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A Solution of the Transport Equation

F. Fonseca

Universidad Nacional de Colombia
Grupo de Ciencia de Materiales y Superficies
Departamento de Física
Bogotá, Colombia

Abstract

In this paper we solve the Transport Equation, known as the TASEP + LK model, using the tanh method, the Riccati solutions and Jacobi elliptic function.

Keywords: Transport equation, Tanh method, Riccati equation and Jacobi elliptic function

1 Introduction

The investigation of non-equilibrium physics, still is one of the open research frontiers in physical systems [1]. So much investigation has been done over the years, and one of the research branches is known as driven diffusive systems [2]. A very useful model to investigate this field, known as the totally asymmetric simple exclusion process (TASEP) [3], describing particle propagation along a one-dimensional lattice.

This work presents the solution of the TASEP + LK, a refined model [3], using solitary wave methods [4]. In section 2, is given the transport equation and its solution using tanh wave method [4]. In section 3, we apply the solution of Riccati equation [5], in order to get solitary wave solutions. In section 4, we find solutions using the Jacobi Elliptic functions [6]-[7]. Finally, in section 5 we present conclusions .

2 Transport equation

The TASEP + LK equation is:

$$\frac{\epsilon}{2}\frac{\partial\rho^2}{\partial x^2} + (2\rho-1)\frac{\partial\rho}{\partial x} + \Omega_A(1-\rho) - \Omega_D\rho = 0 \tag{1}$$

We use the tanh method [4], defining the independent variable:

$$\rho = \tanh(Y) \tag{2}$$

The derivatives of x in terms of Y, are:

$$\frac{d}{dx} = (1-Y^2)\frac{d}{dY}, \quad \frac{d^2}{dx^2} = -2Y(1-Y^2)\frac{d}{dY} + (1-Y^2)^2\frac{d^2}{dY^2} \tag{3}$$

The solutions are postulated [4] as:

$$\rho = \sum_{i=0}^{n} a_i Y^i \tag{4}$$

Replacing in eq. (1)

$$\begin{aligned} &\frac{\epsilon}{2}(-2Y(1-Y^2)\frac{d\rho}{dY} + (1-Y^2)^2\frac{d^2\rho}{dY^2}) \\ &+(2\rho-1)(1-Y^2)\frac{d\rho}{dY} + \Omega_A(1-\rho) - \Omega_D\rho = 0 \end{aligned} \tag{5}$$

Balancing the highest nonlinear term with the highest linear derivative, we get:

$$Y^4\frac{d^2\rho}{dY^2} \to \rho Y^2\frac{d\rho}{dY} \to m+2 = m+m+1 \to m = 1 \tag{6}$$

Then, we get in eq. (4)

$$\rho = a_0 + a_1 Y \to \rho^{'} = a_1 \to \rho^{''} = 0 \tag{7}$$

$$\begin{aligned} &\frac{\epsilon}{2}(-2Y(1-Y^2)a_1) + (2a_0 + 2a_1Y - 1)(1-Y^2)a_1 \\ &+\Omega_A(1-a_0-a_1Y) - \Omega_D a_0 - \Omega_D a_1 Y = 0 \end{aligned} \tag{8}$$

	A	C	F
1	1/2	-1/2	$coth(\xi) \pm cosh(\xi)$, $tanh(\xi) \pm isech(\xi)$
2	1/2	1/2	$sec(\xi) \pm itan(\xi)$
3	-1/2	- 1/2	$csc(\xi) \pm icot(\xi)$
4	1	- 1	$tanh(\xi)$, $coth(\xi)$
5	1	1	$tan(\xi)$
6	-1	-1	$cot(\xi)$

Table 1: Solutions of the Riccati equation [5].

