Contemporary Engineering Sciences, Vol. 10, 2017, no. 6, 291 - 304
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ces.2017.6562

A New Spectral Method Applied to Immobilized Biocatalyst Model Arising in Biochemical Engineering

S. Padma[1], G. Hariharan[2] and Sonia Castellucci[3]

[1,2] Department of Mathematics, School of Humanities & Sciences
SASTRA University, Thanjavur-613 401, Tamilnadu, India

[3]CIRDER, University of Tuscia, Via San Camillo de Lellis, 01100 Viterbo, Italy

Abstract

The mathematical model of immobilized liver esterase and kinetics by flow calorimetry is discussed. In this paper, we have developed an efficient Legendre polynomial based spectral method to the model. To the best our knowledge until there is no rigorous Legendre spectral method has been reported for the above model. Illustrative examples are given to demonstrate the validity and applicability of the proposed method. The numerical results have been compared with modified Adomian decomposition method (MADM) and homotopy analysis method (HAM). Moreover the use of Legendre computational matrix method is found to be simple, accurate, flexible and an efficient tool.

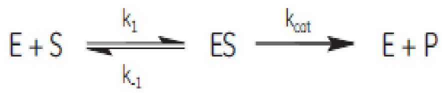
Keywords: Reaction- diffusion equations; Mathematical model; homotopy analysis method; Legendre spectral method; operational matrices; modified Adomian decomposition method

1. Introduction

It has become a powerful tool for having applications in almost all the areas of engineering and sciences such as image coding, edge extraction, weather forecasting, statistical estimation and numerical solutions of ordinary and partial differential equations [1- 2]. Orthogonal functions play an important role in the solution of nonlinear reaction- diffusion equations. The main aim in the current work is to establish the shifted Legendre polynomials and the operational matrix together with collocation method to discretize the nonlinear reaction- diffusion equation into a system of algebraic equations [2]. Hariharan and his coworkers [4 - 6] introduced the wavelet methods for solving differential equations arising in engineering. Recently, Hariharan and Kannan [7] reviewed the wavelet method for the solutions of reaction-diffusion problem. Boigues Muñoz et al. [31] established more accurate macro-models of solid oxide fuel cells through electrochemical and microstructural parameter estimation. Carlini et al. [32] presented mesophilic fermentation of SOMW in a micro pilot-scale anaerobic digester. Beginning from 1991, wavelet technique has been applied to solve differential equations. Wavelets, as very well-localized functions, are considerably useful for solving differential equations and provide accurate solutions. Hariharan and his coworkers [18-20] have introduced the wavelet based approach for solving the differential equations in science and engineering. The wavelet based approximation methods had been compared with Adomian decomposition method and other classical Recently, Rajaraman and Hariharan [25, 26] had introduced the efficient wavelet based spectral methods to singular boundary value problems and nonlinear reaction-diffusion problems. Hariharan and his coworkers [27-30] had developed the wavelet based approximation methods to differential equations.

2. Mathematical formulation of the problem [11 – 13]

The interaction of enzyme with substrate forms an enzyme-substrate complex (ES) and then converted to transition state (ES^*). It is then getting transformed to enzyme-product (EP) which finally splits into enzyme and product. [14]



(1)

where E is the enzyme, S the substrate k_1 is the rate constant for the formation of ES; also k_1 is the rate constant for the disassociation of ES back into $E+S$ and k_{cat} is the catalytic rate constant [13]. Assume that glucoamylase was immobilized in porous particles. The temperature difference between the column input and output, ΔT, is measured by thermistors and registered by a computer. The temperature of the system is 303.15 Kelvin and the buffer solution was continuously pumped through the column with constant flow rate of 1ml/min. Initially the buffer solution is replaced by the substrate solution containing 1 – 200 g/l of MDX in 0.1 M acetate buffer (pH 4.7). Two methods were adopted (i) single flow mode (ii) total recirculation mode. The single flow mode was performed with the switching valve 2 opened to the waste [12]. The equation of substrate balance in the particle is calculated from the substrate concentration gradient [12]. The biocatalyst particles are spherical shaped, the material balance reaction diffusion equation is given by:

