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On Stationary Solutions and Blow-up for a Discrete Model with Nonlocal Diffusion

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Abstract

In this paper, we study some results relatives to a nonlocal diffusion model with Neumann boundary conditions in the discrete case. In particular, aspects relatives to stationary solutions are derived. We also study the blow-up phenomenon of the solutions and we find the blow up rates when the boundary conditions is given by the function u^p with $p > 1$. The asymptotic behavior of the solutions as t goes to infinity is studied. Finally, we show some numerical experiments which illustrate our results.

Keywords: Nonlocal diffusion; Neumann boundary conditions; Discrete model; Blow-up; blows up rates

1 Introduction

Many nonlocal problems are given by variants of equation

$$u_t(x,t) = (J * u)(x,t) - u(x,t) = \int_{\mathbb{R}^N} J(x-y)[u(y,t) - u(x,t)]dy, \qquad (1)$$

where in many cases $J : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a nonnegative function with compact support, $J(0) > 0$, and symmetric radially, $J(-z) = J(z)$, with $\int_{\mathbb{R}^N} J(z)dz = 1$. In (1), $(J * u)(x,t)$ is the usual convolution

$$(J * u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy.$$

The nonlocal diffusion problems have been used to modeling several problems of the applied mathematics, associate with several branch of sciences and engineering, see for instance [1],[2],[3] and references therein. Analytical studies of that models can be found in a lot of references on the theme, however, the following (including its bibliography) can be considered illustrative [4],[5],[6],[7],[8]. A physical description of Eq. (1) can be found in [1] and many of the references mentioned early.

The model

$$\begin{aligned} &u_t(x,t) = \int_{\Omega} J(x-y)(u(y,t)-u(x,t))dy + \int_{\partial\Omega} G(x-y)g(y,t)dS_y. \\ &u(x,0) = u_0(x), \end{aligned} \tag{2}$$

have been studied by the authors in [5]. Results on existence and uniqueness of solutions as well as a comparison principle and others important characteristics of (2) was obtained. In the case $\Omega = [-L, L]$ (2) reduces to

$$\begin{aligned} &u_t(x,t) = \int_{-L}^{L} J(x-y)(u(y,t)-u(x,t))dy + G(x+L)g(-L,t) + G(x-L)g(L,t), \\ &u(x,0) = u_0(x), \end{aligned} \tag{3}$$

with $u_0 \in L^1([-L, L])$ is non-negative function and $J, G : \mathbb{R} \to \mathbb{R}$ are smooths, symmetric and nonnegative functions, $J(z) = J(-z)$, $G(z) = G(-z)$, increasing on $[-L, 0]$ and the compact support in the unitary ball with $\int_{\mathbb{R}} J(x)dx = 1$. Here g is positive and regular function.

As in [9], let $N \in \mathbb{N}$ and $L > 0$. For $h = \frac{L}{N} > 0$, we consider a uniform mesh of the interval $[-L, L]$, $P = \{-L = x_{-N}, \cdots, x_0, \cdots, X_N = L\} = \{x_i = hi,\ -N \leq i \leq N\}$. Approximating the integrals in (3), we obtain the ODEs system

$$\begin{aligned} &u_i'(t) = \sum_{j=-N}^{j=N} hJ(h(i-j))(u_j(t)-u_i(t)) + G(h(i+N))g_-(t) + G(h(i-N))g_+(t) \\ &u_i(0) = u_0(x_i) = u_0(ih), \end{aligned} \tag{4}$$

for $-N \leq i \leq N$. We have used $g_-(t)$ and $g_+(t)$ instead of $g(-Nh,t)$ and $g(Nh,t)$ respectively.

We will denote

$$U(t) = (u_{-N}(t), \cdots, u_N(t)), \quad U_0 = U(0). \tag{5}$$

The discrete model (4) was studied by the authors in [9]. Results such as the existence an uniqueness of solutions, a comparison principle as well the consistence and convergence was obtained.

