Contemporary Engineering Sciences, Vol. 10, 2017, no. 13, 621 - 629
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ces.2017.7760

Solution of Nonlinear Equation Representing a Generalization of the Black-Scholes Model Using ADM

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Abstract

In this work the Adomian decomposition method (ADM) is used to solve the non-linear equation that represents the generalized model of Black-Scholes, that is to say that considers the volatility as a non-constant function. The efficiency of this method is illustrated by investigating the convergence results for this type of models. The numerical results show the reliability and accuracy of the ADM.

Keywords: Adomian decomposition method, Black-Scholes, Volatility

1 Introduction

Under certain assumptions, Black and Scholes published in the early 1970s their article the pricing of options and corporate liabilities in which they presented the partial differential equation of second order parabolic type [1]:

$$\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S_t^2}\sigma^2 S_t^2 + \frac{\partial c}{\partial S_t} r S_t - rc = 0, \tag{1}$$

where $C(S,t)$ is the price of a European call option, t the contract start time, S_t the price of the asset, σ the volatility and r is the interest rate.

This work focuses special attention on the problem of non-constant volatility, since it is the opinion of the experts that the fundamental variable in the Black-Scholes equation [2]. In this paper we propose an alternative view to solve Eq.(1): Adomian Decomposition Method. The advantage of this method lies in its rapid convergence, which implies a smaller number of iterations, but mainly in that it does not require any discretization, linearization or perturbation techniques which could affect the solution of the model real.

2 Description of ADM

The Adomian decomposition method is applied to a general nonlinear equation in the form [3]

$$Lu + Ru + Nu = g \tag{2}$$

Here, the linear terms are decomposed into $L + R$ and the nonlinear terms are represented by Nu. Here, L is the operator of the highest-ordered derivatives with respect to t and R is the remainder of the linear operator. Thus we get

$$Lu = -Ru - Nu + g \tag{3}$$

L^{-1} is regarded as the inverse operator of L and is defined by

$$L^{-1}(\cdot) = \int_0^t \int_0^t (\cdot)\, dt\, dt \tag{4}$$

If L is a second order operator, then L^{-1} is defined by a two-fold indefinite integral

$$L^{-1}Lu = u(x,t) - u(x,0) - t\frac{\partial u(x,0)}{\partial t} \tag{5}$$

Now, operating on both sides of Eq.(3) by using L^{-1} we obtain

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{6}$$

Therefore we have

$$u(x,t) = u(x,0) + t\frac{\partial u(x,0)}{\partial t} + L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{7}$$

The ADM represents the solution of Eq.(7) as a series

$$u(x,t) = \sum_{n=0}^{+\infty} u_n(x,t) \tag{8}$$

Here, the operator Nu (nonlinear) is decomposed as

$$Nu = \sum_{n=0}^{+\infty} A_n \tag{9}$$

Now, substituting (8) and (9) into (7) we obtain

$$\sum_{n=0}^{+\infty} u_n(x,t) = u_0 - L^{-1}R\sum_{n=0}^{+\infty} u_n(x,t) - L^{-1}\sum_{n=0}^{+\infty} A_n \tag{10}$$

where

$$u_0 = u(x,0) + t\frac{\partial u(x,0)}{\partial t} + L^{-1}g \tag{11}$$

Then, consequently we can obtain

$$\begin{aligned} u_1 &= -L^{-1}Ru_0 - L^{-1}A_0 \\ u_2 &= -L^{-1}Ru_1 - L^{-1}A_1 \\ &\vdots \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n \end{aligned} \tag{12}$$

where $u_n(x,t)$ will be determined recurrently, and A_n are the so-called polynomials (Adomian) of $u_0, u_1, \ldots, u_n$ defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{\infty}\lambda_i u_i\right)\right], \quad n = 0, 1, 2, \ldots \tag{13}$$

In this case we obtain

$$\begin{aligned} A_0 &= f(u_0) \\ A_1 &= u_1 f'(u_0) \\ A_2 &= u_2 f''(u_0) + \frac{1}{2!} u_1^2 f''(u_0) \\ &\ \vdots \end{aligned} \tag{14}$$

Now, if we introduce the parameter λ conveniently, we can obtain that

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n \tag{15}$$

where

$$N\left(u(\lambda)\right) = \sum_{n=0}^{\infty} \lambda^n A_n \tag{16}$$

Therefore, expanding by Taylor's series at $\lambda = 0$ we have

$$\begin{aligned} N\left(u(\lambda)\right) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N(u(\lambda)) \right] \lambda^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N\left(\sum_{n=0}^{\infty} \lambda^i u_i \right) \right] \lambda^n \end{aligned} \tag{17}$$

The Adomian's polynomials A_n can be calculated using the recurrence equation

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N\left(\sum_{n=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0} \tag{18}$$

If we are working with systems of differential equations (or algebraic type alike), the nonlinear terms N can be of the form

$$N = N(u_1, u_2, \ldots, u_k, \ldots)$$

where

$$u_k = \sum_{n=0}^{\infty} u_{k_i}$$

3 Generalized Black-Scholes model

In this paper we study the generalized Black-Scholes model, that is, the one that considers volatility as a function that depends on the time, the price of the underlying and the premium of the option since, in the opinion of the experts, volatility is one of the main variables in the model since real markets do not obey linear behavior [4,5].

