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Application of Lagrange Equations in the Analysis of Slider-Crank Mechanisms

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Abstract

This paper presents a two-dimensional movement analysis and the forces acting on a slider-crank mechanism by applying the Lagrangian approach, to develop a mathematical model for the complete description of the kinetic and kinematics of the mechanism. The aim of the research is restricted to a specific scenario: the determination of the force exerted by the slider, which becomes an important parameter in positive displacement machines (such as for compressors and pumps), and acquires greater value in the study of internal combustion engines (ICE). From the Lagrangian analysis, partial differential equations are obtained, relating the force generated by the slider, with three variables: The crank angular position, its variation concerning time–angular velocity–and the angular acceleration. The results show the Lagrangian analysis as a useful methodology of kinetic analysis, which can be used to design processes to find economical options for the manufacture of machines (mainly ICE) that meet the specifications from the kinetic point of view, with reduced losses.

Keywords: Mechanism, Lagrangian, angular velocity, angular acceleration, kinetic analysis

1. Introduction

From all the basic types of mechanisms, the slider-crank mechanism (four-link mechanism with three revolute joints and one prismatic or sliding joint) is a well known one, due to its capability to transform linear displacement to rotational displacement and vice-versa [1], which make these very useful in power transmission appliances, such as internal combustion engines, and to emulate human processes, such as walking, running, and so. To improve these mechanisms (and by extension, the performance of machines and processes that use them), the slider-crank mechanism has undergone through studies focused to analyze three main aspects:

- The kinematic response to an external load (like forces or torque) [2][3]
- The kinetic response to an external load, because this is associated with the general state of stress around the elements that conform it, which compromises the material used in these [4]–[6]
- The influence of geometrical factors, as tolerances, over undesired aspects, such as the damage generated by mechanical vibrations [7]–[9]

To study these aspects, a series of kinematic and kinetic models have been developed, to relate relevant variables (such as position, acceleration and forces) as a function of some reference system (commonly time or angular displacement) [1]. However, the classic mechanics (that is, the base in Newton's 2nd law), can lead to some uncertainties, besides the fact some models are quite long and complicated to formulate and solve, even with numerical methods. To solve these limitations, the Lagrangian mechanics (a reformulation of the classic one) has been used extensively to model many problems where conservative forces are included [10]. For this reason, the present work makes an overview of the modeling of these mechanisms, which helps to predict the system response in some common engineering problems, through the application of Lagrange equations.

2. Methodology

2.1. Development of the study

In the present study, Lagrange equations were applied to a slider-crank mechanism, to develop a mathematical model that relates the angular position, velocity and acceleration of the system, and the force exerted by the slider, all of these modeled as a time-dependent function. Following that, this model was applied to an ICE, to predict the force applied during the slider (or piston) movement, which is the case for other machines, like the reciprocating piston compressors; however, to solve the implicit model generated (as shown below), a series of experimental points, measured at regular time intervals, were used in the solution, where the response for the kinematic and kinetic variables of interest, as same as the force exerted by the mechanism, were calculated, with the help of computer packages, such as MATLAB®.

2.2. Fundamental Equations

The main formulation of the Lagrange equation, which is common in the literature, is the following [10]:

$$\frac{d}{dt}\left(\frac{\partial T_{total}}{\partial \dot{\theta}}\right) - \frac{\partial T_{total}}{\partial \theta} + \frac{\partial V_{total}}{\partial \theta} = W_{nc} \tag{1}$$

Where T_{total} is the total kinetic energy, V_{total} is the total potential energy of the system, and W_{nc} is the virtual work done by a force. For the solution of these derivatives, a system as shown in Figure 1, was studied; for this system, a geometrical analysis relates the main angles of the mechanism with the dimensions of the crank and connecting rod (2 and 3 in Figure 1, respectively), by the expression

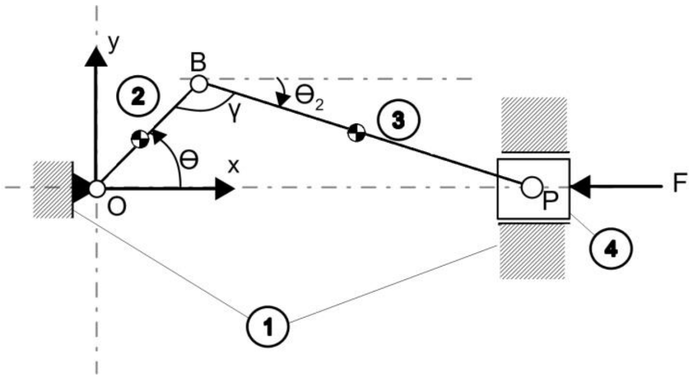

Figure 1. Slider-crank mechanism.

$$\sin\theta_2 = \frac{L_2}{L_3} \cdot \sin\theta \tag{2}$$

And their derivatives with the expression

$$\dot{\theta}_2 = \frac{L_2 * \cos\theta}{L_3 \cdot \sqrt{1 - \frac{L_2^2}{L_3^2} * \sin^2\theta}} \cdot \dot{\theta} \tag{3}$$

