Contemporary Engineering Sciences, Vol. 9, 2016, no. 16, 781 - 790
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ces.2016.6565

An Approximate Analytical Method for Stress Analysis in a Viscoelastic Body with a Circular Inclusion Considering the Geometric and Physical Nonlinearity

Konstantin Moiseevich Zingerman

Fidesys Limited, Office 402, 1 bld. 77, MSU Science Park, Leninskie Gory
Moscow 119234, Russian Federation

Dmitry Alekseevich Shavyrin

Tver State University
Department of Applied Mathematics and Cybernetics
170100, Tver, 33, Zhelyabova st., Russian Federation

Abstract

An analytical solution method is stated for plane problem of the theory of viscoelasticity of the stress-strain state of an infinite viscoelastic body, which has a circular viscoelastic inclusion of other properties, when the stresses are set at infinity, while the strains are finite. The solution uses a small parameter method, the Laplace integral transform, and the Kolosov–Muskhelishvili complex potentials. The effect of physical nonlinearity on the stress-strain state near the viscoelastic inclusion and inside it was assessed using the analytical methods.

Keywords: viscoelasticity theory, plane problem, analytical solution, complex potentials, computer algebra, finite strains, physical nonlinearity

1 Introduction

The problems of stress analysis near inclusions (regions with other properties) are important in mechanics of composites. The analytical solutions of some problems within the framework of linear elasticity are well-known [2, 9]. Some approaches for solution of such problems with account of nonlinear effects are developed in [11, 12]. In these papers, the inclusion is considered as nonlinear, but the matrix is assumed to be linear elastic. The approximate methods of analytical solution for the problem of stress distribution around nonlinear-elastic inclusion in a nonlinear-elastic medium are developed in [13]. These methods are based on linearization procedures (the perturbation technique or the Newton method); the linearized problems are solved using the Kolosov–Muskhelishvili technique [2]. These methods permit one to solve the problems with physical and geometrical nonlinearities accounted for.
The approach to solution of such problems for viscoelastic materials is proposed in [15]. In this paper the plane quasi-static problem of stress distribution near circular inclusion is solved under finite strains. The perturbation technique, the Laplace transform, and the Kolosov–Muskhelishvili complex potentials are used for solution. The materials are assumed to be linear viscoelastic.
The further development of this approach is presented here for nonlinear viscoelastic materials.

2 Problem statement

A stress-strain state of an infinite viscoelastic body (matrix) with a circular viscoelastic inclusion is investigated. Mechanical characteristics of matrix and inclusion materials differ from one another. Normal and shear stresses are prescribed at infinity. This problem is solved in a quasi-static formulation at finite plane strains. Physical nonlinearity is determined by constitutive relations defined in the form of a nonlinear relationship between the second Piola–Kirchhoff stress tensor and the Green strain tensor, which generalizes constitutive relations for the five-constant Murnaghan potential [7, 8] for the case of viscoelasticity under finite strains. In these relations, integral operators of convolution type over time replace the elastic constants. Matrix and inclusion materials are considered compressible. It is assumed, that the perfect contact conditions take place at the interface between the matrix and the inclusion — the displacement vector and normal stresses vector are continuous. It is required to solve the problem of the quasi-static deformation of the body at a given stresses at infinity. Fig. 1 shows a scheme of loading for a body with a circular viscoelastic inclusion. The coordinate system is chosen so that the loading directions coincide with the axes of a Cartesian coordinate system x and y, and the origin coincides with the center of the inclusion.

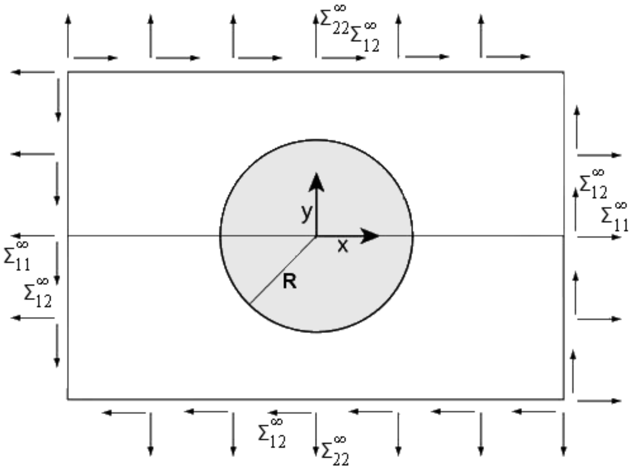

Fig. 1. Scheme of loading for a body with inclusion

The mathematical formulation of the problem is described in the coordinates of non-deformed state. Further, the following symbols are used: u — displacement vector, Ψ — deformation gradient, $\overset{0}{E}$ — Green strain tensor, $\overset{0}{\Sigma}$ — the second Piola–Kirchhoff stress tensor, σ — true stress tensor, $\overset{0}{\nabla}$ — gradient; I — the identity tensor, $\overset{0}{N}$ — the normal to the initial boundary of the inclusion. Values marked with *M* index are related to the matrix, and the *B* index is used for inclusion. If the indices are not specified, the expression is related both to the matrix and to the inclusion.

