necessary to calculate all the constant determinants for the fixed points to be selected. Thus, in the first part of the technique we reduce the number of computational operations, so the total computation time of the technique is reduced.

Specifically, we know that $\overline{z^n} = \overline{z}^n$. Let the polynomial

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_2 s^2 + a_1 s + a_0$$

If $z \in \mathbb{C}$ we have

$$\begin{aligned}\overline{P(z)} &= \overline{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0} \\ &= \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \ldots + \overline{a_2 z^2} + \overline{a_1 z} + a_0 \\ &= a_n \overline{z^n} + a_{n-1} \overline{z^{n-1}} + \ldots + a_2 \overline{z^2} + a_1 \overline{z} + a_0 \\ &= a_n \overline{z}^n + a_{n-1} \overline{z}^{n-1} + \ldots + a_2 \overline{z}^2 + a_1 \overline{z} + a_0 \\ &= P(\overline{z})\end{aligned}$$

Then, in step 2 of the above computational method (DCEI), we determine $\left\lceil \frac{n}{2} \right\rceil$ random fixed complex points. The points are given by

$$x_i = k \cdot i, \quad \text{where } k \in \mathbb{N}$$

For each point x_i we evaluate the values of the determinant $det(A(x_i))$ and we know the value of $det(A(\overline{x_i})) = \overline{det(A(x_i))}$. This computational method is entitled DCEIC (Determinant Calculation with Evaluation-Interpolation technique in Complex basis)

Example 2. *Let the matrix* $A \in \mathbb{R}^{3\times 3}[s]$

$$A(s) = \begin{bmatrix} s & 2s & 1 \\ 3 & s+1 & s \\ 6s & 4 & s+1 \end{bmatrix}$$

From Example 1 the number of required fixed points is $n = d + 1 = 4$.

Step 2: We determine $\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{4}{2} \right\rceil = 2$ *random fixed complex points,* $x_i = i, 3i$.

Step 3: For each complex point and the corresponding conjugate point, we have

$$A(i) = \begin{bmatrix} i & 2i & 1 \\ 3 & i+1 & i \\ 6i & 4 & i+1 \end{bmatrix}, \quad A(3i) = \begin{bmatrix} 3i & 6i & 1 \\ 3 & 3i+1 & 3i \\ 18i & 4 & 3i+1 \end{bmatrix}$$

and $det(A(i)) = 26 - 24i$ *and* $det(A(3i)) = 138 - 384i$, *respectively.*

Step 4: The interpolation set is

x_i	i	$3i$	$\overline{i} = -i$	$\overline{3i} = -3i$
$\det(A(x_i))$	$26 - 24i$	$138 - 384i$	$\overline{26 - 24i} = 26 + 24i$	$\overline{138 - 384i} = 138 + 384i$

Step 5: We calculate the Newton interpolation polynomial in above set

$$P(s) = 13s^3 - 14s^2 - 11s + 12$$

which is the determinant of polynomial matrix A.

4 Performance Tests and Results

In this section we present the performance tests and the results of the computational methods which are presented in previous sections. The computational methods are a) Evaluation-Interpolation for the numerical computation of the determinant of a one-variable polynomial matrix (DCEI) and b) Evaluation-Interpolation in complex basis for the numerical computation of the determinant of a one-variable polynomial matrix (DCEIC).
The performance tests are implemented with the following conditions: a) respect to Degree of a Polynomial Matrix d, b) respect to Dimensions of polynomial matrix and c) each element of polynomial matrix is a random polynomial with specific degree .
The performance tests are implemented in software Matlab which support symbolic operations. In following tables are presented the execution times of each computational method and the execution time of build-in function of Matlab which he is work with symbolic operations.
In Table 1 the execution times for $d = 1, 2, 3, 5, 8$ and dimensions from 2×2 until 10×10 are illustrated

From Table 1 we conclude that

1. The execution times of computational method DCEIC are better from DCEI.

2. While dimensions of matrix increase, the execution time of DCEIC in relation to DCEI go to half according the number of required points of each computational method.

3. The Matlab function works satisfactorily only in combination of small d and small dimensions.

5 Conclusion

A novel computational method for the calculation of the determinant of an one-variable polynomial matrix has been proposed, entitled DCEIC. The computational method is based to Evaluation-Interpolation technique and the existing computational method DCEI. The novel element in method is the different determination of the number of required fixed points with the help of complex basis. The use of complex basis reduce the number of required fixed points to half. The results of performance tests implies that the execution times reduce accordingly to the number of required fixed points of each method.

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Table 1: Execution times respect to d and Dimensions

d	Dimensions	DCEI	DCEIC	Matlab
1	2×2	0.48392	0.36012	0.009630
	3×3	0.97595	0.077477	0.045187
	4×4	0.168504	0.137114	0.216778
	5×5	0.241597	0.190110	1.269331
	6×6	0.313872	0.239553	4.975834
	7×7	0.527195	0.423166	43.950977
2	2×2	0.43590	0.034067	0.005155
	3×3	0.119101	0.082586	0.027394
	4×4	0.267719	0.178970	0.139848
	5×5	0.494341	0.320852	0.879766
	6×6	0.876761	0.617175	6.734698
	7×7	1.232907	0.765439	40.417281
3	2×2	0.044119	0.034433	0.010315
	3×3	0.260651	0.180485	0.057168
	4×4	0.431606	0.304279	0.335720
	5×5	0.697449	0.543164	1.337663
	6×6	1.289666	0.780189	4.227212
5	10×10	15.395493	8.199356	≥ 180
8	8×8	13.568474	7.093312	≥ 180

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