As a result, we obtain a set of algebraic equations, order by order in Y^i (with $i = 0, 1, 2, 3$), in eq. (8). Solving them, we get 2 families of solutions

$$f_1 \rightarrow \left(a_{0_1} = \frac{1}{2}, a_{1_1} = \frac{\epsilon}{2}, \Omega_A = \frac{\epsilon}{2}, \Omega_D = \frac{\epsilon}{2} \right), \tag{9}$$
$$f_2 \rightarrow (a_{1_1} = 0, \Omega_A = -a_{0_1}\epsilon, \Omega_D = (-1 + a_{0_1})e)$$

3 Solution Riccati equation

Also, we use the solutions given by Riccati equation, in order to find solutions for $\rho(x)$, [5]. We suppose a solution:

$$\rho = \sum_{i=1}^{m} a_i F^i \tag{10}$$

where F solves, table (1), the Riccati equation, i.e.

$$F^{'} = CF^2 + A \rightarrow F^{''} = C2FF^{'} = 2CF(CF^2 + A) = 2C^2F^3 + 2ACF \tag{11}$$

Here A, C are constants, table (1). Replacing in eq. (1), taking $n = 1$ from the homogeneous balancing and doing the algebra, we get, $\rho = a_0 + a_1F$. So:

$$\rho^{'} = a_1F^{'} = a_1CF^2 + Aa_1 \rightarrow \rho^{''} = 2a_1CFF^{'} = 2a_1C^2F^3 + 2a_1CAF \tag{12}$$

Replacing in eq. (1)

$$\epsilon a_1C^2F^3 + \epsilon a_1CAF + 2a_0a_1CF^2 + 2a_1a_1CF^3 - a_1CF^2 + a_02Aa_1 \tag{13}$$
$$+a_12Aa_1F - Aa_1 + \Omega_A - \Omega_Aa_0 - \Omega_Aa_1F - \Omega_Da_0 - \Omega_Da_1F = 0$$

Also, we get a set of algebraic equations, order by order in F^i, and solving them, we obtain:

$$f_1 \to a_0 = \frac{1}{2}, a_1 = -\frac{C}{2}, \Omega_A = \frac{C}{2}(1 - 2A + A\epsilon), \Omega_D = \frac{C}{2}(A\epsilon - 1), \quad (14)$$
$$f_2 \to (a_1 = 0, \Omega_A = a_0 AC\epsilon, \Omega_D = AC\epsilon - Aa_0 C\epsilon)$$

Then, we get 12 solutions using Ricatti method.

4 Solution Jacobi elliptic functions

We suppose a solution given by the Jacobi elliptic functions, [6] and [7]. They hold the next relations

$$sn^2(\zeta, k) + cn^2(\zeta, k) = 1, \quad k^2 sn^2(\zeta, k) + dn^2(\zeta, k) = 1 \quad (15)$$

$$dn^2(\zeta, k) - k^2 cn^2(\zeta, k) = k'^2, \quad k'^2 sn^2(\zeta, k) + cn^2(\zeta, k) = dn^2(\zeta, k) \quad (16)$$

$$k' = \sqrt{1 - k^2} \quad (17)$$

Then, we suppose the next solution and its derivatives

$$\rho = Acn(a\zeta, k); \quad \frac{dv}{d\zeta} = -aAsn(a\zeta, k)dn(a\zeta, k); \quad (18)$$
$$\frac{d^2 v}{d\zeta^2} = Aa^2(-2k^2 cn^3(a\zeta, k) - (1 - 2k^2)cn(a\zeta, k))$$

Then, replacing eqs. (18) in eq. (1)

$$\frac{\epsilon}{2}(Aa^2(-2k^2 cn^3(a\zeta, k) - (1 - 2k^2)cn(a\zeta, k))) + Asn(a\zeta, k)dn(a\zeta, k) \quad (19)$$
$$-2Acn(a\zeta, k)Asn(a\zeta, k)dn(a\zeta, k) + \Omega_A(1 - Acn(a\zeta, k))$$
$$-\Omega_D Acn(a\zeta, k) = 0$$

Equating the left hand side of eq. (19) to zero, we get:

$$\begin{aligned}&Asn(a\zeta,k)dn(a\zeta,k)+Aa^2\epsilon k^2cn(a\zeta,k)sn^2(\zeta,k)\\&-k^2Aa^2\epsilon cn(a\zeta,k)-2Acn(a\zeta,k)Asn(a\zeta,k)dn(a\zeta,k)\\&+\Omega_A-A\Omega_Acn(a\zeta,k)-\Omega_DAcn(a\zeta,k)-\frac{\epsilon}{2}(1-2k^2)cn(a\zeta,k)=0\end{aligned}\tag{20}$$

