Contemporary Engineering Sciences, Vol. 10, 2017, no. 18, 881 - 890
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ces.2017.79105

Solution of the Nonlinear Telegraph Equation Using Lattice-Boltzmann and Solitary Wave Methods

F. Fonseca

Universidad Nacional de Colombia
Departamento de Física
Bogotá, Colombia

Abstract

In this work we solved the nonlinear one-dimensional Telegraph equation using lattice-Boltzmann and a $d1q3$ velocity scheme. Also, using the Tanh and Riccati solitary wave methods, we find several families of solutions. In addition, for two colliding nonlinear solutions they holds the superposition principle.

Keywords: Telegraph equation, Lattice-Boltzmann, Tanh and Riccati methods

1 Introduction

In this paper we deal with the nonlinear telegraph equation (NTEq). The telegraph equation is a very important analytical tool that gives the temporal evolution of voltage and current in electromagnetic systems, and with central role in communication devices,[1], and in many problems in engineering and science. Important analytical methods have been developed and established, in order to find solutions to nonlinear differential equations. Among them, we apply the tanh method, [2], which has been extensively used. On the other hand, the technique called, lattice Boltzmann (LB), [3]-[4] has been applied with success in several nonlinear equations as, e.g., Navier-Stokes[5], Poisson[6], Lorenz [7], Korteweg de Vries[8].
This paper is prepared as follows. Section (2), presents the lattice-Boltzmann model. In section (3), the moments of the particle distribution are defined in

order to get the NTEq. Section (4), we obtain the NTEq. In Section (5), we get the equilibrium distribution function. Also, in section (6) we apply the Tanh method, [2], in order to find solutions to the NTEq. At last, in section (7), we present results and conclusions.

2 The lattice Boltzmann model

The lattice Boltzmann (LB) equation is, [3]:

$$f_i(x + e_i\epsilon, t + \epsilon) - f_i(x, t) = \Omega_i(x, t) + \omega_i(x, t) \tag{1}$$

Where $f_i(x, t)$ is the distribution function, e_i is the velocity at position x and time t, and ϵ is the time step. The equilibrium distribution is given by $f_i^{eq}(x, t)$ and τ is a relaxation time, that accounts the rate to the equilibrium. The term $\Omega_i(x, t)$ represents the *B.G.K.*, [4], approximation of the collision term

$$\Omega_i(x, t)) = -\frac{1}{\tau}\left(f_i(x, t) - f_i^{eq}(x, t)\right) \tag{2}$$

We expand the distribution function in a Taylor series, up to third order, we obtain:

$$\begin{aligned} f_i(x + e_i\epsilon, t + \epsilon) - f_i(x, t) &= \epsilon\left(\frac{\partial}{\partial t} + e_i\frac{\partial}{\partial x}\right) f_i + \frac{\epsilon^2}{2}\left(\frac{\partial}{\partial t} + e_i\frac{\partial}{\partial x}\right)^2 f_i \\ &+ \frac{\epsilon^3}{6}\left(\frac{\partial}{\partial t} + e_i\frac{\partial}{\partial x}\right)^3 f_i + O(\epsilon^4) \end{aligned} \tag{3}$$

Doing a perturbative expansion of the distribution function in powers of ϵ, we get:

$$f_i = f_i^{(0)} + \epsilon f_i^{(1)} + \epsilon^2 f_i^{(2)} + \epsilon^3 f_i^{(3)} \tag{4}$$

And assuming:

$$f_i^{(0)} = f_i^{(eq)} \tag{5}$$

Where the temporal scales are defined as:

$$t_0 = t \quad t_1 = \epsilon t \quad t_2 = \epsilon t^2 \quad t_3 = \epsilon t^3 \tag{6}$$

And the perturbative expansion of the temporal derivative operator, in powers of ϵ, is given by:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon^1\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2} + \epsilon^3\frac{\partial}{\partial t_3} \tag{7}$$

