

Stability in Predator-Prey System

Lotka-Volterra Model Incorporating a Prey Refuge

Like the Law of Mass Action

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Abstract

A deterministic continuous-time predator-prey model is analyzed considering the use of refuge by a part of prey population. In earlier works it has been claimed that the prey refuge use exerts a stabilizing effect in the dynamics of the interacting populations. In this work, we show that the above statement it is true assuming that the quantity of prey in refuge considering that this quantity is described by a function proportional to encounters between prey and predators $X_r = \alpha XY$ and we analyze the dynamic properties of such a system through modifying the well-known Lotka-Volterra predator-prey model.

Keywords: Antipredator behavior, Refuge, Stability, Population dynamics, Bifurcation

1 Introduction

According to [1], two refuge types have been considered in the ecological literature: the quantity of hidden prey is proportional to the prey population at instant t and the quantity of hidden prey is a constant number. In [2], they introduce a new form to describe the quantity of prey in refuge considering that this quantity

is described by a monotonic growth function $X_r = \frac{\alpha X}{X + \beta}$ dependent of prey population size $X = X(t)$, α represents the maximum capacity of refuge and where the population's fraction in refuge is falling in the way. The most important and popular interactions between species in ecology environment is the predation interaction, which has been modelled by the predator-prey system and considered extensively in many aspects with many functional responses [3].

In population dynamics the Prey-predator models are of great interest to researchers because they deal with environmental problems such as community's morbidity and how to control it, optimal harvest policy to sustain community. The models proposed considered diverse suppositions to simplify their mathematical descriptions, such as: The populations' homogeneity, homogeneity of environmental, distribution spatial uniform, constant rates of growth, encounters between the species predators and equally probable prey, sizes population clerks exclusively of the time, the species predator feeds exclusively of the species prey, while these feeds of a resource that is in the habitat in big quantities the one which alone it intervenes passively, they are not considered behaviours of the species of physiological, morphological, social type, neither reintroduction of species etc. Uses of refuges, in the nature, many preys respond to the attacks of the predators looking for such space refuges. The effect itself of refuge use on the population growth is complex in nature, but for modelling purposes it can be understood as the reduction of prey mortality due to reduction in predation success. The refuges affect positively the population growth of preys and negatively that of predators. Spatial refuges are found where environmental heterogeneity provides less-accessible sites for predators in which a number of preys can stay, at least temporarily. In this way, some fraction of the prey population is partially protected against predators and we assume that the refuge is a physical location in which prey either live or temporally hide. The Most of the works show the refuge conclusion that stabilizes predator interactions [4].

However, González-Olivares and Ramos-Jiliberto, discard the common conclusion that the use of the shelter by the population of prey always leads to stability as considering the same assumptions in the model Rosenzweig-McArthur obtained which trajectories can oscillate for some parameter values [1].

We denote for $X(t) = X$ and $Y(t) = Y$ the population sizes of preys and predators, respectively for $t \geq 0$, considered as continuous variables that can represent density, biomass, or quantity of each population's individuals. The common Lotka-Volterra predator prey model is

$$X_\varphi : \begin{cases} \frac{dX}{dt} = r\left(1 - \frac{X}{K}\right) X - qXY \\ \frac{dY}{dt} = (pX - c)y \end{cases} \quad (0)$$

$\varphi = (p, q, c, K, r) \in R_+^5$ is a vector of biological parameters, usually the parameters have the following biological meanings:

p : is the efficiency with which predators convert consumed prey into new predators;

q : is the maximal per capita predator consumption rate;

c : is the natural per capita death predator rate;

K : is the prey environmental carrying capacity;

r : is the intrinsic per capita prey growth rate.

The unique positive equilibrium point within the first quadrant is globally asymptotically stable, if and only if $pK - c > 0$.

If $X_r(t) = X_r$, a quantity of prey population that occupies a refuge (heterogeneity of the means), the quantity of preys in the refuge, then the quantity of preys that interaction with the predators is $X - X_r$. The model (0) is transformed

$$X_\varphi : \begin{cases} \frac{dX}{dt} = r\left(1 - \frac{X}{K}\right)X - q(X - X_r)Y \\ \frac{dY}{dt} = (p(X - X_r) - c)Y \end{cases} \quad (1)$$

Which is a Kolmogorov type system, where the functional response is linear [5].

According to [6,7] two refuge types have usually been considered in the ecological literature

I). those that protect a constant fraction of the preys. $X_r = \beta X$

Objection: The fraction of hidden preys is a growing lineal function; this implies that the refuge readiness is bigger while bigger it is the population size.

Then the system is topologically equivalent to the original one [3], changing only the coordinates of the positive equilibrium point.

II). those that protect a fixed quantity of preys. $X_r = \beta$

Objections: The quantity of refuge preys not depend on capacity physical of refuge. The occurrence of a constant number or constant proportions of the prey in refuge seem to be very unlikely in the nature [8].