$$\begin{aligned}&Asn(a\zeta,k)dn(a\zeta,k)+\Omega_A=0\rightarrow sn(a\zeta,k)dn(a\zeta,k)=-\frac{\Omega_A}{A}\\&\rightarrow sn^2(a\zeta,k)=-\frac{2\Omega_A}{Aa^2\epsilon k^2}\rightarrow dn^2(a\zeta,k)=-\frac{\Omega_Aa^2\epsilon k^2}{2A}\end{aligned}\tag{21}$$

$$\begin{aligned}&(Aa^2\epsilon k^2sn(\zeta,k)-2Adn(a\zeta,k))cn(a\zeta,k)Asn(a\zeta,k)=0\\&\rightarrow dn(a\zeta,k)=\frac{a^2\epsilon k^2sn(\zeta,k)}{2}\\&\rightarrow sn(\zeta,k)=\frac{2dn(a\zeta,k)}{a^2\epsilon k^2}\end{aligned}\tag{22}$$

Using eq. (15)

$$k^2\frac{2\Omega_A}{Aa^2\epsilon k^2}+\frac{\Omega_Aa^2\epsilon k^2}{2A}=-1\rightarrow A=-\frac{2\Omega_A}{a^2\epsilon}-\frac{\Omega_Aa^2\epsilon k^2}{2}\tag{23}$$

And

$$\begin{aligned}&(-k^2Aa^2\epsilon-A\Omega_A-\Omega_DA-\frac{\epsilon}{2}(1-2k^2))cn(a\zeta,k)=0\\&k^2Aa^2\epsilon+A\Omega_A+\Omega_DA+\frac{\epsilon}{2}(1-2k^2)=0\end{aligned}\tag{24}$$

Solving equations (23) and (24), and defining $l_1=2a^6-12a^6k^2+24a^6k^4-16a^6k^6-6a^8k^2\Omega_A^2+24a^8k^4\Omega_A^2-24a^8k^6\Omega_A^2-36a^{10}k^6\Omega_A^2+72a^10k^8\Omega_A^2+6a^{10}k^4\Omega_A^4-12a^{10}k^6\Omega_A^4-72a^12k^8\Omega_A^4-2a^{12}k^6\Omega_A^6-6a^8k^2\Omega_A\Omega_D+24a^8k^4\Omega_A\Omega_D-24a^8k^6\Omega_A\Omega_D+12a^10k^4\Omega_A^3\Omega_D-24a^10k^6\Omega_A^3\Omega_D-72a^{12}k^8\Omega_A^3\Omega_D-6a^{12}k^6\Omega_A^5\Omega_D+6a^10k^4\Omega_A^2\Omega_D^2-12a^{10}k^6\Omega_A^2\Omega_D^2-6a^12k^6\Omega_A^4\Omega_D^2-2a^12k^6\Omega_A^3\Omega_D^3$, $l_2=12a^8k^6\Omega_A^2-(-a^2+2a^2k^2+a^4k^2\Omega_A^2+a^4k^2\Omega_A\Omega_D)^2$ and $l_3=-a^2+2a^2k^2+a^4k^2\Omega_A^2+a^4k^2\Omega_A\Omega_D$

$$A_1=\frac{1}{-2\Omega_A-2\Omega_D}(-2k^2(-\frac{2^{1/3}l_2}{3a^6k^4\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}\Omega_A}\tag{25}$$

$$+\frac{\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}}{3\ 2^{1/3}a^6k^4\Omega_A}-\frac{l_3}{3a^6k^4\Omega_A})-\frac{2^{1/3}l_2}{3a^6k^4\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}\Omega_A}$$

$$+\frac{\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}}{3\ 2^{1/3}a^6k^4\Omega_A}-\frac{l_3}{3a^6k^4\Omega_A}-4k^2\Omega_A$$

$$-a^4k^4(-\frac{2^{1/3}l_2}{3a^6k^4\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}\Omega_A}+\frac{\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}}{3(2)^{1/3}a^6k^4\Omega_A}$$

$$-\frac{l_3}{3a^6k^4\Omega_A})^2\Omega_A)$$

$$\epsilon_1=-\frac{2^{1/3}l_2}{3a^6k^4\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}\Omega_A}+\frac{\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}}{3\ 2^{1/3}a^6k^4\Omega_A}-\frac{l_3}{3a^6k^4\Omega_A} \quad (26)$$

