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Some Necessary Conditions for Schur Stability of Polynomials

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Abstract

This paper concerns the Schur stability of polynomials. Firstly necessary conditions are presented for the Schur stability of complex polynomials with fixed coefficients based on the discrete Hermite-Biehler theorem. Then the result is applied to obtain necessary conditions for the robust Schur stability of real interval polynomials.

Keywords: robust Schur stability, Hermite-Biehler theorem, interval polynomial

1 Introduction

Consider a linear discrete system with the characteristic polynomial

$$p(z) = \sum_{i=0}^{n} a_i z^i \quad (a_n > 0). \tag{1}$$

As is well known, the system is stable if $p(z)$ is Schur stable, i.e., $p(z)$ has all its zeros in open unit disc of the complex plane. If some uncertainty exists in the system, then the characteristic polynomial can be expressed by the interval polynomial

$$P(z) = \sum_{i=0}^{n} [a_i^-, a_i^+] z^i \quad (a_n^- > 0). \tag{2}$$

In this case the system is stable if $P(z)$ is Schur stable, i.e., every polynomial with fixed coefficients in $P(z)$ is Schur stable.

When designing a digital control system or filter, a fundamental step is to check the stability of the system, which in turn requires to determine if its characteristic polynomial is Schur stable or not. To this end, one can use the classical Jury test [1] which provides a necessary and sufficient condition for the Schur stability of a real polynomial in a tabular form. For a real interval polynomial, the edge theorem [2] can be used after verifying the Schur stability of all vertex polynomials. Although the Jury test serves as a powerful tool for checking the stability of a real polynomial or interval polynomial, it is required that all the entries of the stability table be calculated in order to finally conclude that a real polynomial is Schur stable. Consequently the computational burden rapidly increases in proportional to the degree of the polynomial. In such cases time and energy can be considerably saved if some quickly checkable sufficient or necessary conditions are used before applying the Jury test.

Two well-known sufficient conditions to guarantee the Schur stability of a polynomial are the monotonic condition and the dominant condition [1]. These two sufficient conditions respectively state that $p(z)$ in (1) is Schur stable if

$$a_n > a_{n-1} > a_{n-2} > \cdots > a_1 > a_0 > 0, \tag{3}$$

or

$$|a_n| > \sum_{i=0}^{n-1} |a_i|. \tag{4}$$

On the other hand, a simple necessary condition for the Schur stability of $p(z)$ in (1) is [3]

$$\Big|\frac{a_i}{a_n}\Big| < C_n^i,\ i = 0, 1, \cdots, n-1, \tag{5}$$

where $C_n^i = \frac{n!}{i!(n-i)!}$.

For the Schur stability of an interval interval polynomial, several necessary conditions are available in the literature [4–7]. According to [4], if $P(z)$ in (2) is Schur stable, then

$$\max\{|a_{n-1}^-|, |a_{n-1}^+| \le 2a_n^-\Big(\log^+ \frac{a_n^-}{a_n^+ - a_n^-} + 4.4\Big). \tag{6}$$

It was also shown in [6] that if $P(z)$ in (2) is Schur stable, then

$$a_i^+ - a_i^- \le 4a_n^-,\ \ i = 0, 1, 2, \cdots, n-1. \tag{7}$$

On the other hand, a necessary condition for the Schur stability of $P(z)$ with $a_n^- = a_n^{+1} = 1$ in (2) was obtained in [7] as follows: let

$$c_i = \frac{a_i^+ + a_i^-}{2},\ \ r_i = \frac{a_i^+ - a_i^-}{2},\ \ i = 0, 1, 2, \cdots, n-1.$$