We say that a solution of (4) blows up in finite time if there exists a finite time $T_h > 0$ such that

$$\lim_{t \nearrow T_h} \|U(t)\|_\infty = \lim_{t \nearrow T_h} \max_k u_k(t) = \infty. \tag{6}$$

The main idea of this work, is to complement the results associated to (4) obtained in [9]. In particular, we will study all referent to stationary solutions for (4), the asymptotic behavior of the solutions as t goes to infinity. When the boundary condition is give by the function u^p with $p > 1$, we analyzed the blow-up ocurrense for the discrete problem (4), moreover we find the blow-up rates. Finally we show some numerical experiments that illustrate our results.

2 Stationary solutions

As was mentioned early, the following results are the continuation of those presented in the reference [9], therefore the space and the norm used here, is the same considered in the mentioned paper. Wit this in mind, in the following lines we will show that the solutions of the problem (4) converges (as $t \to \infty$) to solutions of the respective stationary problem, which is given by equation

$$0 = \sum_{j=-N}^{N} hJ(h(i-j))[\phi_j - \phi_i] + G(h(i+N))g_- + G(h(i-N))g_+. \tag{7}$$

We have the lemma:

Lemma 2.1. *If Eq. (7) has a solution, then*

$$\sum_{i=-N}^{N} (G(h(i+N))g_- + G(h(i-N))g_+) = 0. \tag{8}$$

Proof. In fact, summing to both sides of (7) we have

$$0 = \sum_{i=-N}^{N} \sum_{j=-N}^{N} hJ(h(i-j))\phi_j - \sum_{i=-N}^{N} \sum_{j=-N}^{N} hJ(h(i-j))\phi_i + \sum_{i=-N}^{N} (G(h(i+N))g_- + G(h(i-N))g_+).$$

Using the symmetry of J, we obtain

$$\sum_{i=-N}^{N}\sum_{j=-N}^{N} hJ(h(i-j))\phi_j - \sum_{i=-N}^{N}\sum_{j=-N}^{N} hJ(h(i-j))\phi_i = 0,$$

and therefore we obtain (8). □

Theorem 2.1. *The stationary problem (7) has a solutions if and only if it satisfies (8).*

Proof. In accordance with the previous lemma, we need only to show that the condition is sufficient. Suppose that we have the condition (8) and we put (7) in the equivalent form:

$$\sum_{j=-N}^{N} hJ(h(i-j))\phi_i - \sum_{j=-N}^{N} hJ(h(i-j))\phi_j = G(h(i+N))g_- + G(h(i-N))g_+. \quad (9)$$

We note that, (9) take the form

$$\sum_{j=-N\,(j\neq i)}^{N} hJ(h(i-j))\phi_i - \sum_{j=-N\,(j\neq i)}^{N} hJ(h(i-j))\phi_j = G(h(i+N))g_- + G(h(i-N))g_+. \quad (10)$$

Writing all equations for $-N \leq i \leq N$, we obtain a square system of $2N+1$ equations in the variable ϕ_i, which can be written in the matrix form as follow

$$B\phi = V, \quad (11)$$

being

$$\phi = \begin{pmatrix} \phi_{-N} \\ . \\ . \\ . \\ \phi_N \end{pmatrix}, \qquad V = \begin{pmatrix} v_{-N} \\ . \\ . \\ . \\ v_N \end{pmatrix},$$

where

$$v_i = G(h(i+N))g_- + G(h(i-N))g_+,$$

$-N \leq i \leq N$, and B is matrix of coefficients which is symmetric due to symmetry of J. We consider the homogeneous system associated to (7)

$$B\phi = 0, \quad (12)$$

where each line i is given by

$$\phi_i \sum_{j=-N}^{N} hJ(h(i-j)) - \sum_{j=-N}^{N} hJ(h(i-j))\phi_j = 0,$$

or in equivalent form

$$\sum_{j=-N}^{N} hJ(h(i-j))(\phi_j - \phi_i) = 0. \tag{13}$$

We note that from (13) we have $\phi_i = \phi_j$ for all $|h(i-j)| \leq 1$. In other case, if i satisfy $\phi_i = \max_{-N \leq j \leq N} \phi_j$, then from (13) we have

