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Approximate Analytical Solution for the Black-Scholes Equation by Method of Lines

Christian Duque Sánchez

Universidad Tecnológica de Pereira, Colombia

Harold Duque Sánchez

Universidad Tecnológica de Pereira, Colombia

José Rodrigo Gonzalez

Universidad Tecnológica de Pereira, Colombia

Abstract

The Stochastic Partial Differential Equations are part of a set of non-linear partial differential equations (PDE), which by their random behavior are difficult to solve analytically and numerically; One of them has been known as the Black-Scholes PDE since 1973, which determines the valuation of goods and/or assets called financial options. The development of the present work is to find numerical approximations to the solution of Black-Scholes PDE by the Method of Lines (MOL). The previous was achieved by means of a methodology based on an analytical study of the classic solution of the BlackScholes PDE. Finally, numerical methods and algorithms were applied to the Black Scholes PDE.

Keywords: Stochastic Partial Differential Equations, Black-Scholes Equation, Method of Lines, Financial Options, Numerical Methods, Algorithms

1 Introduction

The theoretical and practical aspects of Brownian motion are subjects in several and diverse fields such as plant biology, mechanical statistics and more precisely in almost the entire financial theory based on stochastic continuous-time models that explain the behavior of random phenomena present in almost 99% of the valuation theory of risks and uncertainties of decisions related to financial portfolio and derivatives [1].

Although, Brownian motion is one of basis for models of economic and financial risks; it cannot represent the behavior of every variable within the financial theory; for example, the valuation of assets that are not initially priced at zero. The description of such return (percentage change) resorts to a process known as geometric Brownian motion, which is obtained with an exponential transformation of standard Brownian motion [1].

In 1973, with the purpose of determining the price of underlying assets driven by a geometric Brownian motion, Fischer Sheffey Black and Myron Samuel Scholes developed an economic model that was later named after them. Their model has been recognized as one of the most important applications in the financial sector during the twentieth century, leading to the Nobel Prize in Economics sciences in 1997. Further and significant development of their model was published in the doctoral thesis Theory of Rational Option Pricing by Merton Howard Miller.

Currently, the study of PDEs is one of the fields within the Applied Mathematics that exhibits increasing interest because they allow to solve a wide variety of problems stemmed from sciences and engineering. Several applications are worthy of mention, among them: 1) Physics based on convection-diffusion equations that studies the transformation phenomena of particles within a physical system, 2) Biological studies dealing with density population of species of fish where species lifetimes are regarded as straight lines while movements of individuals as random motions, 3) Chemical estimations of time-dependent substance distributions along the longitudinal axis of a rectangular reactor and 4) Financial studies on underlying pricing based on Black-Scholes model [2].

Besides the classical solution solved with stochastic calculus [1] other several analytical solutions to Black-Scholes PDE have yielded good results, namely: Lie symmetry analysis [3], Cauchy Problem, construction methods of conservation laws [4], integral transform methods like Mellin [5], Fourier [6, 7], Harper [8] and Adomian decomposition Method [9, 10]. Additionally, numerical methods such as Euler Maruyama and Milstein [11, 12], have solved the Black-Scholes formula. Along these lines, this paper also aims to provide a numerical solution for Black-Scholes PDE using MOL after proposing its mathematical model for a diffusion PDE.

2 Black-Scholes Model

The model has a continuous-time setting of periods $t \in [0, T]$. The two type of assets included in the Model are the following [13]:

- Bonds denoted by $B = (B_t)_{t\in[0,T]}$, which evolves deterministically according to the law:
$$\frac{dB_t}{B_t} = rdt, \quad B_0 = 1,$$
Where r is the interest rate for unit of time.

- Stocks denoted by $S = (S_t)_{t\in[0,T]}$, with random or contingent evolution due to the law:
$$\frac{dS_t}{S_t} = \mu dt + \sigma dW, \quad S_0 = x,$$
Where σ: volatility, μ: expected return and W: Brownian motion.