If $P(z)$ is Schur stable, then
(i) $r_i < 2$, $i = 0, 1, \cdots, n-1$ and

$$r_i \leq 2 - \frac{r_{i+1}^2}{2} - |c_{n-1}| r_{i+1}, \quad i = 0, 1, \cdots, n-2, \tag{8}$$

$$r_i \leq 2 - \frac{r_{i+2}^2}{2} - \frac{2(r_{i+1} + |c_{n-1}| r_{i+2})^2}{4 - r_{i+2}^2}, \quad i = 0, 1, \cdots, n-3, \tag{9}$$

or
(ii) there exists i_0 such that $r_{i_0} = 2$. In this case

$$P(z) = z^{2i_0 - n}(z^{2(n-i_0)} + a_{i_0} z^{n-i_0} + 1), \quad a_{i_0} \in [-2, 2].$$

In this paper we present new necessary conditions for the Schur stability of polynomials. Firstly necessary conditions are derived for the Schur stability of complex polynomials based on the discrete Hermite-Biehler theorem [8]. Then the result is applied to obtain necessary conditions for the Schur stability of real interval polynomials.

A complex polynomial $q(z) = \sum_{i=0}^{n} b_i z^i$ is called self-inversive if

$$q(z) = e^{i\theta} z^n \bar{q}(\bar{z}^{-1}),$$

for some real θ, where 'bar' denotes the complex conjugate. It is easily seen that all the zeros of a self-inversive polynomial $q(z)$ lie on the unit circle or symmetric to the unit circle in the sense that if z_k is a zero of $q(z)$, then so is $\bar{z}_k^{-1}$. The following result will be used in the next section.

Lemma 1.1 [**9**] *All the zeros of a self-inversive polynomial are simple and lie on the unit circle if and only if its derivative is Schur stable.*

2 Main results

Firstly we consider a complex polynomial.

Theorem 2.1 *If a complex polynomial $p(z)$ in (1) is Schur stable, then*

$$\left| \frac{a_{n-i} + \bar{a}_i}{a_n + \bar{a}_0} \right| < C_i^n, \quad 1 \leq i \leq \left[\frac{n}{2}\right], \tag{10}$$

and

$$\left| \frac{a_{n-i} - \bar{a}_i}{a_n - \bar{a}_0} \right| < C_i^n, \quad 1 \leq i \leq \left[\frac{n}{2}\right]. \tag{11}$$

Proof. Let

$$\begin{aligned} p_1(z) &= p(z) + z^n \bar{p}(\bar{z}^{-1}) \\ &= \sum_{i=0}^{n} (a_{n-i} + \bar{a}_i) z^{n-i}, \end{aligned}$$

and

$$\begin{aligned} p_2(z) &= p(z) - z^n \bar{p}(\bar{z}^{-1}) \\ &= \sum_{i=0}^{n} (a_{n-i} - \bar{a}_i) z^{n-i}. \end{aligned}$$

The derivatives of $p_1(z)$ and $p_2(z)$ are respectively given by

$$p_1^{'}(z) = \sum_{i=0}^{n-1} (n-i)(a_{n-i} + \bar{a}_i) z^{n-i-1},$$

and

$$p_2^{'}(z) = \sum_{i=0}^{n-1} (n-i)(a_{n-i} - \bar{a}_i) z^{n-i-1}.$$

Note that $p_1(z)$ and $p_2(z)$ are self-inversive. Furthermore, since $p(z)$ is Schur stable by assumption, all the zeros of $p_1(z)$ and $p_2(z)$ are simple and lie on the unit circle $|z| = 1$ by the Hermite-Biehler theorem [8]. Hence $p_1^{'}(z)$ and $p_2^{'}(z)$ are Schur stable by Lemma 1.1.

