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Problem for the Optimal Control of Cigarette Addiction

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Abstract

The formulation and analysis of a deterministic optimal control problem by the Pontryagin maximum principle is presented, using a cost functional attached to a two-dimensional system of non-linear differential equations that interpret the dynamics of cigarette addiction with variable population, mortality due to tobacco consumption (lung cancer and other pathologies) and control for prevention of addiction to smoking. The problem of contour is solved in MATLAB choosing hypothetical parameter values.

Keywords: Cigarette, Smoking, Deterministic optimal control, Pontryagin maximum principle

1 Introduction

It is a disease caused by excessive consumption of tobacco, not only is a public health problem but it is also a social problem because it has harmful effects on health, not only for people who consume it, but also for those who live with them. This disease, considered a voluntary risk addiction, very difficult to abandon and control, so once the habit is started it is very difficult to leave it, since it becomes part of the life of a person, who despite knowing the harm to his health, he does not realize in exchange for a moment of pleasure. Slowly but effectively, tobacco causes irreversible damage to most organs of the body, generating several chronic and degenerative diseases [1], [2], [3].

Cancer is not the only danger that stalks smokers. Another important pathological consequence of smoking is the increased risk of suffering a myocardial infarction. This risk is evaluated approximately twice as much as that of a non-smoker. The stomach is another organ that suffers the consequences of smoking. Gastro-duodenal and dangerous gastroduodenal is almost three times more frequent in smokers than in non-smokers.

The tobacco epidemic causes chronic obstructive pulmonary disease, chronic bronchitis and emphysema, as well as ischemic heart disease and other diseases of the vascular system, while other pathologies considerably frequent in smokers are cancer of the lip, tongue, mouth, larynx, esophagus and bladder. The habit of smoking decreases more than 30% physical performance both sports and work. In turn, it generates a code of genetic information that the smoker transmits to his offspring [1],[2],[3].

In relation to the optimal control and application of the maximum principle of Pontryagin, deterministic applications are found in Cancer, HIV AIDS, Tuberculosis, Dengue, Malaria and dynamics in population ecology [4], [5], [6], [7], [8], [9], [10], [11], [12], [13],[14], [15], [16], [17], [18], [19].

2 Optimal control problem

A deterministic optimal control problem is formulated and analyzed by the Pontryagin maximum principle, using a functional of direct and indirect costs, attached to a system of non-linear differential equations that interpret the addiction dynamics to smoking that presents the following assumptions: the dynamics of addiction to smoking as an epidemiological dynamics type SI with the total population from the average age of 10 years divided into susceptible persons and smokers, it is considered that the smoker is a smoker during his life, the incidence of addiction to smoking is of the form $\beta \frac{y}{x+y} x$, increase of the susceptible population corresponding to the flow of people who reach the age of ten years, mortality due to tobacco consumption that causes lung cancer and other pathologies, variable total population and natural death of susceptible people and smokers.

The variables and parameters are x: average number of people over 10 years susceptible to being smokers, y: average number of people over 10 smokers, N: variable total population, ρ: flow of susceptible people who reach the age of ten years, β: probability of addiction to smoking, δ: rate of death due to tobacco use due to lung cancer and other pathologies, μ: rate of natural mortality,

$\lambda(x,y)$: the force of addiction to smoking that depends on the smoking population and the susceptible population, $u_1(t)$: control dependent on the time indicated by the prevention of addiction to smoking of susceptible people, τ: fixed time of optimal control and η_i, $i=1,2$: pesos of direct and indirect costs.

The functional objective of direct and indirect costs is proposed:

$$J(\mathbf{x},\mathbf{u}) = \int_0^\tau L(\mathbf{x},\mathbf{u})dt = \int_0^\tau \left\{ \ \eta_1 y(t) + \frac{\eta_2}{2} u_1^2(t) \ \right\} \ dt$$

Attached to the system of differential equations:

$$\frac{dx}{dt} = \rho - (1-u_1(t))\lambda(x,y)x - \mu x \equiv f_1(\mathbf{x},\mathbf{u}) \qquad (1)$$

$$\frac{dy}{dt} = (1-u_1(t))\lambda(x,y)x - (\mu+\delta)y \equiv f_2(\mathbf{x},\mathbf{u}) \qquad (2)$$

where $\rho,\mu,\delta > 0$, $0 < \beta < 1$, $\lambda(x,y) = \beta\frac{y}{x+y}$ and initial conditions $\mathbf{x(0)}=(x(0),y(0))$. It is about finding an optimal control $(\bar{u}_1(t))$ such that:

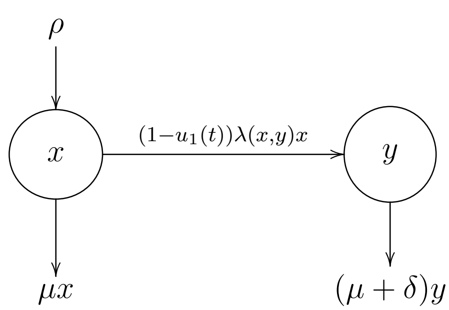

Figure 1: Optimal control dynamics of addiction to smoking.

$$J(\widetilde{u}_1(t)) = \min_{\Gamma} J(u_1(t)) \qquad (3)$$

where,

$$\Gamma = \{u_1(t) \in L^2(0,\tau) : 0 \le u_1(t) \le 1\}$$

2.1 Analysis of the optimal control problem

Given the optimal control problem:

$$\begin{cases} J(\mathbf{x}(t), u(t)) = \displaystyle\int_0^\tau L(\mathbf{x}(t), \mathbf{u}(t))dt \\ \\ \dfrac{d\mathbf{x}}{dt} = F(\mathbf{x}(t), \mathbf{u}(t)), \quad \forall \;\; t \geq 0 \;\; \forall \;\; u \in \Gamma \\ \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{4}$$

The solution exists if the following hypotheses are met:

i) The set of controls and state variables is not empty.

ii) The set of admissible controls Γ is closed and convex.

iii) Each f_i of the system (1) - (2) is continuous, is bounded above by a sum of the bounded control and the state, and can be written as a linear function of $u(t)$ with coefficients depending on time and the state.

iv) There exist constants $\alpha_1, \alpha_2 > 0$ y $\sigma > 1$ such that the integrand $L(\mathbf{x}(t), \mathbf{u}(t))$ of the objective functional J is concave and satisfies

$$L(\mathbf{x}(t), \mathbf{u}(t)) \leq \alpha_1 - \alpha_2 \left(|u_1(t)|^2 + |u_2(t)|^2\right)^{\frac{\sigma}{2}}$$

In this regard, the following proposition is formulated:

Proposition 2.1. *Given the functional objective*

$$J(u_1(t)) = \int_0^\tau L(\mathbf{x}(s), \mathbf{u}(s))ds,$$

where

$$\Omega = \{\mathbf{u} = u_1 : u_1 \;\; es \;\; medible, \;\; 0 \leq u_1(t) \leq 1, t \in [0, \tau]\},$$

subject to equations of state variables (1) - (2) with $\boldsymbol{x} = x_0$ *and* $\lambda(\tau) = 0$, *then there is an optimal control* $\bar{u} = (\bar{u}_1)$ *such that* $\max_{u \in \Gamma} J(u_1) = J(\bar{u}_1)$.

The Hamiltonian function or (Pontryagin function) is of the form:

$$H(\mathbf{x}, \mathbf{u}, \lambda) = L(\mathbf{x}, \mathbf{u}) + \sum_{i=1}^{2} \lambda_i f_i$$

where $\mathbf{x}$ is the vector of state variables, $\mathbf{u}$ is the vector of controls, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is the vector of adjoint or conjugated variables and L is the Lagrangian, that is,

$$H(\mathbf{x}, \mathbf{u}, \lambda) = \eta_1\, y(t) + \frac{\eta_2}{2}\, {u_1}^2(t) + \lambda_1 \left[\rho - \beta(1 - u_1(t))\frac{y}{x+y}x - \mu x\right]$$

$$+\lambda_2 \left[\beta(1 - u_1(t))\frac{y}{x+y}x - (\mu + \delta)y\right] + \upsilon_1 u_1 + \upsilon_2(1 - u_1).$$

Each $\upsilon_i(t)$ $(i = 1, 2)$ are penalty multipliers such that:

$$\upsilon_1 u_1 = 0\,,\ \upsilon_2(1 - u_1) = 0 \tag{5}$$

Applying the first order condition $\dfrac{\partial H}{\partial \mathbf{u}} = 0$, in particular $\dfrac{\partial H}{\partial u_1} = 0$ and the conditions of the penalty multipliers, optimal control is obtained:

$$\bar{u}_1(t) = \min\left(\max\left(0, \left(\frac{\lambda_2 - \lambda_1}{\eta_2}\right)\beta\frac{y}{x+y}x\right), 1\right).$$

The conjugated system (or adjoint system) has the form:

$$\frac{d\lambda}{dt} = -H_x(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u})$$