$$D_e \left[\frac{d^2 S_{sc}}{dr^2} + \frac{2}{r} \frac{dS_{sc}}{dr} \right] - \nu_r = 0 \tag{2}$$

$$\begin{aligned} &S_{sc} = 0 \quad at \quad t = 0, \quad 0 \le r \le 1, \\ &\frac{dS_{sc}}{dr} = 0 \quad at \quad t = 0, \\ &S_{sc} = S_c at \quad r = R_p, \end{aligned} \tag{3}$$

where R_p is the particle radius and $\nu_r = \dfrac{V_m S_{sc}}{K_n + S_{sc} + \left(\dfrac{S_{sc}^2}{K_i} \right)}$ (4)

The parameters are made dimensionless by the following

$$x = \frac{r}{R_p}; \quad U = \frac{S_{sc}}{S_c}; \quad \delta_k = \frac{R_p^2 \nu_m}{\chi_e k_m}; \quad \varsigma = \frac{S_c}{k_m}; \quad \tau = \frac{S_c^2}{k_m k_i}$$

where δ_k is the reaction diffusion parameter, x is the dimensionless distance and $U(x)$ is the dimensionless concentration. Here ς and τ are the saturation parameters.

Eq. (2) becomes

$$\frac{d^2 U}{dx^2} + \frac{2}{x} \frac{dU}{dx} - \frac{\delta_k U}{1 + \varsigma U + \tau U^2} = 0 \tag{5}$$

with the boundary conditions

$$\begin{aligned} &U = 1 \quad at \quad x = 1 \\ &\frac{dU}{dx} = 0 \quad at \quad x = 0 \end{aligned} \tag{6}$$

3. Method of solution using LCMM [3]

The well known Legendre polynomials $P_n(z)$, defined on the interval [-1, 1], have the following properties:

$P_n(z) = (-1)^n P_n(z)$, $P_n(-1) = (-1)^n$, $P_n(1)=1$

The analytic form of the shifted Legendre polynomial $P_n^*(x)$ of degree n is given by;

$$P_n^*(x)=\sum_{k=0}^{n}(-1)^{n+k}\,\frac{(n+k)!}{(n-k)!(k!)^2 L^k}\,x^k \tag{7}$$

Let $\omega_L(x)=1,$ and the weighted space $L^2_{\omega_L}(0,L)$ is defined with the following inner product and norm.

$$(u,v)_{\omega_L}=\int_0^L u(x)\,v\,(x)\,\omega_L(x)\,dx,\quad \|u\|_{\omega_L}=(u,u)^{\frac{1}{2}}_{\omega_L}. \tag{8}$$

The orthogonality condition is of the form [10]

$$\int_0^1 P_i^*(x)\,P_j^*(x)\,dx=\begin{cases}\dfrac{1}{2i+1} & for\quad i=j\\ 0 & for\quad i\neq j\end{cases} \tag{9}$$

The set of the shifted Legendre polynomials forms a complete $L^2_{\omega_L}(0,L)$ orthogonal system and $\|P_n^*(x)\|^2_{\omega_L}=\frac{L}{2}h_n=\frac{L}{2n+1}$ is obtained. The function u(x) which is square integrable in [0,L], may be expressed in terms of shifted Legendre polynomials as;

$$u(x)=\sum_{i=0}^{\infty}c_i\;P_i^*(x),$$ where the coefficients c_i are given by;

$$c_i=\frac{1}{\|P_i^*(x)\|^2_{\omega_L}}\int_0^L u(x)\,P_i^*(x)\,\omega_L(x)\,dx,\qquad i=0,1,2,..... \tag{10}$$

