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On the Stability of Compressed Plate

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Abstract

The problem of the stability of compressed elastomeric plate in the theory of thin plates and shells is considered. The stability of the flat form of equilibrium is investigated. The bifurcation forms of equilibrium are built with numerical methods. The theoretical results are compared with the data of field experiment.

Keywords: elastomers, the stability, bifurcation, the elastic potential

1 Introduction

Professor V.V. Novozhilov was the founder of the non-linear theory of elasticity [1]. In his works the main provisions of nonlinear elasticity theory and the concept of physical and geometrical nonlinearities were formulated. The many theoretical and experimental works are devoted to the study of the elastic loss of stability. In theoretical studies the definition of critical loads and geometries for the above-critical region is carried out. Investigations of the stability of plates and shells with "linear" the physical properties of the material were considered in the works [2-8]. Contemporary methods are applied for the evaluation of critical loads and the numerical solution of nonlinear problems in the nonlinear theory of elasticity [9-14]. In the works the influence of external aggressive environments for the stress-strain state of elastic bodies were considered [15-21]. In the programming environment for mathematical package the stress-deformation state of plates, shells and rods for various types of physical nonlinearity is investigated [8]. Simultaneously with theoretical studies the various methods and algorithms for constructing numerical solutions of nonlinear equations in a neighborhood of the critical points are developed [22-26]. The problem of finding the critical load for compressed rod is not trivial.

The values of critical load are obtained for various cases of the boundary conditions with different approaches. Hooke's law for describe the mechanical properties of the elastic material is applied. For materials with nonlinear mechanical properties of the form of the stress-strain state in the problem of compression rods and plates may be different from results of the solution of the linear problem [3-6, 9-14]. The mechanical properties of various materials at large deformations are nonlinear [27-31]. The best description of their properties provides the potential Ogden [32, 33]. In the work from positions non-linear theory of thin shells the task of finding the critical load to the compressed plate with nonlinear material properties is resolved.

2 Formulation of the problem

In the coordinate system (*x*,*y*,*z*) the problem of compression the infinite length the plate with the load, having a linear density *P*, which applied on the edge is considered (Fig. 1).

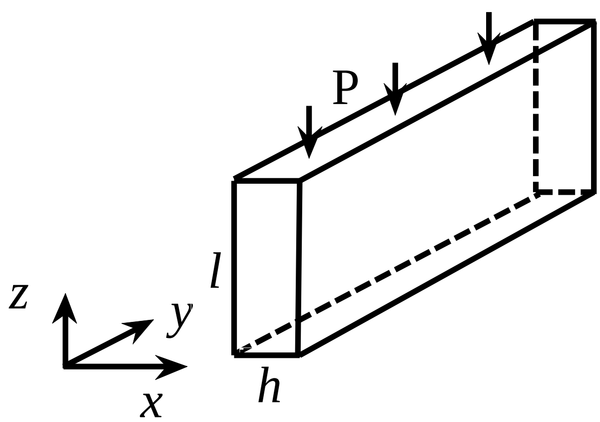

Figure.1. The plate with the load

For small loads the plate is flat. When the load is greater than the critical value, the bend of the plate will occur. The system of resolving equations for the planar infinite length the plate, l width and h^0 thickness, compressible in the plane of the load on the edge with a linear density *P*, takes the form [9, 34]

$$T_s' + \varphi' T_n = 0,\ T_n' - \varphi' T_s = 0,\ M' - \lambda_s T_n = 0, \tag{1}$$
$$x' = \lambda_s \cos\varphi, \quad z' = -\lambda_s \sin\varphi.,$$

where T_n - shear forces, and T_s - the force acting in the direction of meridian, M_s - bending moment; *x* and *z* - the coordinates of points the middle surface, φ - the angle between the axis *z* and the normal to the median surface (Fig. 2 - the force distribution in the middle of the surface).
The differentiation along the arc of meridian of the middle surface is carried out $(()' = d()/ds)$.
Horizontal T_x and vertical T_z forces are calculated by the formulas:

$$T_z = T_n \cos\varphi - T_s \sin\varphi, \quad T_x = T_n \sin\varphi + T_s \cos\varphi.$$