The basic assumptions of Black-Scholes model are [1]:

i. The underlying asset is a stock that does not pay any dividend during the life of the contract.

ii. The Price of the underlying asset is driven by a geometric Brownian motion.

iii. The volatility of the price of underlying asset remains constant through time

iv. Short selling of the underlying is permitted.

v. Market of the underlying is perfectly divisible and liquid, so it is possible to purchase or sell any amount of stock or options or their fractions at any given time.

vi. There are no transaction cost (taxes or brokerage fees).

vii. The market operates uninterruptedly, there are no calendar but trading days.

viii. Agents can lend or borrow money from other investors or the banking system at a constant, risk-free rate (passive and active interest rates are equal).

ix. Market operates under conditions of symmetric information for all the agents.

x. Since there are no risk less arbitrage opportunities, equilibrium of the market is guaranteed.

The pricing of a European-Style option [1][1] for its trading within a future date is a function of different parameters involved in the contract clauses; in other words, the pricing of an option can be expressed as follows [1, p. 204]:

$$C = C(S_t, t; K, T, \sigma, \mu, r), \tag{1}$$

Where: C: Option price, S_t: Stock price, t: Contract lifespan, K: Striking price, T: Expiration date, σ: Volatility, μ: Expected return and, r: Anual interest rate.

The parameters S_t, t, K, T y μ are known values defined by the contract and the exchange rate concurrent to the moment of pricing estimation, while interest rate r depends on the expiration of the contract [15].The volatility σ is a risk indicator derived from the changes in the return rates, hence it should be regarded as a temporal series rather than a single fixed parameter through time [16]; its value can be expressed as a function $\sigma(S, t)$ due to reality of the financial developments. In case of knowing the current price of the option, volatility can be estimated based on market historical data [17].

As S_t and t are relevant variables in the contract, the mention of the other parameters K,T,r,σ y μ) (is optional); thus, the option pricing can be simplified as follows $C = C(S_t, t)$.

During the interval of time $[t, t + dt]$ the underlying asset changes from S_t to $S_t + dS_t$, hence the option changes from $C(S_t, t)$ to $C + dC$, where dC is the margin change of the price estimated from Ito lemma [1, p. 67] considering second order terms to obtain the differential of $C = C(S_t, t)$ for an expansion of Taylor Series (which in case of real variables, it is estimated using only terms of first order due to the negiglible nature of the product of inifinitesimal quantities). Therefore, the dC Equation is obtained using the empirical rules of stochastic differentiation [1, p. 67] as well as a stochastic differential equation:

$$dS_t = \mu_t dt + \sigma_t dW_t \tag{2}$$

[1]European-style option can only be exercised at the expiration date of the contract, whereas American-style option can be exercised at any given time prior to expiration. Although both terms refer to the birthplace of the expressions, this geographical point of view is not longer applicable [14].

which simplifies Ito stochastic integral [18]:

$$dC = \left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S_t}\mu\left(S_t, t\right) + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\sigma^2(S_t, t)\right)dt + \frac{\partial C}{\partial S_t}\sigma\left(S_t, t\right)dW_t. \tag{3}$$

2.1 Portfolio Dynamics

Let ω_1 be the number of units of the underlying asset (stock) at a price S_t and ω_2 be the number of units of purchase option of the underlying (bonds) at a price $C(S_t, t)$. The current value of the portfolio will be given by:

$$\Pi_t = \omega_1 S_t + \omega_2 C(S_t, t).$$

To take into account the fluctuations of the market, the change of value of the portfolio during a time dt will be given by:

$$d\Pi_t = \omega_1 dS_t + \omega_2 dC(S_t, t), \tag{4}$$

After replacing (2) and (3) in the equation (4) and grouping terms:

$$d\Pi_t = \left(\omega_1 + \omega_2\frac{\partial C}{\partial S_t}\right)\mu S_t dt + \left(\omega_1 + \omega_2\frac{\partial C}{\partial S_t}\right)\sigma S_t dW_t + \omega_2\left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\sigma^2 S_t^2\right)dt,$$

Where dt coefficients are referred as to ***trending terms*** and the one with dW_t as ***random term*** which in turn model the risk of the portfolio market. The careful selection of amounts ω_1 and ω_2 allows to annul the stocastich term of the equation, and thus eliminating the aforementioned risk:

$$\omega_1 + \omega_2\frac{\partial C}{\partial S_t} = 0,$$

When the values selected for $\omega_2 = 1$ and $\omega_1 = -\frac{\partial C}{\partial S_t} = -\delta$ respectively, a Delta Hedging es obtained:

$$d\Pi_t^{(\delta)} = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\sigma^2 S_t^2\right)dt. \tag{5}$$