The source term ω_i, [6], [7] and [9] in the LB equation, is chosen at second order in ϵ^2, to be at the same temporal scale of the diffusive processes. Then:

$$\omega_i = -\epsilon^2 \beta \phi_i \tag{8}$$

Replacing eqs. (2)-(5), (7) and (8), in eq. (1), we get at first, second and third order in ϵ, respectively, the next set of equations:

$$\frac{\partial f_i^0}{\partial t_0} + e_i \frac{\partial f_i^0}{\partial x} = -\frac{1}{\tau} f_i^1 \tag{9}$$

$$\frac{\partial f_i^0}{\partial t_1} - \tau(1 - \frac{1}{\tau}) \left(\frac{\partial}{\partial t_0} + e_i \frac{\partial}{\partial x} \right)^2 f_i = -\frac{1}{\tau} f_i^2 - \beta \phi_i \tag{10}$$

$$\begin{aligned} \frac{\partial f_i^0}{\partial t_2} + (1 - 2\tau) \left(\frac{\partial}{\partial t_0} + e_i \frac{\partial}{\partial x} \right) \frac{\partial f_i^0}{\partial t_1} + \\ (\tau^2 - \tau + \frac{1}{6}) \left(\frac{\partial}{\partial t_0} + e_i \frac{\partial}{\partial x} \right)^3 f_i^0 = -\frac{1}{\tau} f_i^3 + (\tau) \left(\frac{\partial}{\partial t_0} + e_i \frac{\partial}{\partial x} \right) \beta \phi_i \end{aligned} \tag{11}$$

3 The Moments of the distribution

The moments of the equilibrium distribution are defined as:

$$\sum_i f_i^{(0)} = \sum_i f_i^{(eq)} = \phi + \frac{\partial \phi}{\partial t} \tag{12}$$

$$\sum_i e_i f_i^{(0)} = 0 \tag{13}$$

$$\sum_l e_{l,i} e_{l,j} f_l^{(0)} = \lambda \phi \delta_{ij} \tag{14}$$

Where ϕ is the field of the Telegraph equation and δ_{ij} is the Kronecker's delta. Also, we assume the higher orders ($k \geq 1$), in the equilibrium distribution function as:

$$\sum_i f_i^{(k)} = 0 \quad ; \quad k \geq 1 \tag{15}$$

Figure 1: Lattice-Boltzmann velocity scheme $d1q3$.

4 The Telegraph equation

Summing on i in eq. (9)

$$\frac{\partial \sum_i f_i^0}{\partial t_0} + \frac{\partial \sum_i e_i f_i^0}{\partial x} = -\frac{1}{\tau}\sum_i f_i^1 \tag{16}$$

Using eq. (13) and (15) in eq. (16), we have:

$$\frac{\partial \sum_i f_i^0}{\partial t_0} = 0 \tag{17}$$

And now summing on i in eq. (10) and multiplying by ϵ. Applying eqs. (13), (15), we have:

$$\epsilon\frac{\partial \sum_i f_i^0}{\partial t_1} - \epsilon\tau(1-\frac{1}{\tau})\frac{\partial^2}{\partial x^2}\sum_i f_i e_i e_j = -\epsilon\beta\sum_i \phi_i \tag{18}$$

Summing eqs. (17) and (18)

$$(\frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1})\sum_i f_i^0 = \epsilon\tau(1-\frac{1}{\tau})\frac{\partial^2}{\partial x^2}\sum_i f_i e_i e_j - \epsilon\beta\sum_i \phi_i \tag{19}$$

Using eq. (7) and (12), we obtain:

$$\frac{\partial}{\partial t}(\phi + \frac{\partial \phi}{\partial t}) = \epsilon\tau(1-\frac{1}{2\tau})\frac{\partial^2}{\partial x^2}\sum_i f_i e_i e_j - \epsilon\beta\sum_i \phi_i \tag{20}$$

And using eq. (14), and defining ($\epsilon\lambda(\tau - \frac{1}{2}) = 1$) and ($\epsilon\beta(b+1) = 1$), $\sum_i \phi_i = (b+1)\phi$. Here b is the dimension of the lattice-Boltzmann velocity cell. Then, we obtain the nonlinear telegraph equation:

$$\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} - \phi^n \tag{21}$$

5 Equilibrium distribution function

We use a $d1q3$, [3], see figure (1), one-dimensional velocity scheme with $e_\alpha = \{0, c, -c\}$. Then, the one particle equilibrium distribution function is defined as:

$$f_i^{(eq)} = \left\{ \begin{array}{ccc} (1-\frac{\lambda}{c^2})\phi + \frac{\partial\phi}{\partial t} & \rightarrow & i=0 \\ \frac{\lambda\phi}{2c^2} & \rightarrow & i=1 \\ \frac{\lambda\phi}{2c^2} & \rightarrow & i=2 \end{array} \right\} \tag{22}$$

The derivative $\frac{\partial\phi}{\partial t}$ is taken with the backward difference discretization scheme

$$\frac{\partial\phi(x,t)}{\partial t} \rightarrow \frac{\phi(x,t)-\phi(x,t-\Delta t)}{\Delta t} \tag{23}$$