The relationship between stress and strain using the elastic potential $\Phi = \Phi(\lambda_1, \lambda_2, \lambda_3)$ is determined. For isotropic material the elastic potential is a function of the main multiplicities of elongations λ_1, λ_2 and λ_3. For the case of compressed plate $\lambda_1 = \lambda_s$, $\lambda_2 = 1$, $\lambda_3 = h/h^0$, where h^0 - the thickness of the no deformed plate and h - the thickness for the deformed state.
In the case of incompressible material the condition of incompressibility: $\lambda_1\lambda_2\lambda_3 = 1$ should be satisfied.

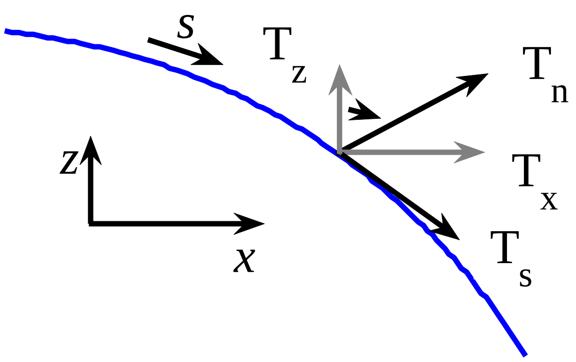

Figure.2. The force of distribution in the middle of the surface

As elastic potential for an incompressible material is reviewed [9, 10, 32]

$$\Phi = \frac{2\mu}{n^2}\left(\lambda_1^n + \lambda_2^n + \lambda_3^n - 3\right),$$

where μ - the "linear" shear modulus, and *n*- no dimension parameter. For this potential, taking into account the condition of incompressibility, the force T_s and bending moment M [10] is expressed in terms of λ_s and φ'.

$$T_s = \frac{2\mu \overset{0}{h}}{n}\frac{1}{\lambda_s}\left(\lambda_s^n - \lambda_s^{-n}\right),\; M = \frac{1}{3}\mu \overset{0}{h}{}^3 \lambda_s^{-n-4}\varphi'. \qquad (2)$$

The conditions of fixing with hinge the plate on the fixed edge $x = 0$ and the horizontal displacement of the edge $x = l$ under load P are considered for boundary conditions:

when $s = 0$: $x = 0$, $z = 0$, $M = 0$, (3)

when $s = l$: $x = \lambda l$, $z = 0$, $M = 0$. (4)

For a given value of λ the load is calculated by the formula: $P = T_s(\lambda_s = \lambda)$. The system of the resolving equations (1) - (2) with the boundary conditions (3) - (4) has a solution:

$x = \lambda s$, $z = 0$, $\varphi = 0$, $M = 0$, $T_n = 0$, $\lambda_s = \lambda = const$,

$T_s = T_s(\lambda) = T = const = P$ (5)

In addition, for compressed plate must satisfy the inequality $\lambda < 1$.

3 On the stability of the plate

Assume that the approximate the solution (5) will be exist:

$\varphi = \delta\varphi$, $M = \delta M$, $T_s = T + \delta T_s$, $x = \lambda s + \delta x$, $z = \delta z$, $\lambda_s = \lambda + \delta\lambda$,

where $\delta\varphi$, δM, δT_s, δx, $z = \delta z$, $\delta\lambda$ - small quantities.

