Contemporary Engineering Sciences, Vol. 12, 2019, no. 4, 259 - 265
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ces.2019.91136

A Solution of the Cylindrical Poisson-Boltzmann Equation

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Abstract

In this paper, we solve a cylindrical Poisson-Boltzmann equation. We use a solitary wave technique known as the Tanh and Ricatti functions. We get several families of solutions.

Keywords: Cylindrical Poisson-Boltzmann equation, Tanh method, Ricatti Functions.

1 Introduction

Electrolytes with cylindrical symmetry are very important in Engineering and Science [1]-[2]. This work presents the solution of the electrostatic potential for the cylindrical Poisson-Boltzmann equation, [2]-[3], applying the tanh method [4] and the Ricatti solutions [5].

2 Cylindrical Poisson-Boltzmann equation

We start from the cylindrical Poisson-Boltzmann equation:

$$\frac{\partial^2 \phi}{dr^2} + \frac{1}{r}\frac{\partial \phi}{dr} = \kappa^2 \sinh(\phi) \tag{1}$$

κ^{-1} is the Debye screening length [3]. Using the transformation [6]

$$e^u = r \tag{2}$$

Then

$$\frac{d^2}{dr^2} = -\frac{1}{r^2}\frac{d}{du} + \frac{1}{r^2}\frac{d^2}{du^2} \tag{3}$$

And replacing in eq. (1)

$$(-\frac{1}{r^2}\frac{d}{du} + \frac{1}{r^2}\frac{d^2}{du^2})\phi + \frac{1}{r^2}\frac{d\phi}{du} = \kappa^2 sinh(\phi) \tag{4}$$

$$\frac{d^2}{du^2}\phi = \kappa^2(e^{2u+\phi} - e^{2u-\phi}) \tag{5}$$

Defining

$$2u + \phi = \psi, \quad 2u - \phi = \psi^{'} \tag{6}$$

And replacing eqs. (6) in eq. (5)

$$\frac{d^2}{du^2}\psi = \kappa^2(e^{\psi} - e^{\psi^{'}}), \quad \frac{d^2}{du^2}\psi^{'} = -\kappa^2(e^{\psi} - e^{\psi^{'}}) \tag{7}$$

Now defining the variables

$$v_0 v = e^{\psi}; \quad w_0 w = e^{\psi^{'}} \tag{8}$$

Then, the first and second derivative for v

$$\frac{d\psi}{du} = \frac{1}{v_0 v}\frac{dv}{du}, \quad \frac{d^2\psi}{du^2} = -\frac{1}{v_0 v^2}(\frac{dv}{du})^2 + \frac{1}{v_0 v}\frac{d^2 v}{du^2} \tag{9}$$

following the same procedure for w

$$\frac{d\psi^{'}}{du} = \frac{1}{w_0 w}\frac{dw}{du}, \frac{d^2\psi^{'}}{du^2} = -\frac{1}{w_0 w^2}(\frac{dw}{du})^2 + \frac{1}{w_0 w}\frac{d^2 w}{du^2} \tag{10}$$

And replacing in eqs. (7)

$$-\frac{1}{v_0 v^2}(\frac{dv}{du})^2 + \frac{1}{v_0 v}\frac{d^2 v}{du^2} = \kappa^2(v_0 v - w_0 w) \tag{11}$$

$$-\frac{1}{w_0 w^2}(\frac{dw}{du})^2 + \frac{1}{w_0 w}\frac{d^2 w}{du^2} = -\kappa^2(v_0 v - w_0 w) \tag{12}$$

So

$$v\frac{d^2 v}{du^2} - (\frac{dv}{du})^2 - \kappa^2 v_0^2 v^3 + \kappa^2 v_0 v^2 w_0 w = 0 \tag{13}$$

$$w\frac{d^2w}{du^2} - (\frac{dw}{du})^2 - \kappa^2 w_0^2 w^3 + \kappa^2 v_0 w_0 w^2 v = 0 \tag{14}$$

Now, we introduce a new independent variable [4]:

$$Y = \tanh(\mu u) \tag{15}$$

Then, the derivatives of u, are:

$$\frac{d}{du} = \mu(1-Y^2)\frac{d}{dY}, \quad \frac{d^2}{du^2} = -2Y\mu^2(1-Y^2)\frac{d}{dY} + \mu^2(1-Y^2)^2\frac{d^2}{dY^2} \tag{16}$$