6 Solitary wave solutions

The general telegraph equation is:

$$\frac{\partial\phi}{\partial t} + \frac{\partial^2\phi}{\partial t^2} + \phi^n = \frac{\partial^2\phi}{\partial x^2} \tag{24}$$

Using

$$\xi = x - at + \xi_0 \tag{25}$$

The derivatives change like:

$$\frac{\partial}{\partial t} = -a\frac{\partial}{\partial\xi}; \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial\xi}; \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial\xi^2}; \quad \frac{\partial^2}{\partial t^2} = a^2\frac{\partial^2}{\partial\xi^2} \tag{26}$$

replacing the derivatives and using $n = 2$, we have:

$$-a\frac{\partial\phi}{\partial\xi} + (a^2-1)\frac{\partial^2\phi}{\partial\xi^2} + \phi^2 = 0 \tag{27}$$

Now, we apply the tanh method [2]. Then, we introduce an independent variable:

$$Y(x,t) = \tanh(\xi) \tag{28}$$

The derivatives of ξ in terms of Y, are:

$$\frac{d}{d\xi} = (1-Y^2)\frac{d}{dY}, \quad \frac{d^2}{d\xi^2} = -2Y(1-Y^2)\frac{d}{dY} + (1-Y^2)^2\frac{d^2}{dY^2} \tag{29}$$

The solutions are postulated [2] as:

$$\phi(\xi) = \sum_{i=1}^{m} a_i Y^i \tag{30}$$

Now, using eqs. (29) in eq. (27), we get:

$$-a(1-Y^2)\frac{d\phi}{dY} + (a^2-1)(-2Y(1-Y^2)\frac{d\phi}{dY} + (1-Y^2)^2\frac{d^2\phi}{dY^2}) + \phi^2 = 0 \tag{31}$$

We balance of the highest-order linear term with the nonlinear term. So, we have:

$$Y^4\frac{d^2\phi}{dY^2} \to \phi^2 \to m+2 = 2m \to m = 2 \tag{32}$$

Then, eq. (31) is:

$$\phi(\xi) = a_0 + a_1 Y + a_2 Y^2 \tag{33}$$

Replacing

$$\begin{aligned} &-a(1-Y^2)(a_1+2a_2Y) - 2(a^2-1)Y(1-Y^2)(a_1+2a_2Y) \\ &+(a^2-1)(1-Y^2)^2 2a_2 + (a_0+a_1Y+a_2Y^2)^2 = 0 \end{aligned} \tag{34}$$

Doing some algebra

$$\begin{aligned} &f_1 \to (a_0 = ia, \quad a_1 = -a, \quad a_2 = 0) \\ &f_2 \to (a_0 = -ia, \quad a_1 = -a, \quad a_2 = 0) \\ &f_3 \to \Big(a_0 = (a^2-1), \quad a_1 = -a, \quad a_2 = 0\Big) \\ &f_4 \to \Big(a_0 = (a^2-1), \quad a_1 = 0, \quad a_2 = 0\Big) \end{aligned} \tag{35}$$

We get four families of solutions.
If we take $n = 3$ in eq. (27)

$$Y^4\frac{d^2\phi}{dY^2} \to \phi^3 \to m+2 = 3m \to m = 1 \tag{36}$$

Then, eq. (27) is:

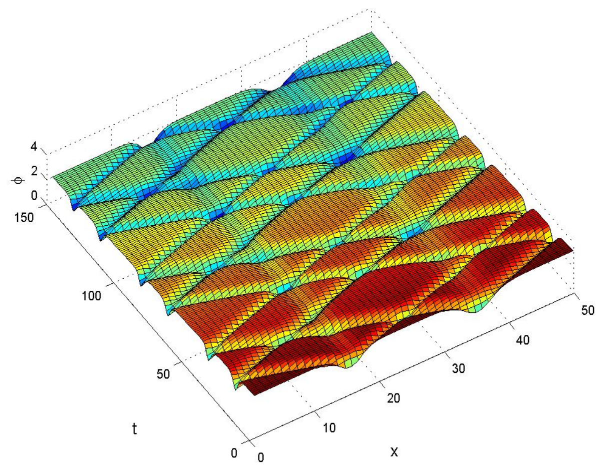

Figure 2: The spatiotemporal Lattice-Boltzmann for $\phi(x,t)$ using a $d1q3$ lattice velocity, for two initial profiles given by eq. (33).

