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A Solution of the Two-dimensional Modified Poisson-Boltzmann Equation

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Abstract

In this work, we find solutions to the modified Poisson-Boltzman (PB) equation using the Tanh and Ricatti solitary wave techniques.

Keywords: Modified Poisson-Boltzman equation, Tanh method

1 Introduction

When in a electrolyte solution is incorporated the finite size of particles, an entropic effect, called excluded volume, we get the modified Poisson-Boltzmann equation [1]-[4]. In the search to find new solutions to non-linear partial differential equations, in recent years, we have witnessed to the creation of a huge number of methods called solitary waves. Among them, we use the tanh [5] and Riccati [6] in order to find solutions to the modified Poisson-Boltzmann equation.

2 The Modified Poisson-Boltzmann equation

We start from the two-dimensional modified Poisson-Boltzmann equation [1]-[4]:

$$\nabla^2\Psi = \frac{\partial^2\Psi}{dx^2} + \frac{\partial^2\Psi}{dy^2} = \frac{8\pi e n_0}{\epsilon}\frac{\sinh\left(\frac{e\Psi}{k_BT}\right)}{1-\phi_0+\phi_0\cosh\left(\frac{e\psi}{k_BT}\right)} \tag{1}$$

$$\psi = \frac{e\Psi}{k_B T} \tag{2}$$

$$\frac{\partial^2 \psi}{dx^2} + \frac{\partial^2 \psi}{dy^2} = \frac{8\pi e^2 n_0}{\epsilon k_B T} \frac{\sinh(\psi)}{1 - \phi_0 + \phi_0 \cosh(\psi)} \tag{3}$$

We define $\kappa^2 = \frac{8\pi e^2 n_0}{\epsilon k_B T}$. Then, we apply the following coordinate transformation:

$$u = x + y \tag{4}$$

The derivatives change like:

$$\frac{\partial}{\partial x} = \frac{d}{du}; \quad \frac{\partial}{\partial y} = \frac{d}{du}; \quad \frac{\partial^2}{\partial x^2} = \frac{d^2}{du^2}; \quad \frac{\partial^2}{\partial y^2} = \frac{d^2}{du^2} \tag{5}$$

Replacing eqs. (5) in eq. (3), we get:

$$2\frac{d^2 \psi}{du^2} = \kappa^2 \frac{\sinh(\psi)}{1 - \phi_0 + \phi_0 \cosh(\psi)} \tag{6}$$

Also, we do the next transformation

$$\psi = 2\tanh^{-1}(v) \tag{7}$$

So, the derivatives are:

$$\frac{d\psi}{du} = \frac{2}{1-v^2}\frac{dv}{du}; \quad \frac{d^2\psi}{du^2} = \frac{4v}{(1-v^2)^2}(\frac{dv}{du})^2 + \frac{2}{1-v^2}\frac{d^2 v}{du^2} \tag{8}$$

And replacing in eq. (6)

$$\begin{aligned}(\frac{4}{1-v^2}\frac{d^2 v}{du^2} + \frac{8v}{(1-v^2)^2}(\frac{dv}{du})^2)(1 - \phi_0 \\ + \phi_0 \cosh(2\tanh^{-1}(v))) = \kappa^2 \sinh(2\tanh^{-1}(v))\end{aligned} \tag{9}$$

Taking into account that

$$\sinh(2\tanh^{-1}(v)) = \frac{2v}{1-v^2}; \quad \cosh(2\tanh^{-1}(v)) = \frac{1+v^2}{1-v^2} \tag{10}$$

Then, replacing eq. (10) in eq. (9)

$$4(1-v^2)(1-v^2+2\phi_0 v^2)\frac{d^2 v}{du^2} + 8v(1-v^2+2\phi_0 v^2)(\frac{dv}{du})^2 = \kappa^2 2v(1-v^2)^2 \tag{11}$$

Now, we introduce a new independent variable [5]:

$$Y = \tanh(\mu u) \tag{12}$$

Then, the derivatives of Y, are:

$$\frac{d}{du} = \mu(1-Y^2)\frac{d}{dY}; \quad \frac{d^2}{du^2} = -2\mu^2 Y(1-Y^2)\frac{d}{dY} + \mu^2(1-Y^2)^2\frac{d^2}{dY^2} \tag{13}$$