The solutions are postulated as [4]:

$$v = \sum_{i=1}^{m} a_i Y^i, \quad w = \sum_{i=1}^{n} b_i Y^i \tag{17}$$

Then replacing

$$\begin{aligned} &-2v\mu^2 Y(1-Y^2)\frac{dv}{dY} + v\mu^2(1-Y^2)^2\frac{d^2v}{dY^2} - (1-Y^2)^2(\frac{dv}{dY})^2 \\ &-\kappa^2 v_0^2 v^3 + \kappa^2 v_0 v^2 w_0 w = 0 \end{aligned} \tag{18}$$

$$\begin{aligned} &-2w\mu^2 Y(1-Y^2)\frac{dw}{dY} + w\mu^2(1-Y^2)^2\frac{d^2w}{dY^2} - (1-Y^2)^2(\frac{dw}{dY})^2 \\ &-\kappa^2 w_0^2 w^3 + \kappa^2 v_0 w_0 w^2 v = 0 \end{aligned} \tag{19}$$

Now, we balance the highest-order linear derivative with the highest order nonlinear terms in eq. (18) and eq. (19). Then, $vY^4\frac{d^2v}{dY^2} \to v^3 \to m = 2$, and $wY^4\frac{d^2w}{dY^2} \to w^3 \to n = 2$. So, replacing in eq. (17)

$$v = a_0 + a_1 Y + a_2 Y^2, \quad w = b_0 + b_1 Y + b_2 Y^2 \tag{20}$$

Replacing in eqs. (18)-(19), we get a set of equations, order by order in Y^i. And doing some algebra, we get:

$$f_1 \to (a_1 = 0, a_2 = -a_0, b_1 = 0, b_2 = -b_0, \mu = -\frac{i\sqrt{a_0 + b_0}kv_0}{\sqrt{2}}, w_0 = -v_0) \tag{21}$$

$$f_2 \to (a_1 = 0, a_2 = -a_0, b_1 = 0, b_2 = -b_0, \mu = \frac{i\sqrt{a_0 + b_0}kv_0}{\sqrt{2}}, w_0 = -v_0) \quad (22)$$

$$f_3 \to (a_0 = -2b_1, a_1 = -2b_1, a_2 = 0, b_0 = \frac{b_1}{2}, b_2 = \frac{b_1}{2}, \quad (23)$$
$$\mu = -\frac{1}{2}\sqrt{b_1}kw_0, v_0 = -\frac{w_0}{2})$$

$$f_4 \to (a_0 = -2b_1, a_1 = -2b_1, a_2 = 0, b_0 = \frac{b_1}{2}, b_2 = \frac{b_1}{2}, \quad (24)$$
$$\mu = \frac{1}{2}\sqrt{b_1}kw_0, v_0 = -\frac{w_0}{2})$$

$$f_5 \to (a_0 = 2b_1, a_1 = -2b_1, a_2 = 0, b_0 = -\frac{b_1}{2}, b_2 = -\frac{b_1}{2}, \quad (25)$$
$$\mu = -\frac{1}{2}i\sqrt{b_1}kw_0, v_0 = -\frac{w_0}{2})$$

$$f_6 \to (a_0 = 2b_1, a_1 = -2b_1, a_2 = 0, b_0 = -\frac{b_1}{2}, b_2 = -\frac{b_1}{2}, \quad (26)$$
$$\mu = \frac{1}{2}i\sqrt{b_1}kw_0, v_0 = -\frac{w_0}{2})$$

$$f_7 \to (a_0 = \frac{a_1}{2}, a_2 = \frac{a_1}{2}, b_0 = -2a_1, b_1 = -2a_1, b_2 = 0, \quad (27)$$
$$\mu = -\frac{1}{2}\sqrt{a_1}kv_0, w_0 = -\frac{v_0}{2})$$

$$f_8 \to (a_0 = \frac{a_1}{2}, a_2 = \frac{a_1}{2}, b_0 = -2a_1, b_1 = -2a_1, b_2 = 0, \quad (28)$$
$$\mu = \frac{1}{2}\sqrt{a_1}kv_0, w_0 = -\frac{v_0}{2})$$

$$f_9 \to (a_0 = -\frac{a_1}{2}, a_2 = -\frac{a_1}{2}, b_0 = 2a_1, b_1 = -2a_1, b_2 = 0, \quad (29)$$
$$\mu = -\frac{1}{2}i\sqrt{a_1}kv_0, w_0 = -\frac{v_0}{2})$$

$$f_{10} \to (a_0 = -\frac{a_1}{2}, a_2 = -\frac{a_1}{2}, b_0 = 2a_1, b_1 = -2a_1, b_2 = 0, \quad (30)$$
$$\mu = \frac{1}{2}i\sqrt{a_1}kv_0, w_0 = -\frac{v_0}{2})$$

Then, we get ten families of solutions.