For $p_1^{'}(z)$, let

$$\begin{aligned} p_1^{'}(z) &= n(a_n + \bar{a}_0) \sum_{i=0}^{n-1} \frac{(n-i)(a_{n-i} + \bar{a}_i)}{n(a_n + \bar{a}_0)} z^{n-1-i} \\ &= n(a_n + \bar{a}_0) \prod_{i=1}^{n-1} (z + z_i) \\ &= n(a_n + \bar{a}_0) \sum_{i=0}^{n-1} e_i(z_1, z_2, \cdots, z_{n-1}) z^{n-1-i}, \end{aligned}$$

where $e_i(z_1, z_2, \cdots, z_{n-1})$, $0 \le i \le n-1$, denote the elementary symmetric functions define by [3]

$$e_i(z_1, z_2, \cdots, z_{n-1}) = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le n-1} z_{j_1} z_{j_2} z_{j_3} \cdots z_{j_i}, \quad 1 \le i \le n-1,$$

with $e_0(z_1, z_2, \cdots, z_{n-1}) = 1$. Since $|z_i| < 1$ for $1 \le i \le n-1$, then

$$|e_i(z_1, z_2, \cdots, z_{n-1})| < C_i^{n-1}, \quad 1 \le i \le n-1,$$

and we have

$$\left|\frac{(n-i)(a_{n-i}+\bar{a}_i)}{n(a_n+\bar{a}_0)}\right| < C_i^{n-1}, \ \ 1 \le i \le n-1,$$

or equivalently

$$\left|\frac{a_{n-i}+\bar{a}_i}{a_n+\bar{a}_0}\right| < \frac{n}{n-i}C_i^{n-1} = C_i^n, \ \ 1 \le i \le n-1.$$

It is easily seen that, for $1 \le i \le [n/2]$, $C_i^n = C_{n-i}^n$ and

$$\left|\frac{a_{n-i}+\bar{a}_i}{a_n+\bar{a}_0}\right| = \left|\frac{a_i+\bar{a}_{n-i}}{a_n+\bar{a}_0}\right|.$$

Then (10) is proved.

Similarly, for $p_2'(z)$, we obtain

$$\left|\frac{(n-i)(a_{n-i}-\bar{a}_i)}{n(a_n-\bar{a}_0)}\right| < C_i^{n-1}, \ \ 1 \le i \le n-1,$$

which is equivalent to (11) and the proof is completed.

Now we consider the real interval polynomial given in (2). To deal with the interval polynomial, we use the following interval arithmetic [10]:

$$\begin{aligned}
[a,b]+[c,d] &= [a+c, b+d], \\
[a,b]-[c,d] &= [a-d, b-c], \\
\frac{[a,b]}{[c,d]} &= [\min\{a/c, a/d, b/c, b/d\}, \max\{a/c, a/d, b/c, b/d\}], \ 0 \notin [c,d].
\end{aligned}$$

Theorem 2.2 *If the interval polynomial $P(z)$ given in (2) is Schur stable, then*

$$\max\{|\alpha_i|, |\beta_i|\} < C_i^n, \ \ 1 \le i \le \left[\frac{n}{2}\right], \tag{12}$$

and

$$\max\{|\gamma_i|, |\delta_i|\} < C_i^n, \ \ 1 \le i \le \left[\frac{n}{2}\right], \tag{13}$$

where

$$\alpha_i = \min\left\{\frac{a_{n-i}^- + a_i^-}{a_n^- + a_0^-}, \frac{a_{n-i}^+ + a_i^+}{a_n^- + a_0^-}, \frac{a_{n-i}^- + a_i^-}{a_n^+ + a_0^+}, \frac{a_{n-i}^+ + a_i^+}{a_n^+ + a_0^+}\right\}, \tag{14}$$

$$\beta_i = \max\left\{\frac{a_{n-i}^- + a_i^-}{a_n^- + a_0^-}, \frac{a_{n-i}^+ + a_i^+}{a_n^- + a_0^-}, \frac{a_{n-i}^- + a_i^-}{a_n^+ + a_0^+}, \frac{a_{n-i}^+ + a_i^+}{a_n^+ + a_0^+}\right\}, \tag{15}$$