The solutions are postulated as:

$$v = \sum_{i=1}^{m} a_i Y^i \tag{14}$$

Then, replacing

$$\begin{aligned} &4\mu^2(1-v^2)(1-v^2+2\phi_0 v^2)(-2Y(1-Y^2)\frac{dv}{dY} \\ &+(1-Y^2)^2\frac{d^2v}{dY^2}) + 8\mu^2 v(1-v^2+2\phi_0 v^2)((1-Y^2)\frac{d}{dY})^2 \\ &-2\kappa^2 v(1-v^2)^2 = 0 \end{aligned} \tag{15}$$

Now, we balance the highest-order linear derivative with the highest order nonlinear terms in eq. (15). Then

$$v^2 Y^4 \frac{d^2 v}{du^2} \to v^5 \to 2m+4+m-2 = 5m \to 3m+2 = 5m \to m = 1 \tag{16}$$

So, eq. (14) is:

$$v = a_0 + a_1 Y \tag{17}$$

Replacing in eq. (15), we get

$$\begin{aligned} &4\mu^2(1-(a_0+a_1Y)^2)(1-(a_0+a_1Y)^2+2\phi_0(a_0+a_1Y)^2)\times \\ &(-2Y(1-Y^2)a_1) + 8\mu^2(a_0+a_1Y)(1-(a_0+a_1Y)^2+2\phi_0(a_0+a_1Y)^2)\times \\ &((1-Y^2)a_1)^2 - 2\kappa^2(a_0+a_1Y)(1-(a_0+a_1Y)^2)^2 = 0 \end{aligned} \tag{18}$$

Defining $l_1 = \left(9+\sqrt{69}\right)$, $l_2 = \left(9-\sqrt{69}\right)$ and $l_3 = \left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)$. Doing some algebra

$$f_1 \to (a_0 = 0, a_1 = -\sqrt{1 + \frac{1}{3}l_3^{1/3} + \frac{\left(\frac{1}{2}l_1\right)^{1/3}}{3^{2/3}}}, \phi_0 = \frac{1}{36}(-6 \tag{19}$$
$$-3 \times 2^{2/3}\left(3l_2\right)^{1/3} + 2^{1/3}\left(3l_2\right)^{2/3} - 3 \times 2^{2/3}\left(3l_1\right)^{1/3} + 2^{1/3}\left(3l_1\right)^{2/3}),$$
$$\kappa^2 = \frac{4}{3}l_3^{1/3}\mu^2 + 2\left(\frac{2}{3}\right)^{2/3} l_1^{1/3}\mu^2)$$

$$f_2 \to (a_0 = 0, a_1 = \sqrt{1 + \frac{1}{3}l_3^{1/3} + \frac{\left(\frac{1}{2}l_1\right)^{1/3}}{3^{2/3}}}, \phi_0 = \frac{1}{36}(-6 \tag{20}$$
$$-3 \times 2^{2/3}\left(3l_2\right)^{1/3} + 2^{1/3}\left(3l_2\right)^{2/3} - 3 \times 2^{2/3}\left(3l_1\right)^{1/3} + 2^{1/3}\left(3\ l_1\right)^{2/3}),$$
$$\kappa^2 = \frac{4}{3}l_3^{1/3}\mu^2 + 2\left(\frac{2}{3}\right)^{2/3} l_1^{1/3}\mu^2)$$

