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Approximate Analytical Solutions for the Black-Scholes Equation by Homotopy Perturbation Method

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Abstract

One of the major problems of the financial market is the valuation of multiple and sophisticated financial products (derivatives, options, swaps, etc). This challenge has promoted the rigorous study of the market using mathematical models. One of them has been know as Black-Scholes model since 1973; it is modeled as a second-order Partial Differential Equation (PDE) of parabolic nature, whose solution determines the pricing of a Europe-style put option. This work consists in finding numeric approximations to the solution by means of a Homotopy Perturbation Method (HPM) that includes numerical methods and algorithms.

Keywords: Black-Scholes Model, Mathematical Models, partial Differential Equation, Europe Put Option, Homotopy Perturbation Method, Financial products, Numerical Methods, Algorithms

1 Introduction

The financing of the rail road system in the United State of America, 1857 can be considered the origin of the modern organized financial market. Nowadays, it is well known that the needs for financing of companies and the uncertainty in respect to future events are key factors in the financial markets.
Since 1970 the financial markets started to experience structural changes
In 1973, Fischer Black, Myron Scholes and Robert Merton developed a formula to determine the price of Europe-style options that eventually was deserving of the Nobel Prize in Economics Sciences in 1997. The Black-Scholes formula is one of the most recognized applications that has been used to model the behavior of the hedging, thus it became a fundamental tool on the decision making process for companies and financial systems worldwide. Nowadays, it is expressed in a second-order Partial Differential Equation (PDE) of the parabolic type
Besides the classical solution solved with stochastic It's calculus , other several analytical have successfully solved the Black-Scholes PDE: Lie symmetry analysis [2], Cauchy Problem, construction methods of conservation laws [3], integral transform methods like Mellin
This paper also aims to provide a numerical approximation to the solution for Black-Scholes PDE using homotopy perturbation method (HPM) that includes numeric methods and algorithms

2 Black-Scholes Model

Although the financial markets coincide with the Black-Scholes, they do not reflect the central hypothesis on continuous time, mostly modern markets affected by discontinuities, [11].

According the ten basic assumptions under which the Black Scholes model operate

$$C = C(S_t, t; K, T, \sigma, \mu, r), \tag{1}$$

Where: C: Option price, S_t: Stock price, t: Contract lifespan, K: Striking price, T: Expiration date, σ: Volatility, μ: Expected return and, r: Annual interest rate.

The parameters S_t, t, K, T and μ are known values defined by the contract and the exchange rate concurrent to the moment of pricing estimation, while interest rate r depends on the expiration of the contract

As S_t and t are relevant variables in the contract, the mention of the other parameters K,T,r,σ y μ) (is optional); thus, the option pricing can be simplified as follows $C = C(S_t, t)$.

The value of the underlying asset S_t (stocks In this case), satisfies the following stochastic differential equation that simplifies It's integral [13]:

$$dS_t = \mu_t dt + \sigma_t dW_t \tag{2}$$

In compliance with the empirical rules of stochastic differentiation

$$\begin{aligned} dC = &\left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S_t}\mu\left(S_t, t\right) + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\sigma^2(S_t, t)\right)dt \\ &+ \frac{\partial C}{\partial S_t}\sigma\left(S_t, t\right)dW_t. \end{aligned} \tag{3}$$

Finally the Black-Scholes PDE is obtained considering portfolio dynamics due to market fluctuations at a time dt and under the assumption of the equilibrium of the market guaranteed by the absence of arbitrage opportunities:

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\sigma^2 S_t^2\right)dt = \left(C - \frac{\partial C}{\partial S_t}S_t\right)rdt$$

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\sigma^2 S_t^2 + \frac{\partial C}{\partial S_t}S_t r - rC = 0 \tag{4}$$

Whose initial and boundary conditions are denoted by:

Boundary conditions:

$$C(0, t) = 0, \qquad C(S_t, t) \approx S_t \qquad \textit{cuando} \quad S_t \to \infty. \tag{5}$$

and

Initial condition:

$$C(S_t, T) = max(S_t - K, 0). \tag{6}$$

2.1 Homotopy Perturbation Method (HPM)

Several non-linear PDE represent problems of science and engineering, among them, stochastic PDE which are hard to solve using analytical or numeric methods due to their random behavior. Some stochastic models have been solved by means of Decomposition Methods such as Adomian (ADM) [14], Variational Iteration (VIDM) [15], Galerkin Discretization (GD)

2.2 Method Description

The non-lineal differential equation is given by:

$$A(u) - f(\tau) = 0, \quad \tau \in \Omega. \tag{7}$$

Whose boundary conditions are:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad \tau \in \Gamma, \tag{8}$$