This kind of hedging is dynamic, since during the period $[t, t+dt]$, the amount $\frac{\partial C}{\partial S_t}$ exhibits correlated variation with S_t and t, Therefore, the value of the portfolio is defined by:

$$\Pi_t^{(\delta)} = C - \delta S_t.$$

If the amount $\Pi_t^{(\delta)}$ is deposited in a bank that pays an interest rate r, the value change of the portfolio during dt, is given by the equation:

$$d\Pi_t^{(r)} = \Pi_t^{(\delta)} r dt = (C - \delta S_t)\, r dt, \tag{6}$$

In this case, dt is the time when interest rate r is applied.
The tenth basic assumption regarding the equilibrium of the market can be expressed as:

$$d\Pi_t^{(r)} = d\Pi_t^{(\delta)} \tag{7}$$

substituting the values of equations (5) and (6) in the equation in the ecuation (7):

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\sigma^2 S_t^2\right) dt = \left(C - \frac{\partial C}{\partial S_t} S_t\right) r dt$$

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\sigma^2 S_t^2 + \frac{\partial C}{\partial S_t} S_t r - rC = 0 \tag{8}$$

Which is known as the Black-Scholes PDE, whose boundary and intial conditions are given by Equations

Boundary conditions:

$$C(0,t) = 0, \qquad C(S_t,t) \approx S_t \qquad cuando \quad S_t \to \infty. \tag{9}$$

and

Initial condition:

$$C(S_t,T) = max(S_t - K, 0). \tag{10}$$

2.2 Methods of Lines (MOL)

MOL is a numerical procedure to solve PDE. Although MOL has been reported to provide accurate numerical solutions, its successful application to new PDE problem depends on the experience and cleverness of the analyst. Rather than being a unique, direct and clearly defined approach, MOL is a general concept that requires specific details of each new PDE problem [19, 20, 21].

The basic idea behind MOL is replacing PDE derivatives with algebraic approximations expressed in terms of one variable (generally time t) instead of two independent variables ((for example x and t), thereby the solutions consists in solving a system of ordinary differential equations ODE that approximates the original PDE, the ODE system can be solved with an ODE integration algorithm. This work uses MATLAB ODE 45 algorithm [22], which is based on Runge- Kutta 4th order method.

2.3 MOL procedure applied to the diffusion Equation

According to [23, p. 21], an example of the diffusion Equation with its corresponding initial and boundary conditions will be given by:

$$\begin{cases} u_t = \beta u_{xx} \\ u(x,0) = sen(2\pi x) + 2x + 1, \quad 0 \le x \le a \\ u(0,t) = 1, \quad u_x(1,t) = 2, \quad 0 \le t \le T \end{cases}, \tag{11}$$

With $\beta = 10^{-5}$, $T = 12000$, $a = 1$.
Firstly the nodes in $x \in [0,a] = [0,1]$ are defined as follows:

$$x_i = i\Delta x, 0 \le i \le n+1, \Delta x = \frac{a}{n+1} = \frac{1}{n+1}. \tag{12}$$

$$0 = x_0 < x_1 < x_2 < ... < x_n < x_{n+1} = a = 1,$$

where each x_n refers to a ***vertical*** semi straight line, whose points are function of t, that is $u_n(t) = u(x_n,t)$. As seen in (12), only variable x, is discretized, and variable t, remains undiscretized and continuous. Thus, the numerical solutions will be obtained along the lines $x = x_n$, $t \in [0, \infty)$.