$$0 = \sum_{j=-N}^{N} hJ(h(i-j))(\phi_j - \phi_i) < 0,$$

because $\phi_j - \phi_i \leq 0$, and then we obtain a contradiction. Therefore, there exists $K \neq 0$ such that (12) can be written as

$$K\, B\tilde{V} = 0,$$

where

$$\tilde{V} = \begin{pmatrix} 1 \\ . \\ . \\ . \\ 1 \end{pmatrix},$$

so that, we conclude (12) has nonzero solutions. (In other words, the kernel of matrix B is given by multiples of $\tilde{V}$). So that, the system (11) has solutions if and only if

$$< V, \tilde{V} >= 0,$$

or in equivalent form, if and only if

$$\sum_{i=-N}^{N} (G(h(i+N))g_- + G(h(i-N))g_+) = 0.$$

Using the hypothesis, we conclude the system (7) has a solution $\phi = (\phi_{-N}, \ldots, \phi_N)$. Taking into account (8) and Corollary 6 in [9] we have $\sum_{j=-N}^{N} (u_0)_j = \sum_{j=-N}^{N} u_0(x_j) = m$, and therefore the solutions ϕ satisfy the condition

$$\sum_{j=-N}^{N} \phi_j = m,$$

which is equivalent to

$$\sum_{j=-N}^{N} \phi_j = \sum_{j=-N}^{N} u_0(x_j). \tag{14}$$

The theorem is proved. □

The following is the most important result of this section,

Theorem 2.2. *Let u be a solution of (3), with g satisfying the condition (8). Let ϕ be the unique solution of the respective stationary problem (7) satisfying (14). Then*

$$u_i(t) \to \phi_i, \tag{15}$$

uniformly as $t \to \infty$.

Proof. Let $w_i(t) = u_i(t) - \phi_i$, so that $w_i'(t) = u_i'(t) - 0$. From condition (8) we obtain as particular case that $G(x_i + L)g_- + G(x_i - L)g_+ = 0$, because G, g_- and g_+ are positive functions. We have

$$\begin{aligned} w_i'(t) &= u_i'(t) - 0 \\ &= \sum_{j=-N}^{N} hJ(h(i-j))(u_j(t) - u_i(t)) - \sum_{j=-N}^{N} hJ(h(i-j))(u_j(t) - u_i(t)) \\ &= \sum_{j=-N}^{N} hJ(h(i-j))(w_j(t) - w_i(t)). \end{aligned} \tag{16}$$

Therefore,

$$\frac{\partial}{\partial t} \sum_{i=-N}^{N} w_i(t) = \sum_{i=-N}^{N} \sum_{j=-N}^{N} hJ(h(i-j))(w_j(t) - w_i(t)) = 0. \tag{17}$$

From this last equation we obtain

$$\sum_{i=-N}^{N} w_i(t) = K,$$

for K any constant. However, for $t = 0$, due to (14) must be $K = 0$ and then

$$\sum_{i=-N}^{N} w_i(t) = 0, \tag{18}$$

from where, using (14) again, we conclude

$$\sum_{i=-N}^{N} u_i(t) = \sum_{i=-N}^{N} \phi_i = \sum_{i=-N}^{N} u_0(x_i). \tag{19}$$

On the other hand, using corollary (4) of [9] we can see that there exists a constant $C > 0$ such that

$$|w_i| \leq C|||(u_0)_i - \phi_i|||, \tag{20}$$

therefore, the functions $w_i(t)$ are uniformly bounded. Each $w_i(t)$ determine an equi-bounded set, then, if we consider the family $\{w_i(t_n)\}$, for any (t_n) increasing sequence there exists a subsequence $\{w_i(t_{n_k})\}$ such that it converges uniformly to a function ξ_i solution of the stationary problem. In fact, in other case, if

$$\sum_{i=-N}^{N} \sum_{j=-N}^{N} hJ(h(i-j))(\xi_j - \xi_i) = k \neq 0$$

then we have

$$\begin{aligned} &|\frac{\partial}{\partial t} \sum_{i=-N}^{N} w_i(t_{nk}) - \sum_{i=-N}^{N} k| \\ &= |\sum_{i=-N}^{N} \sum_{j=-N}^{N} hJ(h(i-j))((w_j(t_{nk}) - \xi_j) - ((w_i(t_{nk}) - \xi_i))|. \end{aligned}$$