It is suggested the solution $u(x)\in C^m[0,L]$ can be approximated in terms of the first (m+1) terms of shifted Legendre polynomials only as;

$$u(x)=\sum_{i=0}^{m}c_i\;P_i^*(x). \tag{11}$$

The method which was suggested is applied when m=2, a system of 3 linear algebraic equations follows, out of this 2 from the initial conditions and one from the main equation using the collocation point $x_0=0.5$ which is the root of $P_1^*(x)=0$ which can be written in the matrix form:

$$U(x)=P(x)\;A, \tag{12}$$

where $P(x)=\left[P_0^*(x)\quad P_1^*(x)\quad P_2^*(x)\right]=\left[1\quad 2x-1\quad 6x^2-6x+1\right],$ (13)

$$A=\left[C_0\quad C_1\quad C_2\right]^T \tag{14}$$

Applying this procedure the given nonlinear differential equation takes the following form:

$$\left(P\underset{(2)}{M} + 4P\underset{(1)}{M} - \frac{\delta_k P}{1+\varsigma PA^T + \tau (LA)^2} \right) A = 0 \tag{15}$$

where $P = (1 \quad 0 \quad -0.5)$, $\underset{(2)}{M} = \begin{pmatrix} 0 & 0 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\underset{(1)}{M} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}$

The initial conditions are

$P(1)A = 1,\quad P(0)\underset{(1)}{M}A = 0$ where $P(0) = (1 \quad -1 \quad 1)$ and $P(1) = (1 \quad 1 \quad 1)$

Eqs. (5) and (6) become the following system of linear algebraic equations

$$\left(1+\varsigma(C_0 - 0.5C_2) + \tau(C_0 - 0.5C_2)^2\right)(12C_1 + 8C_2) - \delta_k (C_0 - 0.5C_2) = 0 \tag{16}$$

$$C_0 + C_1 + C_2 = 1 \tag{17}$$

$$C_1 - 3C_2 = 0 \tag{18}$$

For various values of δ_k, ς *and* τ the wavelet coefficients C_0, C_1 *and* C_2 are obtained.

Using MADM $U(x)$ the substrate concentration,

$$U(x) = 1 - \frac{\delta_k}{6(1+\varsigma+\tau)} + \frac{7\,\delta_k^2}{360(1+\varsigma+\tau)^3} + \left(\frac{\delta_k}{6\,(1+\varsigma+\tau)} - \frac{\delta_k^2\,(1-\tau)}{36(1+\varsigma+\tau)^3} \right)\left(\frac{x^2}{6} \right) + \left(\frac{\delta_k^2\,(1-\tau)}{6(1+\varsigma+\tau)^3} \right)\left(\frac{x^4}{20} \right) \tag{19}$$

The exact solution for the substrate concentration *U(x)* [12] is

$$U(x) = \frac{\sinh\left(\sqrt{\delta_k}\;x\right)}{x\sinh\left(\sqrt{\delta_k}\right)} \tag{20}$$

4. Comparison of dimensionless substrate concentration U(x) with numerical solution for various values of δ_k, ς *and* τ:

Case (i): Consider δ_k=1 and $\varsigma = 0$ *and* $\tau = 0$, in Eq. (5). Using the aforesaid spectral method, one can obtain in the above case, when m=3, a system of 4 linear equations is obtained, two of them from the initial conditions and the other two from the main equation using 2 collocation points x_0 = 0.211325 and x_1=0.788675 which are the roots of $P_2^*(x)=0$.

Then

$$P(x)=\left[P_0^*(x)\quad P_1^*(x)\quad P_2^*(x)\quad P_3^*(x)\right]=\left[1\quad 2x-1\quad 6x^2-6x+1\quad 20x^3-30x^2+12x-1\right] \qquad (21)$$

$$A=\begin{bmatrix}C_0 & C_1 & C_2 & C_3\end{bmatrix}^T \qquad (22)$$

Applying this procedure to Eq.(15) when δ_k =1 and ς = 0 and τ =0

$$\left(P M_{(2)}+4P M_{(1)}-\delta_k P\right)A=0 \qquad (23)$$

where $P_{x_0}=\left[1\quad -0.57735\quad 0\quad 0.38490\right]$ and $P_{x_1}=\left[1\quad 0.57735\quad 0\quad -0.38490\right]$,

$$M_{(1)}=\begin{bmatrix}0 & 2 & 0 & 6\\ 0 & 0 & 6 & 0\\ 0 & 0 & 0 & 10\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad M_{(2)}=\begin{bmatrix}0 & 0 & 12 & 0\\ 0 & 0 & 0 & 60\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$P(0)=\left[1\quad -1\quad 1\quad -1\right]$$