Initial condition

$$u(x_i, t = 0) = f(x_i) = 1. \tag{13}$$

Dirichlet condition

$$u(0,t) = u_o(t) = u_o = g_1(t) = 1. \tag{14}$$

Neumann condition

$$u_x(a,t) = u_x(1,t) = \frac{\partial u_{n+1}}{\partial x} = g_2(t) = 2. \tag{15}$$

Next, based on the ***Forward Divided Differences*** in (x_i, t_j), [24]:

$$\frac{\partial F(x_i,t_j)}{\partial x} \approx \frac{F(x_i+h,t_j) - F(x_i,t_j)}{h}. \tag{16}$$

The Equation (15) can be transformed into:

$$\frac{\partial u(x_i,t)}{\partial x} \approx \frac{u(x_{i+1},t) - u(x_i,t)}{\Delta x} = \frac{u_{i+1} - u_i}{\Delta x} = g_2(t), \tag{17}$$

Solving for u_{i+1}, it is obtained:

$$u_{i+1} = \Delta x g_2(t) + u_i, \tag{18}$$

The ***Central divided differences*** in (x_i, t_j), [24]:

$$\frac{\partial^2 F(x_i,t_j)}{\partial x^2} \approx \frac{F(x_i+h,t_j) - 2F(x_i,t_j) + F(x_i-h,t_j)}{h^2}. \tag{19}$$

are used to obtain the second partial derivative in respect to x

$$\begin{aligned} \frac{\partial^2 u(x_i,t)}{\partial x^2} &\approx \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{(\Delta x)^2}, \\ \frac{\partial^2 u(x_i,t)}{\partial x^2} &\approx \frac{u_{i-1} - 2u_i + u_{i+1}}{(\Delta x)^2}. \end{aligned} \tag{20}$$

After replacing these results in the diffusion Equation $u_t = \beta u_{xx}$, it is obtained:

$$\begin{aligned} \frac{\partial u(x_i,t_j)}{\partial t} &= \beta\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{(\Delta x)^2}\right), \\ \frac{\partial u(x_i,t_j)}{\partial t} &= \frac{\beta}{(\Delta x)^2}\Big(u_{i-1} - 2u_i + u_{i+1}\Big). \end{aligned} \tag{21}$$

Which is an ODE system, whose solution is given by the n functions:

$$u_1(t), u_2(t), ..., u_{n-1}(t), u_n(t), \tag{22}$$

Which in turn are an approximation for $u(x_i, t)$ on the points $i = 1, 2, 3, ..., n$ of the MOL grid, taking into account that functions u_0 and u_{n+1} are the given boundary conditions.

In a case with 5 points in the MOL grid, the points will have the following distribution: $0 = x_0 < x_1 < x_2 < x_3 < x_4 = x_{n+1} = a$, and thereby, when $x = 0$ and $x = 4$ the boundary conditions are:

$$u_0 = u(0,t) = 1 \qquad y \qquad u_{i+1} = 2\Delta x + u_i,$$

Now, it is possible to represent Equation (21) in a matrix form:

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} = \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} &= \frac{\beta}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\ &+ \frac{\beta}{(\Delta x)^2}\begin{pmatrix} {}^{CF}\ \overbrace{u_0}^{Dirichlet} \\ 0 \\ {}_{CF}\ \underbrace{u_4}_{Neumann} \end{pmatrix}, \end{aligned} \tag{23}$$

Where $\frac{\partial \vec{u}}{\partial t} = \begin{pmatrix} \dot{u_1} \\ \dot{u_2} \\ \dot{u_3} \end{pmatrix}$, is an ODE vector, whose solutions $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ provide the temperature at each time t_j.

Using the boundary conditions $u_0 = g_1(t) = 1$ and $u_4 = \Delta x g_2(t) + u_3 = 2\Delta x + u_3$ and replacing s with $\frac{\beta}{(\Delta x)^2}$ the matrix system (23) can be denoted as follows:

$$\frac{\partial \vec{u}}{\partial t} = \begin{pmatrix} \dot{u_1} \\ \dot{u_2} \\ \dot{u_3} \end{pmatrix} = s \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 2\Delta x + u_3 \end{pmatrix}$$

$$\frac{\partial \vec{u}}{\partial t} = \begin{pmatrix} \dot{u_1} \\ \dot{u_2} \\ \dot{u_3} \end{pmatrix} = s \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 2\Delta x \end{pmatrix} \tag{24}$$

Which can be expressed:

$$\dot{\vec{u}} = A\vec{u} + \overrightarrow{b}. \tag{25}$$

Where $A = s \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\overrightarrow{b} = s \begin{pmatrix} 1 \\ 0 \\ 2\Delta x \end{pmatrix}$.