Then, in the linear approximation, the first three equations in (1) are written in the form of:

$$\delta T_s' = 0,\; \delta T_n' - \varphi' T_s = 0,\; \delta M' - \lambda \delta T_n = 0. \qquad (6)$$

Thus, from the expression (2) follows: $\delta M = \frac{1}{3}\mu \overset{0}{h}{}^3 \frac{1}{\lambda^{n+4}}\delta\varphi'$

From the third equation in (6) follows the equation for finding φ:

$\delta\varphi'' + \mu^2 \delta\varphi = 0$, where $\mu^2 = -\frac{6}{n \overset{0}{h}{}^2}\lambda^{n+4}\left(\lambda^n - \lambda^{-n}\right)$

The nontrivial solution of this equation $\delta\varphi = A\sin\mu s$ satisfying the conditions $\delta\varphi(0) = \delta\varphi(l) = 0$, will be exist for values μ, that satisfy the condition $\mu = k\pi / l$ ($k = 1,2...$) or

$$\frac{6}{n}\lambda^{n+4}\left(\lambda^n - \lambda^{-n}\right) = \left(\frac{k\pi \overset{0}{h}}{l}\right)^2 \qquad (7)$$

The left part of this equation has the extremum on $\lambda = \frac{1}{(1+n/2)^{1/2n}} < 1$.

Therefore, for sufficiently the large values of h^0 / l the equation (9) will haven't solutions.

For the case of $n=2$ the extremum is achieved with $\lambda \approx 0.84$ and when the inequality $\frac{\overset{0}{h}}{l} > \frac{\sqrt{0.75}}{\pi} = 0.28$ is valid, the linear approximation of system (1) will have only the trivial solution. Thus, in this model of the compressed plate the stability cannot lose.

4 The numerical solution

The solution of equations (1) - (2) with the boundary conditions (3) - (4) is carried out with numerical methods. The solution of the boundary value problem is reduced to the solution of the Cauchy problem for given conditions on one edge. The integration of differential equations is carried out with the method of Runge-Kutta [12, 13, 22-25]. The boundary conditions on the second edge are applied with the method of "shooting".
Some results of numerical solutions are shown in Fig. 3 as a function "the load - the angle of rotation at the edge". The curve $k=0$ for the rectilinear equilibrium form, the curve $k=1$ for the equilibrium form with the loads greater than the first, but less than the second critical, the curve $k=2$ for the equilibrium form with the loads greater than the second - critical, but less than a third. Thus, with the rectilinear form of equilibrium (Fig. 3, $k=0$) "the bending" the bifurcation forms of equilibrium will be exist. As shown the numerical experiments, "coordinates" of bifurcation points (the critical values of λ) are determined from the relation (7).

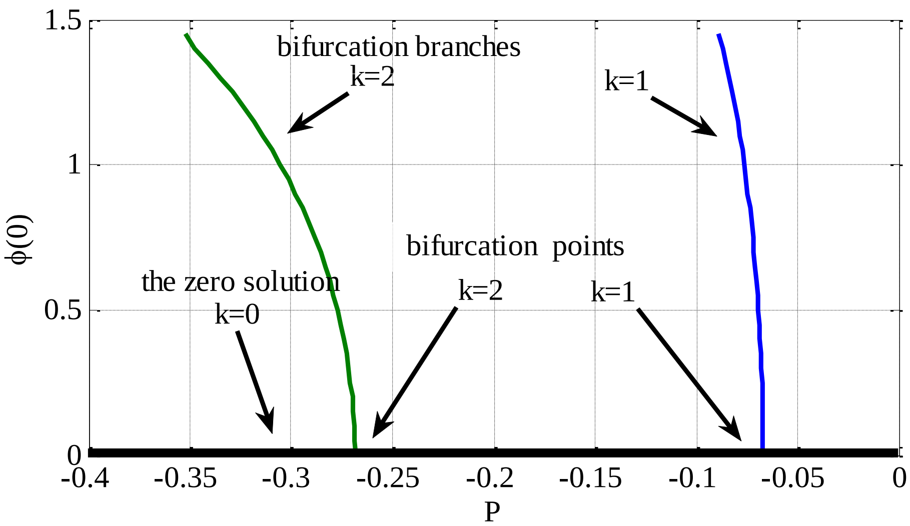

Figure.3. Dependence of the load to the angle of rotation on the edge

The problem of finding solutions of the bifurcation branches is not trivial. In a neighborhood of points the system of resolving equations are badly defined. As a rule, numerical methods for solving the nonlinear differential equations are based on the reduction of the solution "differential task" to an algebraic task [9, 10, 14].