That is,

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= \lambda_1\left[\frac{\beta(1-\bar{u}_1)y^2}{(x+y)^2} + \mu\right] - \lambda_2\,\frac{\beta(1-\bar{u}_1)y^2}{(x+y)^2} \equiv g_1(\mathbf{x}, \mathbf{u}, \lambda)\\
\frac{d\lambda_2}{dt} &= -\eta_1 + \lambda_1\frac{\beta(1-\bar{u}_1)x^2}{(x+y)^2} - \lambda_2\left[\frac{\beta(1-\bar{u}_1)x^2}{(x+y)^2} - (\mu+\delta)\right] \equiv g_2(\mathbf{x}, \mathbf{u}, \lambda)
\end{aligned}$$

with transversality conditions $\lambda_i(\tau) = 0$, $i = 1, 2$.

3 Results and conclusions

The problem of contour is formed by the system of state variables of the dynamics of smoking with their respective initial conditions, and the system conjugated with their respective terminal conditions and optimal control:

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = F(\mathbf{x}, \bar{u}, \boldsymbol{\lambda}) \\ \dfrac{d\boldsymbol{\lambda}}{dt} = G(\mathbf{x}, \bar{u}, \boldsymbol{\lambda}) \\ \mathbf{x}(0) = \mathbf{x}_0 \\ \bar{u}_1 = (\min\left(\max\left(0, \left(\frac{\lambda_2-\lambda_1}{\eta_2}\right)\beta\frac{y}{x+y}x\right), 1\right) \end{cases} \tag{6}$$

It is solved using the MATLAB program with the initial conditions, terminal conditions and hypothetical values of the parameters. Figure 2 shows the effect of control on populations. The population of smokers with control decreases rapidly and at approximately twelve years there are no smokers compared to the population of smokers without control that decreases more slowly, achieving the extinction of smokers in about 30 years. The impact of these behaviors is reflected in the evolution of the susceptible population without control and control. A maximum control until about six years leads to a decrease in the smoking population, from that time the control decreases and the decrease of smokers becomes slow to zero.

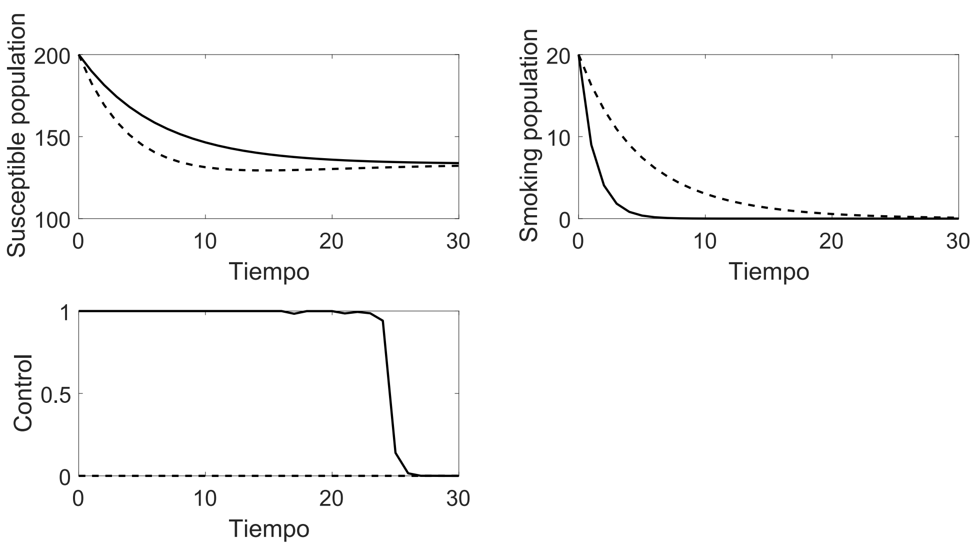

Figure 2: Behavior over time of susceptible people x, smokers y, without control and with control.

This research work contributes to the understanding of the optimal control of smoking by applying the classic principle of Pontryagin's maximum, enriching the applications of deterministic optimal control theory and modeling based on nonlinear differential equations that interpret a dynamic of a social phenomenon like smoking in mathematical epidemiology. It is provided in the integration of areas of knowledge such as dynamic systems, linear algebra, control theory, numerical methods and epidemiological modeling. It should be noted that smoking is a problem of impact on human health, with a high rate of death from diseases such as lung cancer and other pathologies, hence any scientific effort that contributes to its prevention is of great interest.

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