$$\gamma_i = \min\left\{\frac{a_{n-i}^- - a_i^+}{a_n^- - a_0^+}, \frac{a_{n-i}^+ - a_i^-}{a_n^- - a_0^+}, \frac{a_{n-i}^- - a_i^+}{a_n^+ - a_0^-}, \frac{a_{n-i}^+ - a_i^-}{a_n^+ - a_0^-}\right\}, \tag{16}$$

$$\delta_i = \max\left\{\frac{a_{n-i}^- - a_i^+}{a_n^- - a_0^+}, \frac{a_{n-i}^+ - a_i^-}{a_n^- - a_0^+}, \frac{a_{n-i}^- - a_i^+}{a_n^+ - a_0^-}, \frac{a_{n-i}^+ - a_i^-}{a_n^+ - a_0^-}\right\}. \tag{17}$$

Proof. Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be an arbitrary polynomial with fixed coefficients in $P(z)$. Since $P(z)$ is Schur stable by assumption, so is $p(z)$. Then, by Theorem 2.1, we have

$$\left|\frac{a_{n-i} + a_i}{a_n + a_0}\right| < C_i^n, \quad 1 \leq i \leq \left[\frac{n}{2}\right],$$

and

$$\left|\frac{a_{n-i} - a_i}{a_n - a_0}\right| < C_i^n, \quad 1 \leq i \leq \left[\frac{n}{2}\right].$$

Since $p(z)$ is an arbitrary polynomial in $P(z)$, we have

$$\left|\frac{[a_{n-i}^- + a_i^-, a_{n-i}^+ + a_i^+]}{[a_n^- + a_0^-, a_n^+ + a_0^+]}\right| < C_i^n, \quad 1 \leq i \leq \left[\frac{n}{2}\right],$$

and

$$\left|\frac{[a_{n-i}^- - a_i^+, a_{n-i}^+ - a_i^-]}{[a_n^- - a_0^+, a_n^+ - a_0^-]}\right| < C_i^n, \quad 1 \leq i \leq \left[\frac{n}{2}\right].$$

We note that

$$0 \notin [a_n^- + a_0^-, a_n^+ + a_0^+], \quad 0 \notin [a_n^- - a_0^+, a_n^+ - a_0^-],$$

since $P(z)$ is Schur stable by assumption. Then

$$\frac{[a_{n-i}^- + a_i^-, a_{n-i}^+ + a_i^+]}{[a_n^- + a_0^-, a_n^+ + a_0^+]} = [\alpha_i, \beta_i],$$

where α_i and β_i are as given in (14) and (15), respectively, and (12) is proved.

Similarly we have

$$\frac{[a_{n-i}^- - a_i^+, a_{n-i}^+ - a_i^-]}{[a_n^- - a_0^+, a_n^+ - a_0^-]} = [\gamma_i, \delta_i],$$

where γ_i and δ_i are as given in (16) and (17), respectively, and (13) is proved.

3 Examples

Example 3.1 Consider the polynomial

$$p(z) = z^5 + 4.9z^4 + 1.2z + 0.2.$$

(5) does not give any information on the Schur stability of $p(z)$. However, (10) is violated for $i = 1$, and then we can conclude that $p(z)$ is unstable. In fact

$p(z)$ has a zero at $z = -4.8901$.

Example 3.2 Consider the interval polynomial

$$P(z) = z^4 + [2, 2.7]z^3 + [2.3, 3.1]z^2 + [0.4, 1.2]z + [0, 0.2].$$

In this case (6), (7), (8), (9) are all satisfied. On the other hand, (11) is violated for $i = 2$ since

$$\beta_2 = \max\left\{\frac{4.6}{1}, \frac{6.2}{1}, \frac{4.6}{1.2}, \frac{6.2}{1.2}\right\} = 6.2 > C_2^4,$$

and so $P(z)$ is unstable. For example,

$$p(z) = z^4 + 2.6z^3 + 3z^2 + 1.2z + 0.2,$$

has zeros at $z = -1.0252 \pm j0.8417$.

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