This MOL procedure explained in section 2.3. will be applied in Black-Scholes PDE.

3 MOL procedure applied to Black-Scholes PDE

As a preliminary step, Equation (2) is modified by replacing time variable t with $\tau = T - t$, whose expiration date is still T. This implies $C(S_t, t)$ to become $C(S_t, T - \tau)$. These modifications result in the following progressive parabolic equation.

$$\frac{\partial C}{\partial \tau} = \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} + S_t r \frac{\partial C}{\partial S_t} - rC. \tag{26}$$

Whose initial condition represents the value solution for a Europe-style put option, and is denoted by:

$$C_i(0) = max(K - S_i, 0) = f(S_i, K), \quad i = 0, 1, 2, ..., M, \tag{27}$$

The idea of MOL is discretize both the asset value and initial and boundary conditions of Black-Scholes PDE. The width of each interval is given by:

$$\Delta S = h = \frac{S_{max}}{M}, \tag{28}$$

where M is the number of partitions of the asset S and, S_{max} its higher price, thereby variable S can have values $0, \quad h, \quad 2h, ...Mh = S_{max}$, each one of these are supposed to be asset prices, whose intervals have width h.[25] he MOL grid formed by the points $ih, i = 0, 2, ...M$ y $j, j = 0, 1, ...T$, represents values of asset S in a time τ, Additionally, from now on $U_i(\tau)$ will represent the option value in point (i, j).

Using the ***Central divided differences*** in (x_i) [24]

$$f'(x_i) \approx \frac{f(x_i + h) - f(x_i - h)}{2h}. \tag{29}$$

and

$$f''(x_i) \approx \frac{f(x_i + h) - 2f(x_i) + f(x_i - h)}{2h^2}. \tag{30}$$

The right-hand side of the Equation (26) changes to:

$$\begin{aligned} \frac{dC_i(\tau)}{d\tau} &= \frac{1}{2}\sigma^2 S_i^2 \left(\frac{C_{i-1}(\tau) - 2C_i(t) + C_{i+1}(\tau)}{h^2} \right) \\ &+ rS_i \left(\frac{C_{i+1}(\tau) - C_{i-1}(\tau)}{2h} \right) - rC_i(\tau), \end{aligned} \tag{31}$$

with $i = 1, 2, 3, ..., M + 1$. Leading to an ODE system to be solved by means of numeric algorithm.

After discretizing initial condition $C_i(0) = mx(K - S_i, 0) = f(S_i, K)$ [24]:

$$\begin{aligned} C_i(0) &= f(S_i, K) = (ih, K) = max(K - ih, 0), \\ C_i(0) &= K - ih, \qquad i = 1, 2, ...M + 1. \end{aligned} \tag{32}$$

Whose boundary conditions

$$C_0(\tau) = Ke^{-r\tau}, \quad C_M(\tau) = 0, \tag{33}$$

the Equation $C_0(\tau) = Ke^{-r\tau}$ is derived after replacing $S = 0$ in Equations (26) y (10).

Equations (32) and (33) define the put option value along three points of the MOL grid $S = 0$, $S = S_{mx} = hM$ y $t = T$.

The system (31) with its initial boundary conditions can be represented in a matrix form:

$$\frac{d\vec{C}_i(\tau)}{d\tau} = (\mathbf{A} + \mathbf{Y} + \mathbf{O})\vec{U}(\tau) + \vec{B}(\tau), \tag{34}$$

Where $\vec{U}(\tau)$ and $\vec{B}(\tau)$ are defined by the vectors:

$$\vec{U}(\tau) = \begin{pmatrix} U_1(\tau) \\ U_2(\tau) \\ \vdots \\ U_{M-1}(\tau) \end{pmatrix}. \tag{35}$$

and

$$\vec{B}(\tau) = \begin{pmatrix} \left(\frac{\sigma^2 S_1^2}{2h^2} - \frac{rS_1}{2h}\right) Ke^{-r\tau} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{36}$$