Where A is the differential operator, B is the boundary operator, $f(\tau)$ is a known analytical function and Γ differe is the boundary of the domain Ω. The operator A is split into a Linear Operator L and a non-linear Operator N, the corresponding rewriting of Equation (7) will lead to:

$$L(u) + N(u) - f(\tau) = 0, \tag{9}$$

Where

$$A(u) = L(u) + N(u)$$

The main idea behid HPM is to obtain a function

$$v(\tau, p) : \Omega \times [0, 1] \rightarrow R,$$

that satisfies the following Equations:

$$\begin{aligned} H(v, p) = (1 - p)\left[L(v) - L(v_0)\right] \\ + p\left[A(v) - f(\tau)\right] = 0, \end{aligned} \tag{10}$$

with

$$p \in [0, 1], \quad \tau \in \Omega,$$

Which can be formulated as follows:

$$\begin{aligned} H(v, p) &= L(v) - L(v_0) + pL(v_0) - pL(v) + p\left[A(v) - f(\tau)\right] = 0 \\ &= L(v) - L(v_0) + pL(v_0) + p\left[A(v) - L(v) - f(\tau)\right] = 0 \\ H(v, p) &= L(v) - L(v_0) + pL(v_0) + p\left[N(v) - f(\tau)\right] = 0, \end{aligned} \tag{11}$$

where p is known as the ***homotophy parameter*** and v_0 is an approximate initial solution of Equation (7 that satisfies boundary conditions.

Then, Equation (11 can be rewritten to obtain the ***Homotophy Equation*** [17]:

$$\begin{aligned} L(v) - L(v_0) &= -pL(v_0) - p\left[N(v) - f(\tau)\right] \\ L(v) - L(v_0) &= p\left[f(\tau) - N(v) - L(v_0)\right] \\ L(v) - L(v_0) &= p\left[A(v) - L(v_0)\right], \end{aligned} \tag{12}$$

If $p = 0$ is replaced in Equation (10 and $p = 1$ in Equation (11:

$$H(v, 0) = L(v) - L(v_0) = 0, \tag{13}$$

$$H(v, p) = A(v) - f(\tau) = 0. \tag{14}$$

The topological deformation results from the homotophy parameter p increase from 0 to 1, that is the change in $v(\tau, p)$ from $v_0(\tau)$ to $v(\tau)$. Whereas the homotophy Equations are given by $L(v) - L(v_0)$ and $A(v) - f(\tau)$. In HPM method, p is also an expansion parameter necessary to obtain a solution expressed in power series for Equations (10 and (11:

$$v = \sum_{i=0}^{\infty} p^i v_i = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + ... \tag{15}$$

According to HPM, the approximate solution of Equation (7 is obtained when $p \to 1$ in Equation (15:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + ... \tag{16}$$

The series (16) is convergent for most cases and the radius of ceonvergence depends on $A(u) - f(\tau)$ [18].

2.3 HPM procedure applied to Black-Scholes PDE

The Black-Scholes PDE is given by:

$$\frac{\partial C}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + Sr\frac{\partial C}{\partial S} - rC. \tag{17}$$

With initial condition:

$$C(S_t, T) = max(K - S_t, 0) = f(S). \tag{18}$$

For this exercise the initial condition is defined by the following function [19]:

$$f(S) = C_0(S, t) = S + \frac{1}{S^{7/5}}. \tag{19}$$

Replacing Equation (17) in the ***Homotophy Equation*** (12):

$$\frac{\partial C}{\partial t}-\frac{\partial C_0}{\partial t}=p\left[\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2}+Sr\frac{\partial C}{\partial S}-rC-\frac{\partial C_0}{\partial t}\right], \tag{20}$$

Given that

$$\frac{\partial C_0}{\partial t}=\frac{\partial(S+\frac{1}{S^{7/5}})}{\partial t}=0, \tag{21}$$

equation (20) can be rewriten as follows:

$$\frac{\partial C}{\partial t}-\frac{\partial C_0}{\partial t}=p\left[\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2}+Sr\frac{\partial C}{\partial S}-rC\right]. \tag{22}$$

Assuming the following solution by power series:

$$C=C_0+pC_1+p^2C_2+p^3C_3+... \tag{23}$$

After replacing (23) in Equation (22):

$$\begin{aligned}\frac{\partial C}{\partial t}-\frac{\partial C_0}{\partial t}=p\Bigg[&\frac{1}{2}\sigma^2S^2\frac{\partial^2(C_0+pC_1+p^2C_2+p^3C_3+...)}{\partial S^2}\\&+Sr\frac{\partial(C_0+pC_1+p^2C_2+p^3C_3+...)}{\partial S}\\&-r(C_0+pC_1+p^2C_2+p^3C_3+...)\Bigg].\end{aligned}$$