$$\phi(\xi) = a_0 + a_1 Y \tag{37}$$

Replacing

$$\begin{aligned} &-aa_1 + aa_1 Y^2 - 2(a^2-1)a_1 Y + 2(a^2-1)a_1 Y^3 \\ &+a_1^3 Y^3 + 3a_0 a_1^2 Y^2 + 3a_0^2 a_1 Y + a_0^3 = 0 \end{aligned} \tag{38}$$

Doing some algebra, we find eight families of solutions.

$$\begin{aligned} f_1 &\rightarrow \left(a_0 = \sqrt{2(a^2-1)/3},\quad a_1 = \sqrt{2(1-a^2)}\right) \\ f_2 &\rightarrow \left(a_0 = \sqrt{2(a^2-1)/3},\quad a_1 = -\sqrt{2(1-a^2)}\right) \\ f_3 &\rightarrow \left(a_0 = -\sqrt{2(a^2-1)/3},\quad a_1 = \sqrt{2(1-a^2)}\right) \\ f_4 &\rightarrow \left(a_0 = -\sqrt{2(a^2-1)/3},\quad a_1 = -\sqrt{2(1-a^2)}\right) \\ f_5 &\rightarrow \left(a_0 = \sqrt{2(a^2-1)/3},\quad a_1 = -a/(3\sqrt{2(a^2-1)/3})\right) \\ f_6 &\rightarrow \left(a_0 = -\sqrt{2(a^2-1)/3},\quad a_1 = a/(3\sqrt{2(a^2-1)/3})\right) \end{aligned} \tag{39}$$

	A	C	F
1	1/2	-1/2	$coth(\xi) \pm cosh(\xi)$, $tanh(\xi) \pm isech(\xi)$
2	1/2	1/2	$sec(\xi) \pm itan(\xi)$
3	-1/2	- 1/2	$csc(\xi) \pm icot(\xi)$
4	1	- 1	$tanh(\xi)$, $coth(\xi)$
5	1	1	$tan(\xi)$
6	-1	-1	$cot(\xi)$

Table 1: Solutions for eq. (41)[10] .

$$f_7 \to \left(a_0 = \sqrt{2(a^2-1)/3}, \quad a_1 = (\sqrt{2(a^2-1)/3})^3/a\right)$$
$$f_8 \to \left(a_0 = \sqrt{2(a^2-1)/3}, \quad a_1 = -(\sqrt{2(a^2-1)/3})^3/a\right)$$

Also, we apply $\phi(\xi)$ to find solutions, [10]

$$\phi(\xi) = \sum_{i=1}^{m} a_i F^i \tag{40}$$

where F solves, table (1), the Riccati equation, i.e.

$$F^{'} = CF^2 + A. \tag{41}$$

here A, C are constants, table (1). Replacing and taking $n = 3$ in eq. (27), and doing the algebra, we get, $\phi(\xi) = a_0 + a_1 F$. So:

$$a_{1_1} = \frac{aC}{3}, \ \ a_{1_{2,3}} = \pm\sqrt{2(1-a^2)}C, \ \ a_{0_{1,2}} = \pm\sqrt{\frac{2CA(1-a^2)}{3}} \tag{42}$$

$$\begin{aligned} &a_{0_{3,4,5}} = (aa_{1_{1,2,3}}A)^{1/3}, \quad a_{0_{6,7,8}} = -(-1)^{1/3}(aa_{1_{1,2,3}}A)^{1/3}, \\ &a_{0_{9,10,11}} = (-1)^{2/3}(aa_{1_{1,2,3}}A)^{1/3} \end{aligned} \tag{43}$$

$$\begin{aligned} &f_1 \to (a_{0_1}, a_{1_1}), f_2 \to (a_{0_2}, a_{1_1}), f_3 \to (a_{0_1}, a_{1_2}), f_4 \to (a_{0_2}, a_{1_2}) \\ &f_5 \to (a_{0_1}, a_{1_3}), f_6 \to (a_{0_2}, a_{1_3}), f_7 \to (a_{0_3}, a_{1_1}), f_8 \to (a_{0_4}, a_{1_2}) \\ &f_9 \to (a_{0_5}, a_{1_3}), f_{10} \to (a_{0_6}, a_{1_1}), f_{11} \to (a_{0_7}, a_{1_2}), f_{12} \to (a_{0_8}, a_{1_3}) \\ &f_{12} \to (a_{0_9}, a_{1_1}), f_{14} \to (a_{0_{10}}, a_{1_2}), f_{15} \to (a_{0_{11}}, a_{1_2}) \end{aligned} \tag{44}$$