And the matrices **A**, **Y** and **O** are defined:

$$\mathbf{A} = \frac{\sigma^2}{2h^2} \begin{pmatrix} -2S_1^2 & S_1^2 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ S_2^2 & -2S_2^2 & S_2^2 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & S_3^2 & -2S_3^2 & S_3^2 & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & 0 & S_{M-2}^2 & -2S_{M-2}^2 & S_{M-2}^2 \\ \cdots & \cdots & \cdots & 0 & 0 & 0 & S_{M-1}^2 & -2S_{M-1}^2 \end{pmatrix} \tag{37}$$

$$\mathbf{Y} = \frac{r}{2h} \begin{pmatrix} 0 & S_1 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ -S_2 & 0 & S_2 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & -S_3 & 0 & S_3 & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & 0 & -S_{M-2} & 0 & S_{M-2} \\ \cdots & \cdots & \cdots & 0 & 0 & 0 & -S_{M-1} & 0 \end{pmatrix} \tag{38}$$

$$\mathbf{O} = -r \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & 0 & 0 & 1 \end{pmatrix} \tag{39}$$

3.1 Case of MOL application in Black-Scholes PDE

Considering a non-dividend paying Europe-style put option, when the strike price $K = 50$, with a yearly risk-free interest rate $r = 10\%$ and yearly volatility $= 40\%$. Obtain a Matrix that shows results taking into account $M = 20$ (partitions of S) and $\Delta S = h = 5$ (width of each partition ΔS). The price of the option is evaluated in \$5 for each partition from \$0 to \$100, with $S \in [0, 100]$.

The lifespan of the option is 5 months with partitions every half-month, with $N = 10$ (partitions of t) [26, p. 212]. The solution of this problem is elaborated in [27].

The results reports price values of put option C numerically obtained for $M = 20$ partitions of asset S. The trend of the put optin is verified with the behavior of values of C, which tend to zero when values S are higher.

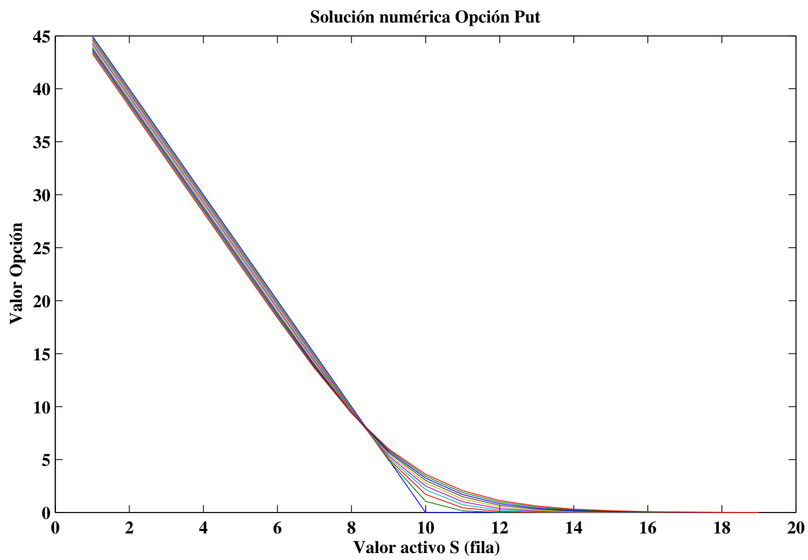

Figure 1: Graphic of numerical solution for put option

In Figure (1) each curve of the numerical solution corresponds to one column of T values, whose data represent the asset partition in the horizontal axis and the value of the option in the vertical axis. All the curves approximately coincide at the same point for partition $M = 8 : 15$, which corresponds to $S = 51$ (asset) and $C = 8 : 8$ (option).

4 Conclusion

- It was verified that the behavior of the solution for the value of a Europe-style put option tends to zero when values S are higher.
- A MOL numeric solution was found for the Black-Scholes PDE using numeric algorithms.

- Volatility σ should be constant in the ODE system modeled by MOL for the Black-Scholes. Otherwise, it would be necessary an appropriate dimensioning of the matrix for its products to be valid.

5 Recommendations

- Due to the non-stationarity of the Black-Scholes PDE, it is recommended to use a time-frequency transform able to reflect changes of frequency in respect to time. Future studies should include Wavelet or Wigner Ville transform to obtain a numeric-analytical solution.

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