The system ceases to be a Kolmogorov type, but the new positive equilibrium point is also globally asymptotically stable.

In [2], it's analyzed the population consequences of refuge use in the Lotka-Volterra model with self-limitation, assuming that the amount of prey in the refuge is

given by the function of the functional response type II Holling, which considers the two objections is used

$$X_r = \frac{\alpha X}{X + \beta}$$

Where α it represents the maximum physical capacity of the refuge β is a half saturation constant of the refuge. Moreover, we have that the per capita fraction of prey population of refuge $\frac{X_r}{X}$ is a decreasing function. With a maximum physical capacity of refuge influences in the number of equilibrium points. The variations of the parameters makes that you of the bifurcation of Hopf and spends from a point of stable equilibrium to a limit cycle. The existence of an equilibrium point within the first quadrant is conditional refuge size and can be: Center, spiral unstable or spiral stable.

On the other hand, modifying the well-known Rosenzweig-MacArthur model, [5]. It has been demonstrated that, for certain parameter constraints, the effect of refuge by a fraction of the prey population implies oscillatory behaviors due the creation of one limit cycle on system, [1].

The simplistic interpretation of the stabilizing role of refugees may not be correct, since for more complex models to describe the refuge use can exert a locally destabilizing effect due to the emergence of a stable limit cycle implying the existence of oscillatory behavior of both populations.

In the present work, we analyze the population consequences of refuge use in the Lotka-Volterra model with self-limitation, assuming that the amount of prey in the refuge is given by the function

$$X_r = \alpha XY$$

Where α rate of encounter of predators with prey. Showing results different from those obtained in [10], both in the number of equilibrium points in the positive quadrant, and in its classification.

2 Mathematical model

Considering the proposed function in the model obtains the vector field $X_\mu^{X_r}$ described by the autonomous system of differential equations:

$$X_\mu^{X_r} : \begin{cases} \frac{dX}{dt} = r(1 - \frac{X}{K})X - q(X - \alpha XY)Y \\ \frac{dY}{dt} = -cY + p(X - \alpha XY)Y \end{cases} \quad (2)$$

$\mu = (c, K, p, q, r, \alpha) \in R_+^6$ Is a vector of biological parameters.

Following the methodology used in [9], we make a reparameterization of the vector field $X_\mu^{X_r}$ or the system (2) including changes of variables and a time rescaling given by the diffeomorphism

$$\varphi : (R^+)^2 \times R \rightarrow (R^+)^2 \times R, \text{ such that}$$

$$(N, P, \tau) \rightarrow (X, Y, t), \text{ where}$$

$$\varphi(N, P, \tau) = (KN, \frac{r}{q}P, \frac{1}{r}\tau) = (X, Y, t), \text{ with } \det D\varphi(X, Y, t) = \frac{K}{q} > 0.$$

The vector field in the new coordinates is $X_\eta = \varphi \circ X_\mu^{X_r}$, and associated second-order differential system is the following Kolmogorov type polynomial:

$$X_\eta = \left\{ \begin{array}{l} \frac{dN}{d\tau} = N(1 - N - (1 - AP)P) \\ \frac{dP}{d\tau} = BP[N(1 - AP) - C] \end{array} \right\} \quad (3)$$

$$A = \frac{\alpha r}{q}, \quad B = \frac{pK}{q} \text{ and } C = \frac{c}{pK}$$

Which has only three parameters, i.e. $\eta = (A, B, C) \in R_+^3$, and the system (3) is topologically equivalent to system (2).

The N-nullclines associated to the system (3) are given by: $N=0$ and $P = \frac{1 \pm \sqrt{1 - 4A(1-N)}}{2A}$, where $1 - 4A(1 - N) \geq 0$, $N \geq 1 - \frac{1}{4A}$ and $A > \frac{1}{4}$.

The P-nullclines associated to the system (2) are given by: $P=0$ and $N = \frac{C}{1-AP}$,

where $1 - AP > 0$.

The Jacobian matrix of system (3) is

$$JX_\eta(N; P) = \begin{bmatrix} AP^2 - P - 2N + 1 & -(1 - 2AP)N \\ B(1 - AP)P & B(N - C - 2ANP) \end{bmatrix}$$

If (N^*, P^*) lies in the first quadrant it has a singularity of system (3)

$$JX_\eta(N^*; P^*) = \begin{bmatrix} AP^{*2} - P^* - 2N^* + 1 & -(1 - 2AP^*)N^* \\ B(1 - AP^*)P^* & B(N^* - C - 2AN^*P^*) \end{bmatrix} = \begin{bmatrix} -N^* & N^* - 2C \\ B(1 - AP^*)P^* & -ABN^*P^* \end{bmatrix}$$

As $AP^{*2} - P^* - N^* + 1 = 0$, then $AP^{*2} - P^* - 2N^* + 1 = AP^{*2} - P^* - N^* + 1 - N^* = -N^*$; and $N^* - AN^*P^* - C = 0$, $B(N^* - C - 2AN^*P^*) = B(N^* - AN^*P^* - C - AN^*P^*) = -ABN^*P^*$ and $-(1 - 2AP^*)N^* = N^* - 2C$.