The equilibrium equation has the following form [4, 7, 15]:

$$\overset{0}{\nabla}\cdot\left[\overset{0}{\Sigma}\cdot\Psi\right]=0, \qquad (1)$$

here

$$\overset{0}{\Sigma}=(1+\Delta)\Psi^{*-1}\cdot\sigma\cdot\Psi^{-1}. \qquad (2)$$

The viscoelasticity law is as follows:

$$\overset{0}{\Sigma}(t) = I\int_{-\infty}^{t} \lambda(t-\tau)\left(\frac{\partial \overset{0}{E}(\tau)}{\partial \tau} : I\right) d\tau + \int_{-\infty}^{t} 2G(t-\tau)\frac{\partial \overset{0}{E}(\tau)}{\partial \tau} d\tau +$$

$$+ I\int_{-\infty}^{t}\left\{3C_3(t-\tau)\left(\frac{\partial\left[\overset{0}{E}(\tau) : I\right]^2}{\partial \tau}\right) + C_4(t-\tau)\left(\frac{\partial\left[\overset{0}{E}(\tau)\right]^2}{\partial \tau} : I\right)\right\} d\tau +$$

$$+ \int_{-\infty}^{t}\left\{2C_4(t-\tau)\left(\frac{\partial\left[\left(\overset{0}{E}(\tau) : I\right)\overset{0}{E}(\tau)\right]}{\partial \tau}\right) + 3C_5(t-\tau)\left(\frac{\partial\left[\overset{0}{E}(\tau)\right]^2}{\partial \tau}\right)\right\} d\tau, \qquad (3)$$

Here, relaxation kernels λ — volumetric and G — shearing, and C_3, C_4, C_5 — non-linear relaxation kernels.

$$\lambda(t) = \lambda_0 + \lambda_1 e^{-\alpha t},\ G(t) = G_0 + G_1 e^{-\beta t},$$
$$C_j(t) = C_{j_0} e^{-f_j t} \quad (j = 3,4,5). \qquad (4)$$

Modules λ_i and G_i at $i = 0{,}1$ and α, β, C_{3_0}, C_{4_0}, C_{5_0}, f_3, f_4 and f_5 take different values for the matrix and for the inclusion.
In the particular case, in which $C_j(t) = 0$ $(j = 3,4,5)$. the model of viscoelasticity is the model of standard linear solid [1, 3] generalized for finite strains.
Kinematic relations are as follows

$$\overset{0}{E} = \frac{1}{2}\left(\Psi \cdot \Psi^* - I\right),\ \Psi = I + \overset{0}{\nabla} u. \qquad (5)$$

The formulation of the problem also includes conditions at infinity

$$\left.\overset{0}{\Sigma}_M\right|_{\infty} = \overset{0}{\Sigma}{}_M^{\infty}, \qquad (6)$$

as well as the continuity conditions for the displacement vector u and for the normal stress vector $\overset{0}{N} \cdot \overset{0}{\Sigma} \cdot \Psi$ at the interface of matrix and inclusion

$$\left.\overset{0}{N} \cdot \overset{0}{\Sigma}_M \cdot \Psi_M\right|_{\Gamma} = \left.\overset{0}{N} \cdot \overset{0}{\Sigma}_B \cdot \Psi_B\right|_{\Gamma},$$
$$\left.u_M\right|_{\Gamma} = \left.u_B\right|_{\Gamma}. \qquad (7)$$

3 Solution method

To solve the problem, the perturbation method (the method of small parameter) is used. A small parameter μ is chosen as follows

$$\mu = \max_{i,j} \left| \overset{0}{\Sigma}{}^{\infty}_{ij} \right| / G_0^M \tag{8}$$

and for all the values involved in the formulation of the problem a series expansion is written by this parameter . For example, for a displacement vector u such an expansion can be written as follows

$$u = u^{(0)} + u^{(1)} + \ldots u^{(j)} \sim \mu^{j+1} \tag{9}$$

Here and below, the superscript in parentheses denotes the approximation number. As a result, the solution of the nonlinear problem is reduced to sequential solution of linearized problems.