The iteration methods for solving the systems of nonlinear algebraic equations don't provide convergence to the required solution in a neighborhood of these points. In this case, a preliminary assessment of the region of existence of bifurcation points is required. Special methods and algorithms for constructing solutions, which usually are not provided in most math packages are applied. One of the methods implemented for solving the system of equations (1) - (4) is de-idealization of initial system of equations [9, 10, 14].
The method is based on the idea that the solution for the non-ideal system will be tends to the solution of the ideal system when non-ideal parameters tends to zero.

5 The experiment on compression of the rectangular plate

For compare the theoretical results with the experimental is set up the experiment for axial compression the long rubber plate with the transverse dimensions 272×160 mm and the thickness of 40 mm. The plate is mounted vertically on the fixed platform, so that one edge of the plate remained fixed. On the second edge of plate the load is applied along the entire length. Compression of the plate is carried out along the side, with the length 160 mm.
The edges of the plates are not rigidly fixed. We suppose that the plate is placed on the hinges. For small deflection the form of upper edge of plate is almost rectilinear.
When the deflection approximately $0.05l$ the plate is very convex. This deformation of the plate with a loss of stability of a rectilinear form of equilibrium is associated. Snapshot of deformed plates during loading is performed. After processing snapshots, the form of the middle surface is built. The calculation of form is based on the method described above. The shear modulus is determined from experiments on uniaxial stretching patterns [27]. The fig. 4 shows the form of middle surface (solid line). The experimental data are marked with a symbol *.

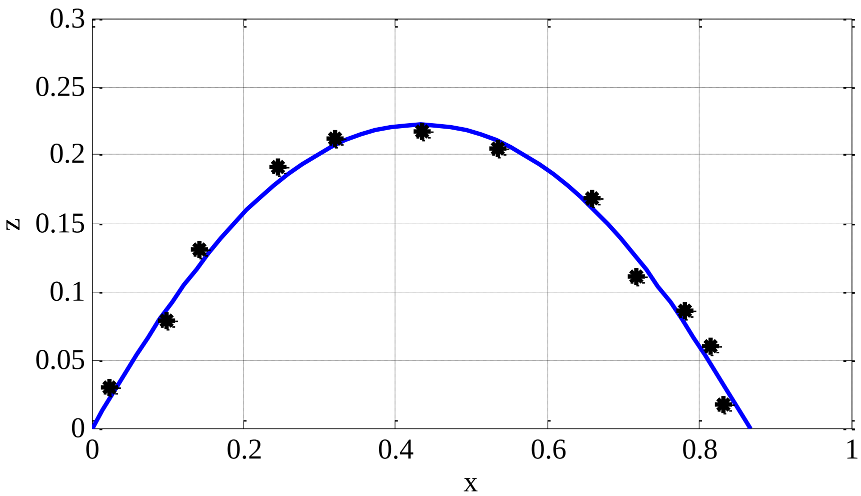

Figure.4. Form of middle surface

In the experiments, "the bifurcation form of equilibrium" is implemented.
The form of bending in the first approximation is a sinusoid for $k = 1$ (7).

6 Conclusion

In consequence of consideration of the nonlinear material can be obtained results different from the results for the linear version of the law of elasticity (Hooke's law). The algorithms used for solving problems of linear theory of shells for building solutions in the above-critical region can be applied.
The boundaries of the area the existence of solutions is necessary to pre-estimate.

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