The initial conditions are

$P(1)A=1, \qquad P(0)M_{(1)}A=0$ where $P(0)=(1\quad -1\quad 1\quad -1)$ and

$P(1)=(1\quad 1\quad 1\quad 1)$

Eq. (5) becomes the following system of linear algebraic equations for the two collocation points x_0 and x_1

$$\left[1+\varsigma\left[C_0-0.57735C_1+0.3849C_3\right]+\tau\left[C_0-0.57735C_1+0.3849C_3\right]^2\right]\left[12C_1-34.641C_2\right]-\delta_k\left[C_0-0.57735C_1+0.3849C_3\right]=0 \qquad (24)$$

$$\left[1+\varsigma\left[C_0+0.57735C_1-0.3849C_3\right]+\tau\left[C_0+0.57735C_1-0.3849C_3\right]^2\right]\left[12C_1+34.641C_2\right]-\delta_k\left[C_0+0.57735C_1-0.3849C_3\right]=0 \qquad (25)$$

If δ_k =1 and ς = 0 and τ =0, Eq. (25) and (26) become

$$\left[12C_1 + 34.641C_2\right] - \delta_k\left[C_0 - 0.57735C_1 + 0.3849\,C_3\right] = 0 \quad (26)$$

$$\left[12C_1 - 34.641C_2\right] - \delta_k\left[C_0 + 0.57735C_1 - 0.3849\,C_3\right] = 0 \quad (27)$$

Using the initial conditions,

$$C_1 - 3C_2 + 8C_3 = 0 \quad (28)$$

$$C_0 + C_1 + C_2 + C_3 = 1 \quad (29)$$

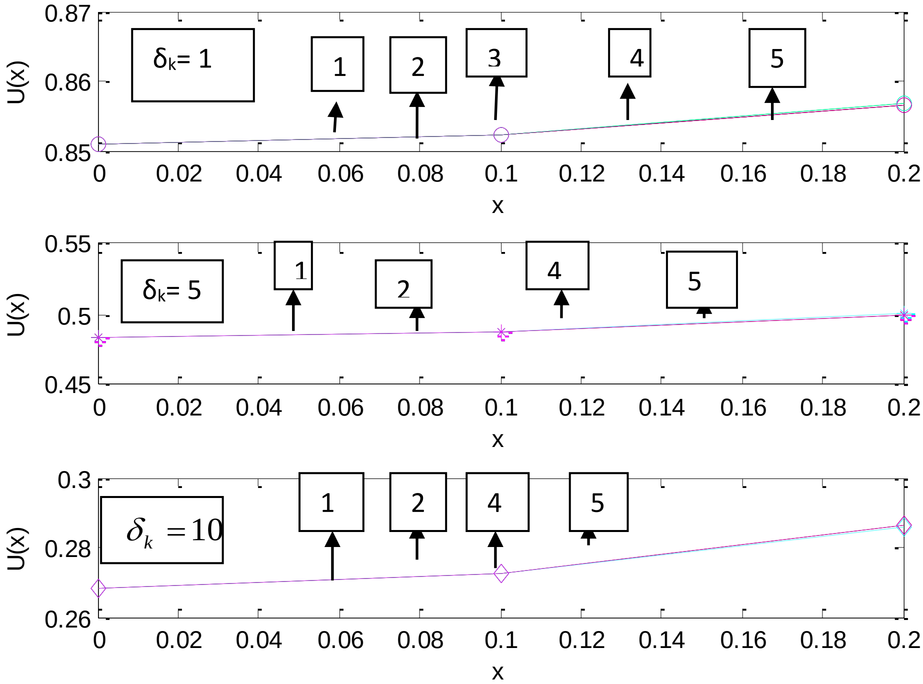
Consider δ_k =1, δ_k =5 and δ_k =10 in Eq. (5) together with the initial conditions gives

$U(x) = 0.0248\ x^3 + 0.142286\,x^2 - 0.000166x + 0.8509$,
$U(x) = 0.4062x^2 + 0.0028x + 0.4834$ and
$U(x) = 0.44576x^2 - 0.00002x + 0.26829$respectively by LCMM.