Theorem 3.1 *The generalized Black-Scholes model with non-constant volatility*

$$\sigma^*(S, t, C_s, C_{ss}) = \frac{\sigma}{1 - \rho S \lambda(S) C_{ss}},$$

can be expressed as

$$\begin{cases} C_t(S,t) + \frac{1}{2}\sigma^2 S^2 C_{SS}(S,t)(1 + 2\rho S C_{ss}(S,t)) + rSC_s(S,t) - rC(S,t) = 0 \\ C(S,T) = f(S), \qquad S \in [0, \infty], \end{cases} \tag{19}$$

and its solution can be written roughly as:

$$\begin{aligned} C \approx\ & f(S) - \left[\frac{1}{2}\sigma^2 S^2 f''(S) + rSf'(S) - rf(S) + \rho\sigma^2 S^3 \left[f''(S)\right]^2\right] t + \left[\frac{1}{2}\sigma^2 S^2 [1 + 4\rho S f''(S)]\right. \\ & \left[\sigma^2 f''(S) + 2\sigma^2 S f'''(S) + \frac{1}{2}\sigma^2 S^2 f^{(iv)}(S) + rf''(S) + rSf'''(S) + 6\rho\sigma^2 S \left[f''(S)\right]^2 + \right. \\ & \left. 12\rho\sigma^2 S^2 f'''(S) f''(S) + 2\rho\sigma^2 S^3 \left[f'''(S)\right]^2 + 2\rho\sigma^2 S^3 f^{(iv)}(S) f''(S)\right] + r\left[\frac{1}{2}\sigma^2 S^2 f''(S) - \right. \\ & \left.\left. rSf'(S) + rf(S) + 2\rho\sigma^2 S^3 [f''(S)]^2 + \frac{1}{2}\sigma^2 S^3 f'''(S) + rS^2 f''(S) + 2\rho\sigma^2 S^4 [f''(S)]^2\right]\right] \frac{t^2}{2}. \end{aligned}$$

Proof. The linear partial differential equation proposed by Black-Scholes to value the premium of a European call option is given by [6]:

$$\begin{cases} \frac{1}{2}\sigma^2 S^2 C_{SS}(S,t) + rSC_S(S,t) + C_t(S,t) - rC(S,t) = 0 \\ C(S,T) = \max\{S - k, 0\}, \end{cases} \tag{20}$$

The volatility that characterizes a non-linear version of the Black-Scholes model is given by

$$\sigma^* = \frac{\sigma}{1 - \rho S \lambda(S) C_{ss}}, \tag{21}$$

Where σ is the traditional volatility, ρ is a constant that emerges from financial markets and λ is the price of risk, which is a positive function that describes the liquidity of the market. Now, substituting Eq.(21) into equation Eq.(20) we get

$$\begin{cases} \frac{1}{2}\sigma^2 S^2 \frac{C_{SS}(S,t)}{[1-\rho S\lambda(S)C_{ss}]^2} + rSC_S(S,t) + C_t(S,t) - rC(S,t) = 0 \\ C(S,T) = f(S). \end{cases} \tag{22}$$

Liu and Yong tested the existence and uniqueness of a classical solution of this equation. For the case $\lambda(S) = 1$, $\|\rho SC_{ss}\| < \epsilon$, with $\epsilon > 0$ and the fact that $\frac{1}{(1-F)^2} \approx 1 + 2F + O(F)^3$ we get Eq. (22), i.e.

$$\begin{cases} \frac{1}{2}\sigma^2 S^2 C_{SS}(S,t)\left(1 + 2\rho SC_{SS}(S,t)\right) + rSC_S(S,t) + C_t(S,t) - rC(S,t) = 0 \\ C(S,T) = f(S). \end{cases} \tag{23}$$

Now, according to the decomposition method we have

$$\sum_{n=0}^{\infty} C_n = C_0 - L^{-1}R\sum_{n=0}^{\infty} C_n - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{24}$$

where the A_n are given by

$$\begin{aligned} A_0 &= C_{0,SS}^2, \\ A_1 &= \frac{d}{d\lambda}\left[(C_{0,SS} + C_{1,SS}\lambda)^2\right]_{\lambda=0} = \left[2\left(C_{0,SS} + C_{1,SS}\lambda\right)C_{1,SS}\right]_{\lambda=0} = 2C_{0,SS}C_{1,SS}, \\ &\vdots \\ A_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(C_{0,SS} + C_{1,SS}\lambda + C_{2,SS}\lambda^2 + C_{3,SS}\lambda^3 + C_{4,SS}\lambda^4 + \cdots + C_{n,SS}\lambda^n\right)^2\right]_{\lambda=0}. \end{aligned}$$