After performing the indicated operations to determine the powers of p, the solution $C(S;t)$ of Black-Scholes PDE is given by the approximation:

$$C(S,t)=C_0(S,t)+C_1(S,t)+C_2(S,t)+C_3(S,t)+... \tag{24}$$

Where $C_1(S,t),\quad C_2(S,t)$ and $C_3(S,t)$ are given by the following equations:

$$\begin{aligned}C_1(S,t)&=\left(\frac{42}{25}\sigma^2\frac{1}{S^{7/5}}-\frac{12}{5}\frac{r}{S^{7/5}}\right)t\\&=\frac{1}{S^{7/5}}\left(\frac{42}{25}\sigma^2-\frac{12r}{5}\right)t\end{aligned} \tag{25}$$

$$\begin{aligned}C_2(S,t)&=\left(\frac{42}{25}\sigma^2-\frac{12r}{5}\right)^2\frac{1}{S^{7/5}}\frac{t^2}{2}\\&=\left(\frac{42}{25}\sigma^2-\frac{12r}{5}\right)C_1(S,t)\frac{t}{2}\end{aligned} \tag{26}$$

$$C_3(S,t) = \left(\frac{42}{25}\sigma^2 - \frac{12r}{5}\right)^3 \frac{1}{S^{7/5}} \frac{t^3}{2\times 3}$$
$$= \left(\frac{42}{25}\sigma^2 - \frac{12r}{5}\right) C_2(S,t)\frac{t}{3} \tag{27}$$

Replacing (19), (25), (26) and (27) in Equation (24):

$$\begin{aligned} C(S,t) = S + \frac{1}{S^{7/5}} + \frac{1}{S^{7/5}}\left(\frac{42}{25}\sigma^2 - \frac{12r}{5}\right)t \\ + \left(\frac{42}{25}\sigma^2 - \frac{12r}{5}\right)^2 \frac{1}{S^{7/5}}\frac{t^2}{1\times 2} \\ + \left(\frac{42}{25}\sigma^2 - \frac{12r}{5}\right)^3 \frac{1}{S^{7/5}}\frac{t^3}{1\times 2\times 3} + \ldots \end{aligned} \tag{28}$$

which can be written in terms of summation:

$$C(S,t) = S + \frac{1}{S^{7/5}}\left(1 + \sum_{n=1}^{\infty}\left(\frac{42}{25}\sigma^2 - \frac{12r}{5}\right)^n \frac{t^n}{n!}\right) \tag{29}$$

2.4 Results of HPM method applied to Black-Scholes PDE

In this section, different numeric results are shown for different values of both volatility σ and initial condition $C_0(s,t)$
The first case where volatility $\sigma = 0.2$ and initial condition $C_0(s,t) = S + \frac{1}{S^{7/5}}$ is illustrated in Figure (1):

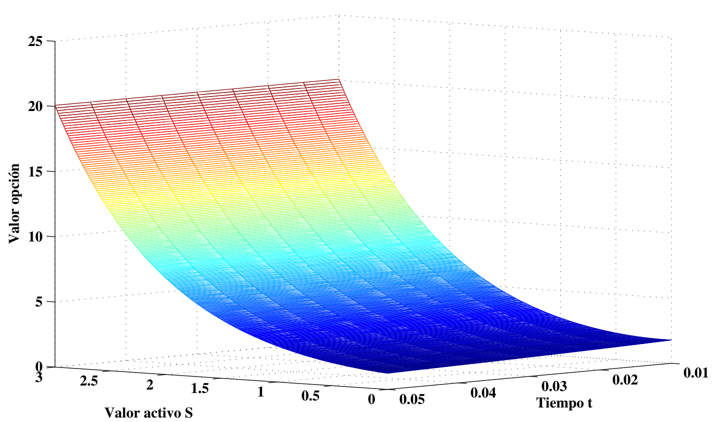

Figure 1: Case of volatility $\sigma = 0.2$ and initial condition $C_0(s,t) = S + \frac{1}{S^{7/5}}$

The first case where volatility $\sigma = t^2r^2$ and initial condition $C_0(s,t) = S + \frac{1}{S^{7/5}}$ is illustrated in Figure (2):