$$f_3 \to (a_0 = 0, a_1 = -(1 - \frac{1}{6}l_3^{1/3} + \frac{il_3^{1/3}}{2\sqrt{3}} - \frac{\left(\frac{1}{2}l_1\right)^{1/3}}{2 \times 3^{2/3}} - \frac{il_1^{1/3}}{2 \times 2^{1/3}3^{1/6}})^{1/2}, \tag{21}$$
$$\phi_0 = -\frac{1}{72}i(-12i + 3 \times 2^{2/3}3^{5/6}l_2^{1/3} + 3 \times 2^{1/3}3^{1/6}l_2^{2/3} + 3i2^{2/3}\left(3l_2\right)^{1/3}$$
$$-i2^{1/3}\left(3l_2\right)^{2/3} - 3 \times 2^{2/3}3^{5/6}l_1^{1/3} - 3 \times 2^{1/3}3^{1/6}l_1^{2/3} + 3i2^{2/3}\left(3l_1\right)^{1/3}$$
$$-i2^{1/3}\left(3l_1\right)^{2/3}), \kappa^2 = -\frac{2}{3}l_3^{1/3}\mu^2 + \frac{2il_3^{1/3}\mu^2}{\sqrt{3}} - \left(\frac{2}{3}\right)^{2/3} l_1^{1/3}\mu^2 - \frac{i2^{2/3}l_1^{1/3}\mu^2}{3^{1/6}})$$

$$f_4 \to (a_0 = 0, a_1 = (1 - \frac{1}{6}l_3^{1/3} + \frac{il_3^{1/3}}{2\sqrt{3}} - \frac{\left(\frac{1}{2}l_1\right)^{1/3}}{2 \times 3^{2/3}} - \frac{il_1^{1/3}}{2 \times 2^{1/3}3^{1/6}})^{1/2}, \tag{22}$$
$$\phi_0 = -\frac{1}{72}i(-12i + 3 \times 2^{2/3}3^{5/6}l_2^{1/3} + 3 \times 2^{1/3}3^{1/6}l_2^{2/3} + 3i2^{2/3}\left(3l_2\right)^{1/3}$$
$$-i2^{1/3}\left(3l_2\right)^{2/3} - 3 \times 2^{2/3}3^{5/6}l_1^{1/3} - 3 \times 2^{1/3}3^{1/6}l_1^{2/3} + 3i2^{2/3}\left(3l_1\right)^{1/3}$$
$$-i2^{1/3}\left(3l_1\right)^{2/3}), \kappa^2 = -\frac{2}{3}l_3^{1/3}\mu^2 + \frac{2il_3^{1/3}\mu^2}{\sqrt{3}} - \left(\frac{2}{3}\right)^{2/3} l_1^{1/3}\mu^2 - \frac{i2^{2/3}l_1^{1/3}\mu^2}{3^{1/6}})$$

$$f_5 \to (a_0 = 0, a_1 = -(1 - \frac{1}{6}l_3^{1/3} - \frac{il_3^{1/3}}{2\sqrt{3}} - \frac{\left(\frac{1}{2}l_1\right)^{1/3}}{2 \times 3^{2/3}} + \frac{il_1^{1/3}}{2 \times 2^{1/3}3^{1/6}})^{1/2}, \tag{23}$$
$$\phi_0 = \frac{1}{72}i(12i + 3 \times 2^{2/3}3^{5/6}l_2^{1/3} + 3 \times 2^{1/3}3^{1/6}l_2^{2/3} - 3i2^{2/3}\left(3l_2\right)^{1/3}$$
$$+i2^{1/3}\left(3l_2\right)^{2/3} - 3 \times 2^{2/3}3^{5/6}l_1^{1/3} - 3 \times 2^{1/3}3^{1/6}l_1^{2/3} - 3i2^{2/3}\left(3l_1\right)^{1/3}$$
$$+i2^{1/3}\left(3l_1\right)^{2/3}), \kappa^2 = -\frac{2}{3}l_3^{1/3}\mu^2 - \frac{2il_3^{1/3}\mu^2}{\sqrt{3}} - \left(\frac{2}{3}\right)^{2/3} l_1^{1/3}\mu^2 + \frac{i2^{2/3}l_1^{1/3}\mu^2}{3^{1/6}})$$

	A_1	C_1	F
1	1/2	-1/2	$coth(\xi) \pm cosh(\xi)$, $tanh(\xi) \pm isech(\xi)$
2	1/2	1/2	$sec(\xi) \pm itan(\xi)$
3	-1/2	- 1/2	$csc(\xi) \pm icot(\xi)$
4	1	- 1	$tanh(\xi)$, $coth(\xi)$
5	1	1	$tan(\xi)$
6	-1	-1	$cot(\xi)$