The right side of this last equality is zero, which is a contradiction since $\sum_{i=-N}^{N} w_i(t) = 0$. Therefore must be $k = 0$, and then

$$\sum_{i=-N}^{N} \sum_{j=-N}^{N} hJ(h(i-j))(\xi_j - \xi_i) = 0.$$

Moreover, due to (14) ξ_i must be constant and taking into account (18), the constant must be zero. Finally we have

$$w_i(t) \to 0,$$

uniformly as $t \to \infty$. In this way we obtain (15). □

Corollary 2.1. *In the case that we not have (8), then, if u is solution to (4) we conclude that each u_i must be unbounded function.*

Proof. We suppose that $\sum_{i=-N}^{N}(G(h(i+N))g_- + G(h(i-N))g_+) > 0$. From Corollary 6 on [9], we obtain for t large, the total mass is large as we want, and therefore we can conclude that $u_i(t)$ is an unbounded function. □

3 Blow-up and blow-up rates

In this section, we considerer $g = u^p(x,t)$ in the problem (3). We will show that for $p > 1$, the solution of (3) blows up in finite time $T_h > 0$, moreover find the blow-up rates. We consider the following problem:

$$\begin{aligned} & u_i'(t) = \sum_{j=-N}^{j=N} hJ(h(i-j))(u_j(t)-u_i(t)) + G(h(i+N))u^p(-Nh,t) + G(h(i-N))u^p(Nh,t), \\ & u_i(0) = u_0(x_i) = u_0(ih), \end{aligned} \tag{21}$$

for $-N \le i \le N$ and $p > 0$.

Theorem 3.1. *Let $u_i(0)$ be nonnegative and nontrivial.*
(i) If $p > 1$ then the corresponding solution to (21) blows up in finite time.
(ii) If $p \le 1$ then every solution to (21) is global.

Proof. (*i*) By simplicity, we will put u_{-N}^p instead of $u^p(-Nh,t)$, u_N^p instead of $u^p(Nh,t)$, G_{-N} instead of $G(h(i+N)$ and G_N instead of $G(h(i-N)$. From (21) we have

$$\begin{aligned} u_N'(t) &= \sum_{j=-N}^{j=N} hJ(h(N-j))(u_j(t)-u_N(t)) + G_{-N}\,u_{-N}^p + G_N\,u_N^p, \\ u_{-N}'(t) &= \sum_{j=-N}^{j=N} hJ(h(-N-j))(u_j(t)-u_{-N}(t)) + G_{-N}\,u_{-N}^p + G_N\,u_N^p. \end{aligned} \tag{22}$$

Therefore

$$\begin{aligned} u_N'(t) &\ge -C_1u_N + C_2(u_{-N}^p + u_N^p), \\ u_{-N}'(t) &\ge -\tilde{C}_1u_{-N} + C_2(u_{-N}^p + u_N^p). \end{aligned} \tag{23}$$

Let $p > 1$. We defined $m(t) = u_{-N} + u_N$ for $t \ge 0$. By (23) and applying Jensen inequality , we obtain

$$m'(t) \geq -Cm(t) + \tilde{C}_2 m(t)^p \geq C_3 m(t)^p, \tag{24}$$

for C_3 any positive constant. As $p > 1$ and $m(0) > 0$ we have that $m(t)$ cannot be global, therefore $m(t)$ blows up in finite time $T_h > 0$. As consequence, u_N or u_{-N} have to blow-up.