In the Lagrange equation formulation, a conservative force F applied in point P, generates a virtual work W_p defined in a differential form as [10]

$$\delta W_p = F \cdot \delta r_p \tag{4}$$

And from some calculations, the position of point P (δr_p) can be expressed as a function of the angle θ, with the formula

$$\delta r_p = \delta\theta \cdot \sin\theta \cdot \left(-L_2 - \frac{L_2^2 \cos\theta}{L_3 \cdot \sqrt{1 - \frac{L_2^2}{L_3^2} * \sin^2\theta}} \right) \tag{5}$$

Replacing (5) in (4), and integrating to include all the angular displacement θ, a general expression is obtained, as follows:

$$W_p = F \cdot \sin\theta \left(-L_2 - \frac{L_2^2 \cos\theta}{L_3 \cdot \sqrt{1 - \frac{L_2^2}{L_3^2} * \sin^2\theta}} \right) \tag{6}$$

Using the expressions found in the literature about rigid body motion [11], the total kinetic and potential energy can be calculated by the equations

$$T_{total} = \frac{1}{2}\left(I_2\dot{\theta}^2 + I_3\dot{\theta}_2^2 + m_4 v_4^2\right) \tag{7}$$

$$V_{total} = g \cdot \sin\theta \cdot [m_2 r_{O2c} + m_3(L_3 + r_{B3c})] \tag{8}$$

Where I_2 and I_3 are the inertia moments of elements 2 and 3, around their centre of rotation (points O and B, respectively). Applying partial derivation in (7) and (8), and replacing in (1) along with (6), we can obtain the expression relating the force applied and the angular displacement of the mechanism:

$$\left(I_2 + \frac{I_3 L_2^2 \cos^2\theta}{L_3^2 - L_2^2 \sin^2\theta} + \frac{m_4 L_2^2 \sin^2\theta\left(L_3^2 - L_2^2\right)^2}{(L_3 + L_2\sin\theta)^2(L_3 - L_2\sin\theta)^2}\right)\ddot{\theta} +$$
$$\left(\frac{m_4 L_2^2 \cos\theta\sin\theta(L_3 - L_2)^2\left(L_3^2 - L_2^2\sin^2\theta\right)(L_3 + L_2)^2}{(L_3 + L_2\sin\theta)^3(L_3 - L_2\sin\theta)^3} - \frac{I_3 L_2^2\left(L_3^2 - L_2^2\right)\sin 2\theta}{2(L_3 + L_2\sin\theta)^2(L_3 - L_2\sin\theta)^2}\right)\dot{\theta}^2 + g \cdot$$
$$\cos\theta\ \left(m_2 r_{O2c} + m_3(L_3 + r_{B3c})\right) = F \cdot \sin\theta \left(-L_2 - \frac{L_2^2 \cos\theta}{L_3 \cdot \sqrt{1 - \frac{L_2^2}{L_3^2} * \sin^2\theta}} \right) \tag{9}$$

Due of the complexity of (9), some tools can be used to simplify the process; one of these implies the direct estimation of the angle γ (through experimental measurements) and obtaining the angle θ with the expression

$$180° - \theta - \arcsin\left(\frac{L_2}{L_3}\sin\theta\right) = \gamma \tag{10}$$

3. Results and discussion

3.1. Experimental data

To solve (9) for the force applied, it's necessary to know the angular position and its derivatives, which are implicit in the same expression; to solve this issue, a series of experimental points corresponding to the variation of angle γ respect to time were measured at regular intervals in a Diesel engine; its geometric characteristics are shown in Table 1.

Crank length L_2 [mm]	47
Crank mass m_2 [kg]	15.3351
Crank inertia moment I_2 [kg mm^2]	27698.81
Connecting rod length L_3 [mm]	155.83
Connecting rod mass m_3 [kg]	0.8602
Connecting rod inertia moment I_3 [kg mm^2]	5807.55
Slider mass m_4 [kg]	0.5341
Distance from rod c.g. to rotation centre r_{b3c} [mm]	51.07

Table 1. Characteristics of the engine studied

As shown in Figure 2, the data follows a sinusoidal pattern, and the symmetry of the plot makes evident that there isn't an eccentricity between the slider and the rotation center of the crank (point O).

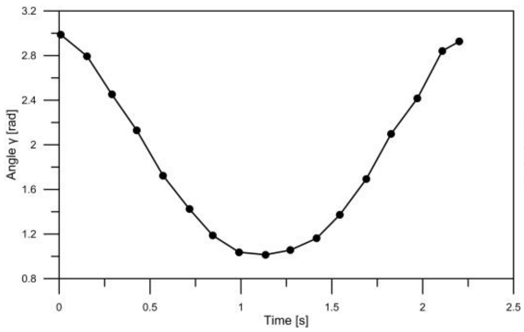

Figure 2. Experimental data used in the analysis.