A solution of the linearized problem for each approximation is determined by Kolosov–Muskhelishvili method [9, 13, 14] using the algorithms described in [4 - 6, 13]. The Laplace integral transform [3] is used for solution. For external loads and relaxation kernels we use their Laplace transforms. Complex potentials are determined in the form of the truncated Laurent series in terms of powers of $z = x + iy = re^{i\theta}$. Substituting the potentials, presented in the form of series, to the boundary conditions (conditions at infinity and an ideal contact conditions), we obtain a system of linear algebraic equations to find expressions for the terms of the series as functions of the Laplace transforms of loads and relaxation kernels. Most of the coefficients obtained are equal to zero. By substituting the expressions for potentials to formulas for stresses and displacement [9], we can obtain expressions for the Laplace transforms of stresses and displacements for each approximation.

The following notation is used below. The tilde is used to mark up the parts of the first approximation of corresponding values, which are determined by the zero approximation.

For the first approximation the solution is as follows. If a variable s is used as the argument of a function, it is assumed that this function is the Laplace transform. The operator L denotes the direct Laplace transform.

1. The deformation gradient $\Psi^{(0)}$ is determined as follows:

$$\Psi^{(0)} = \overset{0}{\nabla} u^{(0)}. \tag{10}$$

2. The relative change in volume during the transition of the body from the initial to the final state is determined as follows:

$$\Delta^{(0)} = \Psi^{(0)} : I \tag{11}$$

3. The Green strain tensor $\overset{0}{\tilde{E}}{}^{(1)}$ is determined as follows:

$$\overset{0}{\tilde{E}}{}^{(1)} = \frac{1}{2} \Psi^{(0)} \cdot \Psi^{(0)*}. \tag{12}$$

4. The Laplace transformation is applied to the Green strain tensor, and the Laplace transform of correction for the second-order effects for the second Piola–Kirchhoff stress tensor $\overset{0}{\tilde{\Sigma}}{}^{(1)}$ is determined:

$$\overset{0}{\tilde{\Sigma}}{}^{(1)}(s) = I\lambda(s)\left(\overset{0}{\tilde{E}}{}^{(1)}(s):I\right) + 2G(s)\overset{0}{\tilde{E}}{}^{(1)}(s) + \\ + I\left\{3C_3(s)L\left(\left[\overset{0}{E}{}^{(0)}:I\right]^2\right) + C_4(s)L\left(\left[\left(\overset{0}{E}{}^{(0)}\right)^2:I\right]\right)\right\} + \\ + \left\{2C_4(s)L\left(\left[\overset{0}{E}{}^{(0)}:I\right]\overset{0}{E}{}^{(0)}\right) + 3C_5(s)L\left(\left(\overset{0}{E}{}^{(0)}\right)^2\right)\right\}. \tag{13}$$

5. The fictitious mass forces vector $f^{(1)}$ is determined as follows:

$$f^{(1)}(s) = -\overset{0}{\nabla}\cdot\left(\overset{0}{\tilde{\Sigma}}{}^{(1)}(s) + L\left[\overset{0}{\Sigma}{}^{(0)}\cdot\Psi^{(0)}\right]\right) \tag{14}$$

6. A particular solution of the inhomogeneous equation using the vector $f^{(1)}$ is obtained as follows:

$$u_{н.}^{(1)}(s) = \frac{1}{4G(s)(\lambda(s)+2G(s))}\cdot \\ \cdot\left[(\lambda(s)+3G(s))\int\int f^{(1)}(s)dzd\bar{z} - (\lambda(s)+G(s))\int\int \bar{f}^{(1)}(s)dzd\bar{z}\right] \tag{15}$$

7. The correction for the second-order effects for the strain tensor $\overset{0}{E}{}^{(1)}$ is determined as follows:

$$\overset{0}{E}{}^{(1)}(s) = \overset{0}{\tilde{E}}{}^{(1)}(s) + \frac{1}{2}\left(\overset{0}{\nabla}u_{н.}^{(1)}(s) + \overset{0}{\nabla}u_{н.}^{(1)*}(s)\right). \tag{16}$$