$$A_2=\frac{1}{-2\Omega_A-2\Omega_D}(-2k^2(\frac{\left(1+i\sqrt{3}\right)l_2}{3\ 2^{2/3}a^6k^4\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}\Omega_A} \quad (27)$$

$$-\frac{\left(1-i\sqrt{3}\right)\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}}{6(2)^{1/3}a^6k^4\Omega_A}-\frac{l_3}{3a^6k^4\Omega_A})$$

$$+\frac{\left(1+i\sqrt{3}\right)l_2}{3\ 2^{2/3}a^6k^4\left(l_1+\sqrt{\ l_1^2+4l_2^3}\right)^{1/3}\Omega_A}-\frac{\left(1-i\sqrt{3}\right)\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}}{6\ \ 2^{1/3}a^6k^4\Omega_A}$$

$$-\frac{l_3}{3a^6k^4\Omega_A}-4k^2\Omega_A-a^4k^4(\frac{\left(1+i\sqrt{3}\right)l_2}{3(2)^{2/3}a^6k^4\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}\Omega_A}$$

$$-\frac{\left(1-i\sqrt{3}\right)\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}}{6\ 2^{1/3}a^6k^4\Omega_A}$$

$$-\frac{l_3}{3a^6k^4\Omega_A})^2\Omega_A),$$

$$\epsilon_2 = \frac{\left(1+i\sqrt{3}\right) l_2}{3\ 2^{2/3} a^6 k^4 \left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3} \Omega_A} - \frac{\left(1-i\sqrt{3}\right)\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}}{6\ 2^{1/3} a^6 k^4 \Omega_A} - \frac{l_3}{3a^6k^4\Omega_A} \quad (28)$$

$$A_3 = \frac{1}{-2\Omega_A - 2\Omega_D}(-2k^2(\frac{\left(1-i\sqrt{3}\right) l_2}{3\ 2^{2/3} a^6 k^4 \left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3} \Omega_A} - \frac{\left(1+i\sqrt{3}\right)\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}}{6\ 2^{1/3} a^6 k^4 \Omega_A} - \frac{l_3}{3a^6k^4\Omega_A}) + \frac{\left(1-i\sqrt{3}\right) l_2}{3\ 2^{2/3} a^6 k^4 \left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3} \Omega_A} - \frac{\left(1+i\sqrt{3}\right)\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}}{6(2)^{1/3} a^6 k^4 \Omega_A} - \frac{l_3}{3a^6k^4\Omega_A} - 4k^2\Omega_A - a^4k^4(\frac{\left(1-i\sqrt{3}\right) l_2}{3(2)^{2/3} a^6 k^4 \left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3} \Omega_A} - \frac{\left(1+i\sqrt{3}\right)\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}}{6(2)^{1/3} a^6 k^4 \Omega_A} - \frac{l_3}{3a^6k^4\Omega_A})^2\Omega_A) \quad (29)$$

$$\epsilon_3 = \frac{\left(1-i\sqrt{3}\right) l_2}{3\ 2^{2/3} a^6 k^4 \left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3} \Omega_A} - \frac{\left(1+i\sqrt{3}\right)\left(l_1+\sqrt{l_1^2+4l_2^3}\right)^{1/3}}{6\ 2^{1/3} a^6 k^4 \Omega_A} - \frac{l_3}{3a^6k^4\Omega_A} \quad (30)$$

So, the solutions, with $i = 1, 2, 3$, are:

$$\rho_i = A_i cn(a\zeta, k) \quad (31)$$

5 Conclusions

We solved the TASEP + LK equation using the tanh method, Jacobi elliptic functions, and Ricatti solitary wave solutions. We obtain several families of solutions. The solutions are:

$$\rho(x)_i = a_{0_i} + a_{1_i}\tanh(x), \;\; \rho(x)_i = a_{0_i} + a_{1_i}F(x), \;\; \rho(x)_i = A_i cn(a\zeta, k) \quad (32)$$

As a future work, we can extend the method to investigate two-dimensional configurations and extended versions of the model.

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