	A_1	C_1	F
1	1/2	-1/2	$coth(\xi) \pm cosh(\xi)$, $tanh(\xi) \pm isech(\xi)$
2	1/2	1/2	$sec(\xi) \pm itan(\xi)$
3	-1/2	- 1/2	$csc(\xi) \pm icot(\xi)$
4	1	- 1	$tanh(\xi)$, $coth(\xi)$
5	1	1	$tan(\xi)$
6	-1	-1	$cot(\xi)$

Table 1: Solutions for eqs. (7), [5] .

3 Solitary wave method 2, Solutions Riccati equation

We use the method in [5], to get solutions for eqs. (13)-(14). So:

$$\Phi = \sum_{i=1}^{n} a_i F^i \tag{31}$$

where F solves, table (1), the Riccati equation, i.e.

$$F^{'} = (C_1 F^2 + A_1), \quad F^{''} = 2C_1 F(C_1 F^2 + A_1) \tag{32}$$

here A_1 and C_1 are constants, table (1). Replacing in eqs. (13)-(14), and balancing nonlinear terms, we have $n = 1$. Then, eq. (25) is, $v = (a_0 + a_1 F + a_2 F^2)$ and $w = (b_0 + b_1 F + b_2 F^2)$. Therefore, the derivatives are:

$$v^{'} = (a_1 + 2a_2 F)F^{'} = (a_1 + 2a_2 F)(C_1 F^2 + A_1), \;\; v^{''} = ((2a_2 F^{'})F^{'} \tag{33}$$
$$+(a_1 + 2a_2 F)F^{''}) = (2a_2(C_1 F^2 + A_1)^2 + (a_1 + 2a_2 F)2C_1 F(C_1 F^2 + A_1))$$

Replacing in eqS. (13)-(14), we obtain a group of equations, order by order in F^i. And doing some algebra, we get:

$$f \leftarrow (a_1 = 0, a_2 = \frac{a_0 C_1}{A_1}, b_0 = \frac{2A_1 C_1 - a_0 \kappa^2 w_0^2}{\kappa^2 w_0^2}, \tag{34}$$
$$b_1 = 0, b_2 = \frac{2A_1 C_1^2 - a_0 C_1 k^2 w_0^2}{A_1 \kappa^2 w_0^2}, v_0 = -w_0)$$

Then, we get 6 families of solutions, g_i, using Ricatti method [5].

4 Conclusions

We solved the cylindrical Poisson-Boltzmann equation for one rod-shaped electrolyte. Using the Tanh we find ten families of solutions and applying Ricatti functions, we get several 6 families of solutions. As a future work, we can investigate solutions for spherical and cylindrical with different charge configurations. The solutions are:

$$\phi = \frac{1}{2}\ln\left(\frac{(a_0 + a_1\tanh(\mu u) + a_2\tanh^2(\mu u))}{(b_0 + b_1\tanh(\mu u) + b_2\tanh^2(\mu u))}\right), \quad u = \ln(r) \tag{35}$$

$$\phi = \frac{1}{2}\ln\left(\frac{(a_0 + a_1F(u) + a_2F^2(u))}{(b_0 + b_1F(u) + b_2F^2(u))}\right), \quad u = \ln(r) \tag{36}$$

Acknowledgements. This research was supported by Universidad Nacional de Colombia in Hermes project (42840).

References

[1] Choi, S., Ann, J., Do, J., Lim, S., Park, C., & Shin, D. W. Application of rod-like Li6PS5Cl directly synthesized by a liquid phase process to sheet-type electrodes for all-solid-state lithium batteries, *Journal of the Electrochemical Society*, **166** (3) (2019), A5193-A5200. https://doi.org/10.1149/2.0301903jes

[2] C.B. Passos, P.S. Kuhn, M.C. Barbosa, DNA denaturation in the rodlike polyelectrolyte model, *Physica A*, **413** (2014), 481-488. https://doi.org/10.1016/j.physa.2014.06.059

[3] D. Andelman, in: *Soft Condensed Matter Physics in Molecular and Cell Biology*, Ed. by W. Poon and D. Andelman, Scottish Graduate Series: SUSSP 59", Taylor & Francis, New York, 2006, pp. 97-122.

[4] W. Malfliet and W. Hereman, The Tanh Method: I. Exact solutions of Nonlinear Evolution and Wave Equations, *Physica Scripta*, **54** (1996), 563-568. https://doi.org/10.1088/0031-8949/54/6/003

[5] Ömer Faruk Gözükizil and Şamil Akçağil, The tanh-coth method for some nonlinear pseudoparabolic equations with exact solutions, *Advances in Difference Equations*, **2013** (2013), 143. https://doi.org/10.1186/1687-1847-2013-143

[6] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, New York, Dover Publications, 1967.

Received: December 5, 2019; Published: December 24, 2019