8. The generalized stress tensor $\overset{0}{\Sigma}{}^{(1)}$ is as follows:

$$\overset{0}{\Sigma}{}^{(1)}(s) = \overset{0}{\tilde{\Sigma}}{}^{(1)} + \lambda\left(\overset{0}{\nabla}\cdot u_{н.}^{(1)}(s)\right)I + G\left(\overset{0}{\nabla}u_{н.}^{(1)}(s) + u_{н.}^{(1)}(s)\overset{0}{\nabla}\right). \tag{17}$$

9. The stress tensor $\overset{0}{\Sigma}{}_{M}^{\infty(1)}$ at infinity is determined as follows:

$$\overset{0}{\Sigma}{}_{M}^{\infty(1)}(s) = -\left(\overset{0}{\tilde{\Sigma}}{}^{(1)}(s) + L\left[\overset{0}{\Sigma}{}^{(0)}\cdot\Psi^{(0)}\right]\right)\Bigg|_{\infty} \tag{18}$$

10. Using complex potentials, we will find the solution of the homogeneous system of equations similar to the zero approximation. These potentials are represented as follows:

$$\varphi_M^{(1)}(z) = \frac{\overset{0}{\Sigma}{}_{\mathbf{M}I}^{\infty\,(1)}}{4}\sum_{k=-1}^{\infty} a_k^{(1)}z^{-k}\ ,\ \psi_M^{(1)}(z) = \frac{\overset{0}{\Sigma}{}_{\mathbf{M}II}^{\infty\,(1)}}{2}\sum_{k=-1}^{\infty} b_k^{(1)}z^{-k} \\ \varphi_B^{(1)}(z) = \frac{\overset{0}{\Sigma}{}_{\mathbf{M}I}^{\infty\,(1)}}{4}\sum_{k=0}^{\infty} c_k^{(1)}z^{k}\ ,\ \psi_B^{(1)}(z) = \frac{\overset{0}{\Sigma}{}_{\mathbf{M}II}^{\infty\,(1)}}{2}\sum_{k=0}^{\infty} d_k^{(1)}z^{k}\ . \tag{19}$$

By substituting the potentials to the boundary conditions and solving of a system of linear algebraic equations, we will find the expressions for coefficients of the

series through loads and relaxation kernels. Calculations have shown that in this case it is possible to amount to a finite number of summands in sums (19) to $k = 9$ inclusively with no loss of accuracy.
By substituting the expressions for potentials (19) to formulas [9, 10] relating the potentials with stresses and displacements we obtain expressions for the Laplace transforms of stresses and displacements. By applying the inverse Laplace transform we will find the solution as a function of time.
The solution is found in the analytical form, but considering the awkwardness of expressions, let us confine by the numerical results only.

4 Calculation results

In order to solve the problem, we developed the software in Maple Computer Algebra System environment and obtained expressions for the stress and displacement in analytical form as functions of coordinates and time. The first two approximations were calculated.
Calculations were performed for the following values of viscoelastic constants:

$\alpha_M = \beta_M = \alpha_B = \beta_B$, $\lambda_0^M / G_0^M = 1.5$, $\lambda_1^M / G_0^M = 14$, $G_1^M / G_0^M = 4$, $C_{3_0}^M / G_0^M = -6$, $C_{4_0}^M / G_0^M = -8$, $C_{5_0}^M / G_0^M = -3$, $\lambda_0^B / G_0^M = 15$, $\lambda_1^B / G_0^M = 140$, $G_0^B / G_0^M = 10$, $G_1^B / G_0^M = 40$, $C_{3_0}^B / G_0^M = -60$, $C_{4_0}^B / G_0^M = -80$, $C_{5_0}^B / G_0^M = -30$, $f_3^M = f_4^M = f_5^M = f_3^B = f_4^B = f_5^B = \alpha_M / 10$. In this case, the inclusion is more

rigid than the matrix. The loads along the x and y axes are simultaneously applied at infinity at the moment $t = 0$: tension stress $\overset{0\ \infty}{\Sigma}_{11} = G_0^M$ along the x axis and compression stress $\overset{0\ \infty}{\Sigma}_{22} = -G_0^M$ along the y axis.

Figure 2 shows the distribution of stress Σ_{11} along the x axis at time points $t \cdot \alpha_M = 0$ and $t \cdot \alpha_M = 30$. Fig. 3 shows a stress variation with time at the center of inclusion. The solid line in the graph corresponds to the linear solution, the dashed line — to the solution with account for geometric nonlinearity, and the dashed-dotted line — to the solution decision with account for physical nonlinearity.