Then, we find fifteen families of solutions. Also, replacing and taking $n = 2$ in eq. (27), and doing the algebra, we get, $\phi(\xi) = a_0 + a_1 F + a_2 F^2$. Then:

$$\begin{aligned} a_2 &= 6C(a^2-1), a_1 = \frac{4a_2 Ca}{2(a^2-1)+2a_2} \\ a_{0_{1,2}} &= \pm(aa_1 A - (a^2-1)2Aa_2 C) \\ a_{0_3} &= \frac{2aa_1 C - (a^2-1)(6a_2 CA + 2a_2 C) - a_1^2}{2a_2} \end{aligned} \tag{45}$$

$$f_{16} \to (a_{0_1}, a_1, a_2), f_{17} \to (a_{0_2}, a_1, a_2), f_{18} \to (a_{0_3}, a_1, a_2) \tag{46}$$

Then, we get 108 solutions using Ricatti method.

7 Conclusions

This work presents the solution of the one-dimensional nonlinear telegraph equation using the LB, Tanh, Riccati methods. In fig. (2), we present two initial profiles, that hold the superposition principle of those nonlinear solutions eqs. (33) and (37). The solutions are:

$$\phi(\xi)_i = a_{0,i} + a_{1,i} \tanh(x - at + \xi_0) + a_{2,i} \tanh(x - at + \xi_0)^2 \tag{47}$$

$$\phi(\xi)_j = a_{0,j} + a_{1,j} \tanh(x - at + \xi_0) \tag{48}$$

Where i goes from $i = 1, ..., 4$ and j goes from $i = 1, ..., 8$.

$$\phi(\xi)_k = a_{0,k} + a_{1,k} F + a_{2,k} F^2 \tag{49}$$

$$\phi(\xi)_j = a_{0,l} + a_{1,l} F \tag{50}$$

Where k goes from $i = 1, ..., 90$ and l goes from $i = 1, ..., 18$.

Acknowledgements. This research was supported by Universidad Nacional de Colombia in Hermes project (32501).

References

[1] Oliver Heaviside, *Electrical Papers*, Chelsea Publishing Company, Incorporated, 1970.

[2] W. Malfliet and W. Hereman, The Tanh Method: I. Exact solutions of Nonlinear Evolution and Wave Equations, *Physica Scripta*, **54** (1996), 563-568. https://doi.org/10.1088/0031-8949/54/6/003

[3] D. A. Wolf-Gladrow, *Lattice-Gas Cellular Automata and Lattice Boltzmann Models: An Introduction*, Springer, Berlin, 2000. https://doi.org/10.1007/b72010

[4] P. Bathnagar, E. Gross, A. Krook, A Model for Collision Processes in Gases. I. Small Amplitude Processes in Charged and Neutral One-Component Systems, *Phys. Rev.*, **94** (1954), 511-525. https://doi.org/10.1103/physrev.94.511

[5] X. He, Li-Shi Luo, Lattice Boltzmann Model for the Incompressible Navier-Stokes Equation, *Journal of Statistical Physics*, **88** (1997), no. 3-4, 927-944. https://doi.org/10.1023/b:joss.0000015179.12689.e4

[6] Zhenhua Chai, Baochang Shi, A novel lattice Boltzmann model for the Poisson equation, *Applied Mathematical Modelling*, **32** (2008), 2050-2058. https://doi.org/10.1016/j.apm.2007.06.033

[7] G. Yan and L. Yuan, Lattice Bhatnagar-Gross-Krook model for the Lorenz attractor, *Physica D: Nonlinear Phenomena*, **154** (2001), 43-50. https://doi.org/10.1016/s0167-2789(01)00224-x

[8] Guangwu Yan, Jianying Zhang, A higher-order moment method of the lattice Boltzmann model for the Korteweg-de Vries equation, *Mathematics and Computers in Simulation*, **79** (2009), 1554-1565. https://doi.org/10.1016/j.matcom.2008.07.006

[9] J. M. Buick and C. A. Greated, Gravity in a lattice Boltzmann model, *Phys. Rev. E*, **61** (2000), 5307-5320. https://doi.org/10.1103/physreve.61.5307

[10] E.S. Fahmy, Exact solution of the generalized time-delayed Burgers equation through the improved tanh-function method. http://faculty.ksu.edu.sa/72323/Publications/Paper.pdf

Received: September 21, 2017; Published: October 15, 2017