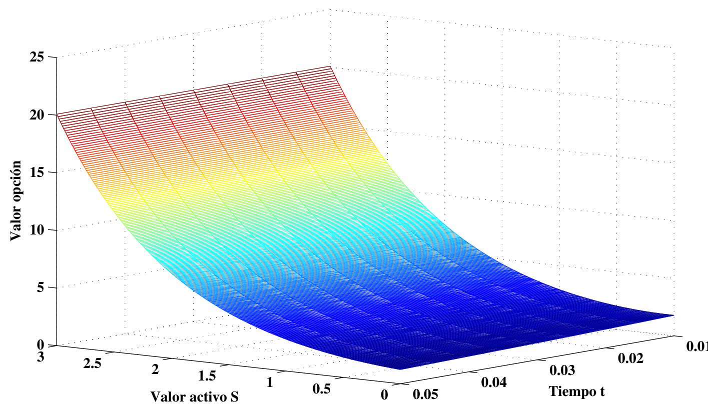

Figure 2: Case of volatility $\sigma = t^2r^2$ and initial condition $C_0(s,t) = S + \frac{1}{S^{7/5}}$

The second case where volatility $\sigma = 0.2$ and initial condition $C_0(s,t) = St^2 + \frac{t}{S^{7/5}}$ is illustrated in Figure (3):

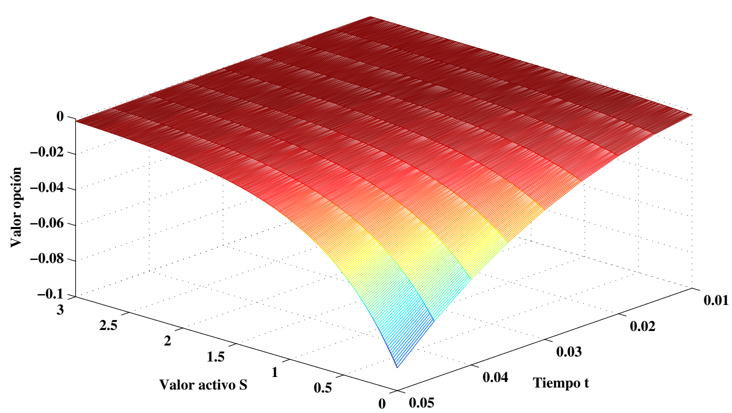

Figure 3: Case of volatility $\sigma = 0.2$ and initial condition $C_0(s,t) = St^2 + \frac{t}{S^{7/5}}$

The third case where volatility $\sigma = t^2r^2$ and initial condition $C_0(s,t) = St + \frac{1}{S^{7/5}}$ is illustrated in Figure (4):

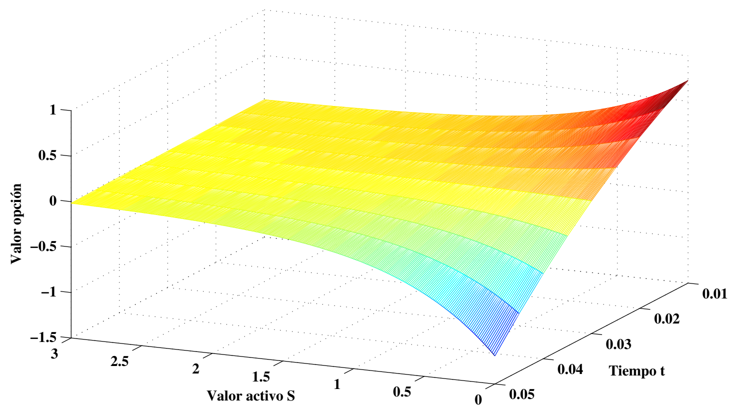

Figure 4: Case of volatility $\sigma = t^2 r^2$ and initial condition $C_0(s,t) = St + \frac{1}{S^{7/5}}$

Option values does not exhibit variations with changes in volatility (cases illustrated in Figures (1) and (2). But it is reported a correlated variation with changes of initial conditions (cases illustrated in Figures (3) and (4). Hence, HPM method depends on initial conditions of the problem, which was analitically demostrated throughout section 2.3.

3 Conclusion

- A HPM numeric solution was found for the Black-Scholes PDE using numeric algorithms.
- It is reported that the application of HPM method to Black-Scholes PDE depends on initial conditions of the problem. The dependency of the partial derivative on the initial conditions affects the robustness or lightness of the HPM method. Given its recursive nature HPM method depends mainly on the initial condition.
- The solution of the Black-Scholes PDE obtained by means of HPM method using the numeric method allowed to simulate the solution with variations of different parameters like volatility σ and interes rate r. The volatility was expressed in terms of time and interest rate.

4 Recommendations

- Due to the non-stationarity of the Black-Scholes PDE, it is recommended to use a time-frequency transform able to reflect changes of frequency in respect to time. Future studies should include Wavelet or Wigner Ville transform to obtain a numeric-analytical solution.

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