Table 1: Comparison between the MADM and LCMM for Eq.(5) and (6) when $\varsigma = 0\, and\ \tau = 0$

δ_k = 1						δ_k =5			
x	Simulation	MADM	LCMM (m = 2)	LCMM (m = 3)	Exact Solution	Simulation	MADM	LCMM	Exact Solution
0.0001	0.8509	0.8509	0.85089	0.85089	0.8509	0.4835	0.4835	0.4834	0.48346
0.1	0.8523	0.8523	0.85233	0.85233	0.8523	0.4875	0.4875	0.4877	0.487535
0.2	0.8566	0.8566	0.85675	0.85675	0.8566	0.4998	0.4998	0.5002	0.499774

	δ_k =10			
x	Simulation	MADM	LCMM	Exact Solution
0.0001	0.2681	0.2682	0.26829	0.26819
0.1	0.2727	0.2727	0.27274	0.272686
0.2	0.2864	0.2864	0.28612	0.286435



1. Simulation 2. MADM 3. HAM 4. LCMM 5. Exact Solution.

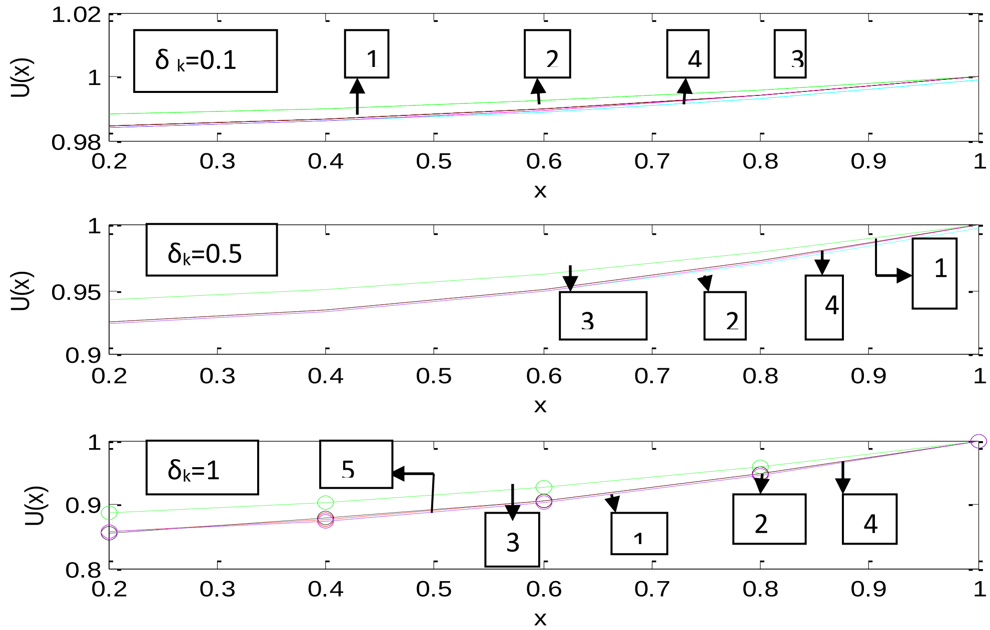
Fig 1: Comparison between the LCMM and MADM δ_k =1, δ_k =5 and δ_k =10 for $\varsigma = 0$ *and* $\tau = 0$

Case (ii): Comparison between the MADM and LCMM for Eq.(5) and (6) when $\varsigma = 0.1$ *and* $\tau = 0.001$. Consider δ_k =0.1, δ_k =0.5 and δ_k =1 in Eq. (5) together with the initial conditions, we obtain $U(x) = 0.01534x^2 + 0.98348$and *U(x)*= $0.07599x^2 + 0.9218$and *U(x)* $= 0.1468x^2 + 0.8509$by using LCMM.