(ii) Let $p \leq 1$. We consider the ODE problem

$$\begin{cases} z'(t) = z(t) \\ z(0) = \max\{u_i(0), 1\} \end{cases} \tag{25}$$

We observe that as $p \leq 1$ and $z(t) > 1$ for all $t > 0$ then $z(t) \geq z^p(t)$. Therefore $z(t)$ is a super-solution of (21). Thus $u_i(t)$ is global by Comparison principle, see [9]. □

Let $u_k(t) = \max_{-N \leq i \leq N} u_i(t)$. By (21), we have that

$$\begin{aligned} u_k'(t) &= \sum_{j=-N}^{j=N} hJ(h(i-j))(u_j(t) - u_k(t)) + G(h(i+N))u^p(-Nh,t) + G(h(i-N))u^p(Nh,t), \\ &\leq C_1 u_k^p(t). \end{aligned} \tag{26}$$

Integrating this last equation from t to T_h, we have that

$$\int_{u_k(t)}^{u_k(T_h)} \frac{ds}{s^p} \leq (T_h - t),$$

and as $u_k(t) \to \infty$ where $t \to T_h$, we have that

$$u_k(t) \geq \frac{K}{(T-t)^{\frac{1}{p-1}}}. \tag{27}$$

Lemma 3.1. *Let*

$$\begin{cases} v_i(t) = \frac{A}{(T-t)^{\frac{1}{p-1}}}, \\ (v_0)_i \geq (u_0)_i \end{cases}$$

If $A = \frac{1}{(C(p-1))^{\frac{1}{p-1}}}$ *whit* $C = \{G_{-N} + G_N\}$ *then* v_i *is a super-solution of (21).*

Proof. In fact, a direct calculation show us that $A = \frac{1}{(C(p-1))^{\frac{1}{p-1}}}$ and therefore v satisfies (21). So that v_i is a super-solution of (21). □

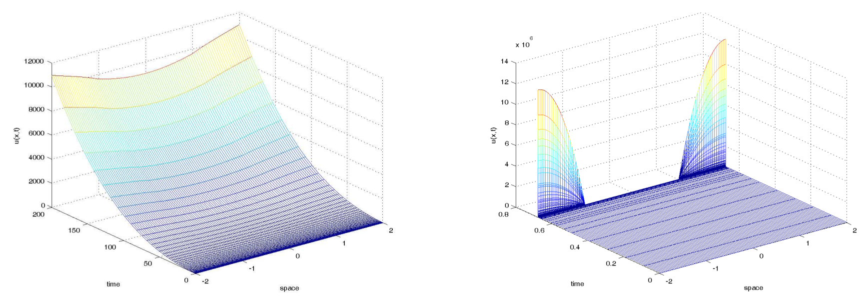
The following theorem establishes the blow up rates of the solution of (21). Its demonstration is an immediate consequence of (3.1) and Lemma 3.1.

Theorem 3.2. *Let* $p > 1$ *and* $u_i(t)$ *the solution of (21). Then exists positives constants* $K,\ A$ *such that*

$$\frac{K}{(T-t)^{\frac{1}{p-1}}} \leq \max_i u_i \leq \frac{A}{(T-t)^{\frac{1}{p-1}}}.$$

4 Discussion and comments

Using the MatLab Program, we illustrate the solutions of (21). We have take the following expressions and values: $L = 2$, $N = 100$, $u_0(x) = \max\{0, 1 - \frac{x}{2}\}$, $J(r) = G(r) = \max\{0, (\frac{3}{4})(1 - r^2)\}$. In figure 1, the case u_1, correspond to $p = \frac{2}{3}$, and we note the solutions is globally. The case u_2, correspond to $p = 3$. We note that the solution blows-up in finite time, which is in accordance with the mentioned results.



(a) $u_1(x,t) = u^{\frac{2}{3}}$ (b) $u_2(x,t) = u^3$

Figure 1: $u(x,t)$

5 Conclusion

Some important characteristics for a nonlocal diffusion model in the discrete case have been studied. We have found a necessary and sufficient condition for existence of stationary solutions. We have described the asymptotic behavior of the solutions when t goes to infinity. We found sufficient condition for the blow-up of the solutions and find the blows-up rates when the frontier data is given by the function u^p with $p > 1$.

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