Table 1: Solutions for eq. (11), [6] .

$$f_6 \to (a_0 = 0, a_1 = (1 - \frac{1}{6} l_3^{1/3} - \frac{i l_3^{1/3}}{2\sqrt{3}} - \frac{\left(\frac{1}{2} l_1\right)^{1/3}}{2 \times 3^{2/3}} + \frac{i l_1^{1/3}}{2 \times 2^{1/3} 3^{1/6}})^{1/2}, \quad (24)$$

$$\phi_0 = \frac{1}{72} i (12i + 3 \times 2^{2/3} 3^{5/6} l_2^{1/3} + 3 \times 2^{1/3} 3^{1/6} l_2^{2/3} - 3i 2^{2/3} (3l_2)^{1/3}$$
$$+ i 2^{1/3} (3l_2)^{2/3} - 3 \times 2^{2/3} 3^{5/6} l_1^{1/3} - 3 \times 2^{1/3} 3^{1/6} l_1^{2/3} - 3i 2^{2/3} (3l_1)^{1/3}$$
$$+ i 2^{1/3} (3l_1)^{2/3}), \kappa^2 = -\frac{2}{3} l_3^{1/3} \mu^2 - \frac{2i l_3^{1/3} \mu^2}{\sqrt{3}} - \left(\frac{2}{3}\right)^{2/3} l_1^{1/3} \mu^2 + \frac{i 2^{2/3} l_1^{1/3} \mu^2}{3^{1/6}})$$

Then, we get 6 families of solutions

3 Solitary wave method 2, Riccati equation

We use the method in [6], in order to find solutions for eq. (11). So:

$$\Phi = \sum_{i=1}^{n} a_i F^i \quad (25)$$

where F solves, table (1), the Riccati equation, i.e.

$$F^{'} = (C_1 F^2 + A_1), \quad F^{''} = 2C_1 F (C_1 F^2 + A_1) \quad (26)$$

here A_1 and C_1 are constants, table (1). Replacing in eq. (11), and balancing nonlinear terms, we get $n = 1$. Then, eq. (25) is, $v = (a_0 + a_1 F)$. So, the first derivative is:

$$v^{'} = (a_1 F^{'}) = a_1 (C_1 F^2 + A_1) \quad (27)$$
$$v^{''} = (a_1 F^{''}) = 2a_1 C_1 F (C_1 F^2 + A_1)$$

Replacing in eq. (11)

$$
\begin{aligned}
&4(1-(a_0+a_1F)^2)(1-(a_0+a_1F)^2+2\phi_0(a_0+a_1F)^2)2a_1C_1F(C_1F^2\\
&+A_1)+8(a_0+a_1F)(1-(a_0+a_1F)^2+2\phi_0(a_0+a_1F)^2)(a_1(C_1F^2+A_1))^2\\
&-\kappa^2 2(a_0+a_1F)(1-(a_0+a_1F)^2)^2=0
\end{aligned} \tag{28}
$$

Then, we get a set of equations, order by order in F^i. And doing some algebra, we get:

$$
g_1 \to \left\{a_0=0, a_1=-\frac{i\sqrt{C_1}}{\sqrt{A_1}}, \kappa=0\right\}, g_2 \to \left\{a_0=0, a_1=\frac{i\sqrt{C_1}}{\sqrt{A_1}}, \kappa=0\right\} \tag{29}
$$

Then, we get 12 families of solutions, g_i, using Ricatti method.

4 Conclusions

We solved the modified Poisson-Boltzmann equation using the Tanh and Riccatti methods. We find 6 families of solutions using tanh and 12 Riccati methods. As a future work, we can extend the method to investigate spherical and cylindrical geometries. The solutions are:

$$
\Psi(x,y)=2\frac{k_BT}{e}\tanh^{-1}\left(a_1\tanh\left(\mu(x+y)\right)\right) \tag{30}
$$

$$
\Psi(x,y)=2\frac{k_BT}{e}\tanh^{-1}\left(a_1F(x+y)\right) \tag{31}
$$

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