δ_k =0.1						δ_k = 0.5				
x	Simulation	MADM	HAM	LCMM	Exact solution	Simulation	MADM	HAM	LCMM	Exact Solution
0.2	0.9844	0.9844	0.9884	0.9841	0.98418	0.9251	0.9254	0.9429	0.9248	0.9244
0.4	0.9864	0.9864	0.9899	0.98593	0.9861	0.9345	0.9347	0.9502	0.9339	0.9336
0.6	0.9897	0.9897	0.9924	0.9890	0.9894	0.9502	0.9504	0.9623	0.9491	0.9491
0.8	0.9943	0.9943	0.9958	0.9933	.99405	0.9725	0.9726	0.9793	0.9704	0.9712
1.0	1.0000	1.0000	1.0000	0.9988	1.0000	1.0000	1.0000	1.0000	0.9978	1.0000

Table 2: Comparison between the MADM, HAM and LCMM for Eq. (5) and (6)

δ_k =1					
x	Simulation	MADM	HAM	LCMM	Exact Solution
0.2	0.8580	0.8554	0.8880	0.8568	0.856602
0.4	0.8754	0.8787	0.9021	0.8743	0873792
0.6	0.9049	0.9057	0.9257	0.9037	0.902900
0.8	0.9471	0.9475	0.9591	0.9448	0.944632
1.0	1.0000	1.0000	1.0000	0.9977	1.000000



1. Simulation 2. MADM 3. HAM 4. LCMM 5. Exact Solution
Fig 2: Comparison between the LCMM, HAM and MADM for δ_k =0.1, δ_k =0.5 and δ_k =1 and for $\varsigma = 0.1$ *and* $\tau = 0.001$

5. Results and discussion

Our proposed numerical solutions have been compared with modified Adomian decomposition method and homotopy analysis method. The results of MADM and HAM are well satisfied with LCMM for various values of the reaction diffusion parameter δ_k. Figures (1- 3) show the comparison between the LCMM and MADM. The analytical expression of the concentration of substrate *U(x)* increases when the dimensionless reaction diffusion parameter δ_k decreases. It attains maximum when $\delta_k = 0.1,\ \varsigma = 0.1, \tau = 0.001$. It should be noted that the time-independent concentration of substrate *U(x)* is less when δ_k =10 and error is also more. For larger M, we get the results closer to the real values. Tables (1-6) show the comparison between the LCMM, HAM and MADM for various values of δ_k. The proposed computation method is very efficient and convenient for solving nonlinear differential equations.

6. Conclusion

Since the Legendre Computational matrix method approximates the nonlinear initial and boundary value problems, it is very easy to implement and extend it to various nonlinear reaction-diffusion problems. In this section, an efficient LCMM has been used to investigate the kinetics of immobilized liver esterase by flow calorimetry. Our results have been compared with MADM and HAM results. Good agreement with the exact solution is observed. The results arrived are useful for design and optimization of immobilized liver esterase by flow calorimetry.

Acknowledgements. This work was supported by the DRDO-NRB project (NRB-322/SC/13-14), Government of India. Our hearty thanks are due to Prof. R. Sethuraman, Vice-Chancellor, SASTRA University, Dr. S. Vaidhyasubramaniam, Dean / Planning and development and Dr. S. Swaminathan, Dean / Sponsored research for their kind encouragement and for providing good research environment.

References

[1] G. Hariharan, An Efficient Legendre Wavelet-Based Approximation Method for a Few Newell–Whitehead and Allen–Cahn Equations, *The Journal of Membrane Biology*, **247** (2014), no. 5, 371-380. https://doi.org/10.1007/s00232-014-9638-z

[2] Ali Davari, Abozar Ahmadi, New Implementation of Legendre Polynomials for solving Partial Differential Equations, *Applied Mathematics*, **4** (2013), 1647-1650. https://doi.org/10.4236/am.2013.412224

[3] Mohamed Meabed Khader and Ahmed Saied Hendy, A Legendre Computational Matrix Method for Solving High-Order Fractional Differential Equations, *Walailak Journal of Science and Technology*, **11** (2014), no. 4, 289-305.