In this way the terms C_n that make up the general solution are

$$\begin{aligned} C_0 &= f(S), \\ C_1 &= -L^{-1}R(C_0) - L^{-1}A_0 \\ &= -\int_0^t \left[\frac{1}{2}\sigma^2 S^2 f''(S) + rSf'(S) - rf(S)\right] d\tau - \int_0^t \rho\sigma^2 S^3 [f''(S)]^2 d\tau \\ &= -\left[\frac{1}{2}\sigma^2 S^2 f''(S) + rSf'(S) - rf(S) + \rho\sigma^2 S^3 \left[f''(S)\right]^2\right] t, \end{aligned}$$

and so on. So that an approximate analytical solution for Eq.(19) has the form:

$$
\begin{aligned}
C \approx\ & f(S) - \left[\frac{1}{2}\sigma^2 S^2 f''(S) + rSf'(S) - rf(S) + \rho\sigma^2 S^3\left[f''(S)\right]^2\right] t + \left[\frac{1}{2}\sigma^2 S^2[1+4\rho S f''(S)]\right. \\
& \left[\sigma^2 f''(S) + 2\sigma^2 S f'''(S) + \frac{1}{2}\sigma^2 S^2 f^{(iv)}(S) + rf''(S) + rSf'''(S) + 6\rho\sigma^2 S\left[f''(S)\right]^2 + \right. \\
& \left. 12\rho\sigma^2 S^2 f'''(S) f''(S) + 2\rho\sigma^2 S^3\left[f'''(S)\right]^2 + 2\rho\sigma^2 S^3 f^{(iv)}(S) f''(S)\right] + r\left[\frac{1}{2}\sigma^2 S^2 f''(S) - \right. \\
& \left.\left. rSf'(S) + rf(S) + 2\rho\sigma^2 S^3\left[f''(S)\right]^2 + \frac{1}{2}\sigma^2 S^3 f'''(S) + rS^2 f''(S) + 2\rho\sigma^2 S^4\left[f''(S)\right]^2\right]\right]\frac{t^2}{2}.
\end{aligned}
$$

The following is a simulation of the Black-Scholes model to obtain the value of a European call option in which [7]

$$
\sigma = 0.033, \quad \rho = -0.02, \quad r = 0.04, \quad f(S) = 3S + 125\sqrt{S} + 75. \tag{25}
$$

In this case, the Adomian's polynomial are given by

$$
\begin{aligned}
A_0 &= \frac{976.5625}{S^3} \\
A_1 &= \frac{38.7966t}{S^3} \\
A_2 &= \frac{0.7706t^2}{S^3} \\
&\vdots
\end{aligned}
$$

and the first terms that make up the solution are

$$
\begin{aligned}
C_0 &= 3S + 125\sqrt{S} + 75 \\
C_1 &= -1.7015 \times 10^{-2}\sqrt{S}t + 4 \times 10^{-2} S\left(3 + \frac{62.5}{\sqrt{S}}\right)t - 6.1269 \times 10^{-2} \\
C_2 &= 4 \times 10^{-2} t + 0.5\left[-3.3799 \times 10^{-4}\sqrt{S} + 4 \times 10^{-2} S\left(\frac{1.2414}{\sqrt{S}} + 0.12\right)\right]t^2 \\
&+ 4.2249 \times 10^{-4} t^2 \\
&\vdots
\end{aligned}
$$

with which the approximate analytical solution is obtained

$$\begin{aligned} C(S,t) &\approx 3S + 125\sqrt{S} + 75 - 1.7015 \times 10^{-2}\sqrt{S}t + 4 \times 10^{-2}S\left(3 + \frac{62.5}{\sqrt{S}}\right)t - 6.126 \\ &\times 10^{-2} + 4 \times 10^{-2}t + 0.5\left[-3.379 \times 10^{-4}\sqrt{S} + 4 \times 10^{-2}S\left(\frac{1.241}{\sqrt{S}} + 0.12\right)\right]t^2 \\ &+ 4.2249 \times 10^{-4}t^2 + 4 \times 10^{-2}t + 0.33333\left[-3.3569 \times 10^{-6}\sqrt{S} + 2\right. \\ &\times \left. 10^{-2}S\left(\frac{2.4660 \times 10^{-2}}{\sqrt{S}} + 4.8 \times 10^{-3}\right)\right]t^3 + 5.5949 \times 10^{-6}t^3 - 4 \times 10^{-2}t \\ &+ 0.25\left[-2.222 \times 10^{-8}\sqrt{S} + 1.333 \times 10^{-2}S\left(\frac{2.449 \times 10^{-4}}{\sqrt{S}} + 9.6 \times 10^{-5}\right)\right]t^4 \\ &+ 5.5568 \times 10^{-8}t^4. \end{aligned}$$

4 Conclusion

The Adomian decomposition method yielded an efficient technique for solving the equation representing the Black-Scholes model, since it provides an approximate analytical solution for which numerical simulations show that it is It converges rapidly, so that the model and solution technique are two valuable tools for valorisation of a European purchase option.

Acknowledgements. We would like to thank the referee for his valuable suggestions that improved the presentation of this paper and our gratitude to the Department of Mathematics of the Universidad Tecnológica de Pereira (Colombia) and the group GEDNOL.

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Received: July 26, 2017; Published: August 21, 2017