Considering the isolated means, where there is not the reintroduction of preys X such that overcome the prey environmental carrying capacity K .

3 Main results

For system (3) we have that

Lemma 1

The set $\Omega = \left\{ (N, P) \in (R_0^+)^2 / 0 \leq N \leq 1, 0 \leq P < \frac{1}{A} \right\}$ is an invariant region of the vector field.

Proof:

- a) As the system (3) is of Kolmogorov type, then the N -axis and P -axis are invariant sets.
- b) If $N = 1$, we have that $\frac{dN}{d\tau} = -(1 - AP)P < 0$, and the trajectories point in region Ω .

Lemma 2

The trajectories are bounded.

Proof: Using Poincaré compactification.

Lemma 3

The system (3) has three equilibrium points:

- I. $(N_1^*, P_1^*) = (N = 0, P = 0)$
- II. $(N_2^*, P_2^*) = (N = 1, P = 0)$
- III. $(N_3^*, P_3^*) = (N^*, P^* = \frac{N^*(1-N^*)}{c})$

Proof: Considering the equations that define to X_η .

Lemma 4

- a) The singularity $(N_1^*, P_1^*) = (N = 0, P = 0)$ is a saddle point for all parameter value.
- b) The singularity $(N_2^*, P_2^*) = (N = 1, P = 0)$ is a saddle point.

Proof: Evaluating the Jacobian matrix we have:

$$a) \quad DX_{\eta}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -BC \end{pmatrix}$$

The eigenvalues: $\lambda_1 = 1 > 0$, $\lambda_2 = -BC < 0$.

Then the singularity is a saddle point for all parameter value.

$$b) \quad JX_{\eta}(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & B(1-C) \end{bmatrix}$$

The eigenvalues: $\lambda_1 = -1 < 0$, $\lambda_2 = B(1-C)$.

As $C = \frac{c}{pK} < 1$, $\lambda_2 > 0$ (1,0) is saddle point.

Theorem 3.1

For system (3) we have the following result:

The singularity $(N_3^*, P_3^*) = (N^*, P^* = \frac{N^*(1-N^*)}{C})$ is a stable nodal sink point.

Proof: Evaluating the Jacobian matrix at (N^*, P^*) , we have:

$$JX_{\eta}(N^*; P^*) = \begin{bmatrix} AP^{*2} - P^* - 2N^* + 1 & -(1-2AP^*)N^* \\ B(1-AP^*)P^* & B(N^* - C - 2AN^*P^*) \end{bmatrix} = \begin{bmatrix} -N^* & N^* - 2C \\ B(1-AP^*)P^* & -ABN^*P^* \end{bmatrix}$$

Where

$$DetJX_{\eta}(N^*, P^*) = AB(N^*)^2 P^* + B(2C - N^*)(1 - AP^*)P^* > 0$$

And the Trace is negative the behavior of singularity depends on the trace and this depends on

$$TraceJX_{\eta}(N^*, P^*) = -N^*(1 + ABP^*) < 0.$$

4. Conclusions

The preys react fleeing the refuge due to the presence of a certain amount of predator. The knowledge of the impact of prey refuge use by a fraction of prey-population is relevant in the context of bioeconomic and conservation management, because it helps in regulating the harvesting activity in the ecosystem and management of reserves or non-take zones; also it is essential for conservation of endangered species creating protected areas (reserves) for preserving them [9]. For this reason, other models for interaction between species must be considered.

In this work, we have analyzed a differential system derived of the classical Lotka-Volterra model for predator-prey interaction, considering that a fraction of

prey population search refuge to protect from predation; we use a new function to express the fraction of prey population in cover. This work was shown in Theorem 3.1, the assertion that regular use refuge stabilizes prey predator-prey. If the equilibrium point exists within the region of invariance in the Lotka-Volterra model with the proposed refuge function, this is a stable attractor focus for any variation of the parameters. The stability of the equilibrium point within the first quadrant in the Lotka-Volterra model with this refuge function that includes the density of predators and prey shows the same conclusion as in the previous works that "the refuge stabilizes the predatory interactions". But different from Almanza-Vásquez, 2007; but the function studied by him included the size of the refuge.

Therefore, when we add the refuge function to the model, the stable equilibrium point of the original model is closer to the point (1,0), which indicates that stability is obtained with more prey and fewer predators. The result obtained is equivalent to that obtained by Almanza-Vásquez when the capacity of the shelter is low, which is stable at the point of equilibrium within the first quadrant.

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