[4] G. Hariharan, K. Kannan, An Overview of Haar Wavelet Method for Solving Differential and Integral Equations, *World Applied Sciences Journal*, **23** (2013), no. 12, 01-14.

[5] G. Hariharan, S. Padma, P. Pirabaharan, An Efficient Wavelet Based Approximation Method to Time Fractional Black-Scholes European Option Pricing Problem Arising in Financial Market, *Applied Mathematical Sciences*, **7** (2013), no. 69, 3445 - 3456. https://doi.org/10.12988/ams.2013.35261

[6] S. Padma, G. Hariharan, K. Kannan, An efficient wavelet based approximation method for a few second order differential equations arising in science and engineering, *International Journal of Engineering and Technology*, **5** (2013), no. 3, 3092 - 3100.

[7] G. Hariharan, K. Kannan, Review of wavelet methods for the solution of reaction-diffusion problem in science and engineering, *Applied Mathematical Modeling*, **38** (2013), 799 - 813. https://doi.org/10.1016/j.apm.2013.08.003

[8] M. Razzaghi, S. Yousefi, Legendre Wavelets Method for the solution of nonlinear problems in the calculus of variations, *Mathematical and Computer Modelling*, **34** (2001), 45 - 54. https://doi.org/10.1016/s0895-7177(01)00048-6

[9] Khosrow Maleknejad, Asyieh Ebrahimzadeh, Optimal Control of Volterra Integro-Differential Systems Based on Legendre Wavelets and Collocation Method, *International Journal of Mathematical, Computational, Natural and Physical Engineering*, **8** (2014), no. 7, 1034 - 1038.

[10] F. Mohammadi, M.M. Hosseini, A new Legendre wavelet operational matrix of derivative and its applications in solving the singular ordinary differential equations, *Journal of the Franklin Institute*, **348** (2011), 1787 - 1796. https://doi.org/10.1016/j.jfranklin.2011.04.017

[11] Vladimir Stefuca, Ingrid Cipakova, Peter Gemeiner, Investigation of immobilized glucoamylase kinetics by flow calorimetry, *Thermochimica Acta*, **378** (2001), 79 - 85. https://doi.org/10.1016/s0040-6031(01)00589-5

[12] S. Muthukaruppan, R. Senthamarai, L. Rajendran, Modelling of Immobilized Glucoamylase Kinetics by Flow Calorimetry, *International Journal of Electrochemical Sciences*, **7** (2012), 9122 - 9137.

[13] David J. Fink, Tsungyen Na, Jerome S. Schultz, Effectiveness factor calculations for immobilized enzyme catalysts, *Biotechnology and Bioengineering*, **15** (1973), 879 - 888. https://doi.org/10.1002/bit.260150505

[14] H. Graebner, R. Kirchner, J. Lerchner, G. Wolf, Flow-calorimetry-a tool for investigation of reactions with complete conversion?, *Thermochimica Acta*, **337** (1999), 39 - 44. https://doi.org/10.1016/s0040-6031(99)00171-9

[15] V. Štefuca and M. Polakovic, Monitoring of Soluble Enzyme Activity by Enzyme Flow Microcalorimetry, *Chem. Papers*, **53** (1999), no. 6, 362 - 365.

[16] Swaminathan Sevukaperumal, Alagu Eswari, Lakshmanan Rajendran, Solution of non-linear boundary value problems in immobilized glucoamylase kinetics, *Natural Science*, **5** (2013), no. 4, 478 - 494. https://doi.org/10.4236/ns.2013.54061

[17] Fukang Yin, Tian Tian, Junqiang Song, Min Zhu, Spectral methods using Legendre wavelets for nonlinear Klein\Sine-Gordan equations, *Journal of Computational and Applied Mathematics*, **275** (2015), 321- 334. https://doi.org/10.1016/j.cam.2014.07.014

[18] Fukang Yin, Junqiang Song, Fengshun Lu, A coupled method of Laplace transform and Legendre wavelets for nonlinear Klein-Gordan equations, *Mathematical Methods in Applied Sciences*, **37** (2013), 781 - 792. https://doi.org/10.1002/mma.2834

[19] Masood Roodaki, Zahra Jafari Behbahani, Two Dimensional Legendre wavelets and their applications to integral equations, *Indian Journal of Science and Technlogy*, **8** (2015), 157 - 164. https://doi.org/10.17485/ijst/2015/v8i2/58075

[20] Naser Aghazadeh, Y. Gholizade Atani and P. Noras, The legendre wavelet method for solving singular integro-differential equations, *Computational Methods for Differential Equation*, **2** (2014), 62 - 68.