The correction for counting the non-linear effects for stress tensor components does not exceed 44 %, and for the displacement vector (not shown) — 5 %.
The distribution of stresses and strains within the inclusion in the zero approximation is uniform, which is consistent with the results of solving the linear problem of elastic inclusion in an elastic medium [2].

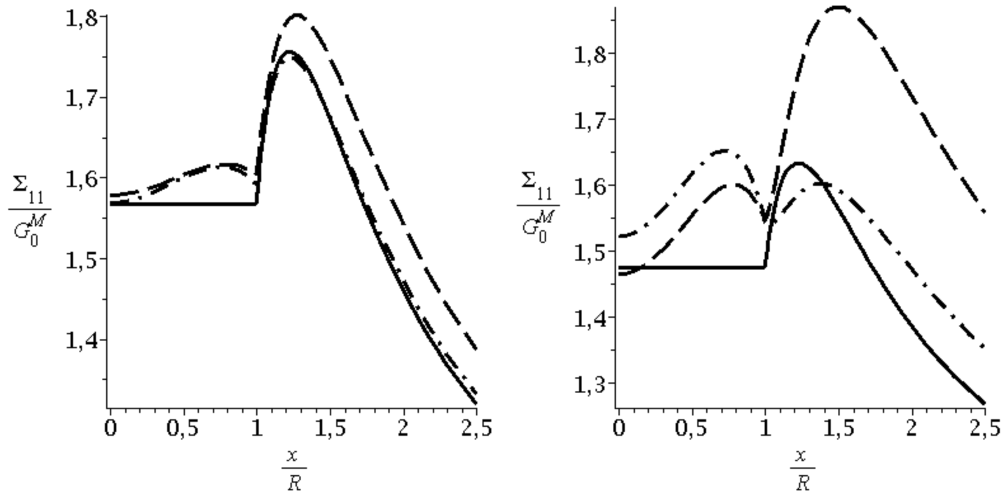

Fig. 2. Distribution of Σ_{11} along the *x* axis at time points $t \cdot \alpha_M = 0$ and $t \cdot \alpha_M = 30$

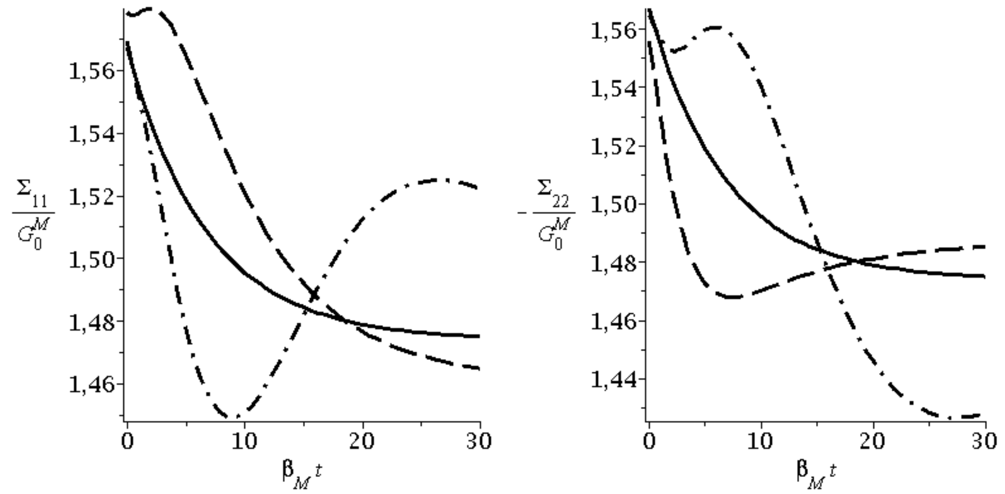

Fig. 3. Change of stresses Σ_{11} and Σ_{22} at the center of inclusion

5 Conclusion

Thus, method, algorithm and software for the approximate analytical solution of the plane problem of the quasi-static deformation of an infinite viscoelastic body with a circular viscoelastic inclusion are developed for finite strains. The estimation of non-linear effects is done; the study of change of stresses and strains against the time is performed. The results of solution of this problem can be used to describe the effective characteristics of viscoelastic composites, including rubbers with nanoscale filler particles, as well as for the analysis and verification of numerical solutions.

Acknowledgements. This work was financially supported by the Ministry of Education and Science of the Russian Federation in the framework of the agreement No. 14.579.21.0076 for a grant (unique identifier of the project RFMEFI57914X0076). Investigations were carried out within the Fidesys company — a grantee of the Ministry of Education and Science of the Russian Federation.

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Received: June 13, 2016; Published: July 26, 2016