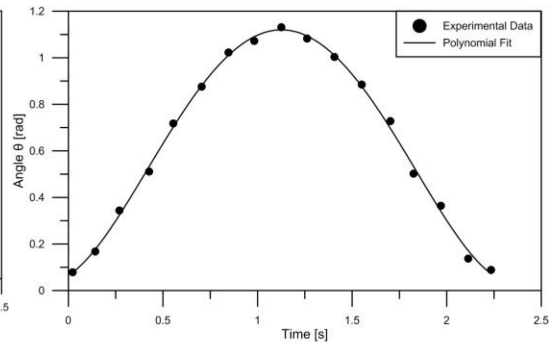

Figure 3. Crank angular position as a function of time.

3.2. Angular variables of the mechanism

From the experimental values of the angle γ, the expression (10) is used to calculate the values of the angular position θ at the measured points. Figure 3 shows the plot of the time dependence of the angular position; to extrapolate this results over any possible time value, the data were fitted to a 4th order polynomial with the form

$$\theta = 0.0649 + 0.522 \cdot t + 2.166 \cdot t^2 - 2.129 \cdot t^3 + 0.472 \cdot t^4 \tag{11}$$

with a R^2 coefficient of 0.9974, which indicates a near-perfect fit. From these values, derivatives $\dot{\theta}$, $\ddot{\theta}$ were easily calculated and plotted for the experimental points, as shown in Figure 4 and 5. As the result follows the same sinusoidal trend as the original data for the angle γ, it can be established that these are in concordance with (9), as the equation is mainly composed of periodic terms, being the polynomial (11) an approximation (easier to derivate) of the periodic behaviour of the system.

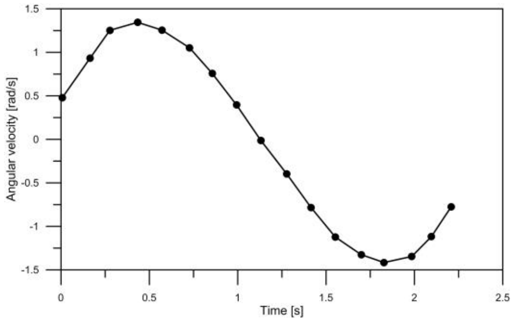

Figure 4. Crank angular velocity as a function of time.

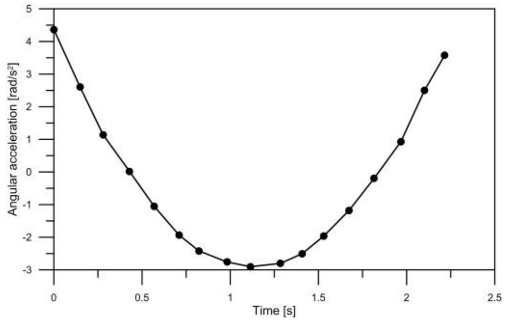

Figure 5. Crank angular acceleration as function of time.

3.3. Calculation of the force exerted by the slider

Knowing the angular position, the angular velocity and acceleration, the force exerted by the slider were calculated using the expression (9); the result is shown in Figure 6. From the results, a clear reduction of the force is visible when the slider moves toward the negative x-axis, with a rise after the crank moves again to the positive direction. From this behavior, it can be deducted that the maximum force values are obtained in the points where the slider changes its displacement direction, and the velocity drops to near zero (and where the acceleration reaches peak values, too). However, the change in the force follows a more accentuated slope than other variables, which is closely related with its inertial properties and can originate long-term damage in this kind of mechanisms.

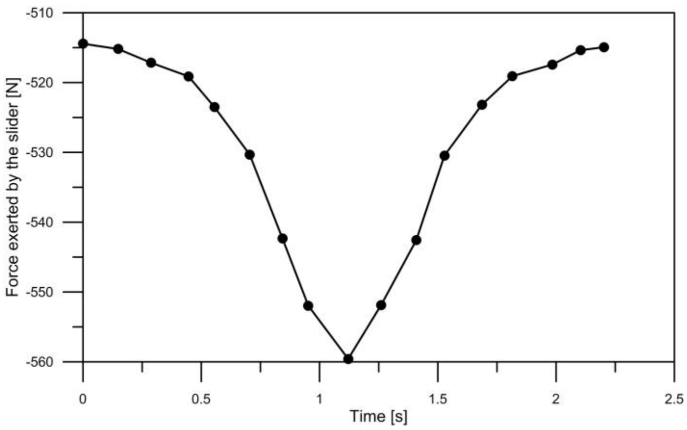

Figure 6. Force exerted by the slider, as a function of time.

4. Conclusions

Between the conclusions of this work, the most essential was to demonstrate the usefulness of the Lagrangian analysis for the kinetic analysis of systems above the Newtonian approach, mainly when conservative forces are included, and the design becomes hard to model in an analytical way. Besides that, a by-product of this work is a useful methodology of kinetic analysis which can be used to perform geometry-driven optimization, to apply the results to design and optimization processes, which can lead to finding economical options for the manufacture of machines that meet the specifications from the kinetic point of view. It was also found that the action of the force on the system is cyclical, depending on the variation of the theta angle, which is why it is necessary to project a fatigue analysis when a deeper design is required, to take in account the nature of the movement on the stress state of the material selection and the estimation of the average life of the mechanism.

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