[21] Zhijun Meng, Lifeng Wang, Hao Li and Wei Zhang, Legendre wavelets method for solving fractional integro-differential equations, *International Journal of Computer Mathematics*, **92** (2015), 1275 - 1291. https://doi.org/10.1080/00207160.2014.932909

[22] P.K. Sahu, S. Saha Ray, Legendre wavelets operational method for the numerical solutions of nonlinear Volterra integro-differential equations system, *Applied Mathematics and Computation*, **256** (2015), 715 - 723. https://doi.org/10.1016/j.amc.2015.01.063

[23] S. Dhawan, S. Arora, S. Kumar, Wavelet based numerical scheme for differential equations, *International Journal of Differential Equations and Applications*, **12** (2013), no. 2, 85 - 94. https://doi.org/10.12732/ijdea.v12i1.932

[24] Vincent Michael Raj Margret PonRani, Lakshmanan Rajendran and Raju Eswaran, Analytical Expression of the Substrate Concentration in Different Part of Particles with Immobilized Enzyme and Substrate Inhibition Kinetics, *Analytical and Bioanalytical Electrochemistry*, **3** (2011), no. 5, 507 - 520.

[25] R. Rajaraman, G. Hariharan, An efficient wavelet based spectral method to singular boundary value problems, *Journal of Mathematical Chemistry*, **53** (2015), 2095 - 2113. https://doi.org/10.1007/s10910-015-0536-0

[26] G. Hariharan, R. Rajaraman, A new coupled wavelet-based method applied to the nonlinear reactions-diffusion equation arising in mathematical chemistry, *J. Math. Chem.*, **51** (2013), no. 9, 2386 - 2400. https://doi.org/10.1007/s10910-013-0217-9

[27] G. Hariharan, K. Kannan, An overview of Haar wavelet method for solving differential and integral equations, *World Applied Sciences Journal*, **23** (2013), no. 12, 1 - 14.

[28] M. Mahalakshmi, G. Hariharan, K. Kannan, The wavelet methods to linear and nonlinear reaction-diffusion model arising in mathematical chemistry, *J. Math. Chem.*, **51** (2013), 2361 - 2385. https://doi.org/10.1007/s10910-013-0216-x

[29] P. Pirabaharan, R. David Chandrakumar, G. Hariharan, An efficient wavelet based approximation method for estimating the concentration of species and effectiveness factors in porous catalysts, MATCH *Communications in Mathematical and in Computer Chemistry*, **73** (2015), no. 3, 705-727.

[30] M. Mahalakshmi, G. Hariharan, An Efficient Wavelet Based Approximation Method to Steady state Reaction-Diffusion Model Arising in Mathematical Chemistry, *The Journal of Membrane Biology*, **247** (2014), no. 3, 263 - 271. https://doi.org/10.1007/s00232-014-9631-6

[31] C. Boigues Muñoz, D. Pumiglia, S.J. McPhail, D. Montinaro, G. Comodi, G. Santori, M. Carlini, F. Polonara, More accurate macro-models of solid oxide fuel cells through electrochemical and microstructural parameter estimation - Part I: Experimentation**,** *Journal of Power Sources*, **294** (2015), no. 6, 658 - 668. https://doi.org/10.1016/j.jpowsour.2015.06.118

[32] M. Carlini, S. Castellucci, S. Cocchi, Mesophilic fermentation of SOMW in a micro pilot-scale anaerobic digester, *Advanced Materials Research*, **827** (2014), 84 - 90. https://doi.org/10.4028/www.scientific.net/amr.827.84

Received: May 31, 2016; Published: February 22, 2017