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Harmonic Wave Excitation in a Semi Infinite Medium

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Abstract

In the present paper, we obtained the most general solution of the one-dimensional partial differential equation for harmonic wave excitation in a semi infinite medium, by computing the symmetry groups using the general prolongation formula for their infinitesimal generators of a groups of transformations based on the technique given by Olver([4], [5]) in explicit form. In the recent year the authors Bao, Wei and Zhao [1], Kurus [3], Ramahi and Seydou [6], Ranosava [7] Sneddon and Read [8] worked for the solution of Helmholtz equation.

Keywords: Axial Displacement, Wave Number, Commutation-Relation

1. Introduction

1.1 The harmonic wave excitation problem in a semi-infinite medium:

The harmonic wave excitation problem in a semi-infinite medium the governing equation is the Helmholtz equation in the standard form as

$$u_{xx} + k^2 u = 0 \qquad (1.1.1)$$

Where u is an axial displacement of the rod and k is the wave number, [2].

2. Main Result

Let us consider one-dimensional Helmholtz equation for harmonic wave excitation in a semi-infinite medium $u_{xx} + k^2 u = 0$ (2.1)
which is the second order differential equation with one independent variables and one dependent variable.

Lemma 1: Let $v = \xi(x, u)\, \partial_x + \phi(x, u)\, \partial_u$ (2.2)
be a symmetry of Helmholtz equation (2.1). Then the smooth coefficient functions ϕ and ξ are given by $\phi = \beta\, u + \alpha$ where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are functions and ξ independent of u.

Proof: Firstly we determine the second prolongation of v (see Oliver[5]),

$$pr^{(2)}v = v + \phi^{x}\, (\partial/\partial u_x) + \phi^{xx}\, (\partial/\partial u_{xx}) \qquad (2.3)$$

By using infinitesimal criterion of invariance the equation (2.1) takes the form

$$\phi^{xx} + \mathrm{k}^2\, \phi = Q(u_{xx} + \mathrm{k}^2\, u\,) \qquad (2.4)$$

where $Q(x, u^{(2)})$. By substituting the values of ϕ^{xx} and ϕ in equation (2.4) and equating the coefficients of the terms in the first and second order partial derivatives of u, the determining equations for the symmetry group of the one-dimensional Helmholtz equation are found as follows see Table 1

Table 1: The Determine Table

Monomial	Coefficient	Equation Number
1	$\phi_{xx} + \mathrm{k}^2\phi - \mathrm{k}^2 Qu = 0$	1
u_x	$\left(2\phi_{xu} - \xi_{xx}\right) = 0$	2
u_x^2	$\phi_{uu} - 2\xi_{xu} = 0$	3
u_x^3	$-\xi_{uu} = 0$	4
u_{xx}	$\phi_u - 2\xi_x = Q$	5
$u_x u_{xx}$	$-3\xi_u = 0$	6

The requirement for equation (6) is that ξ independent of u, equation (3) gives $\phi = \beta\, u + \alpha$ where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are functions.

Lemma 2: The most general infinitesimal symmetry of the one-dimensional Helmholtz equation in a semi-infinite medium has coefficient function of the form $\xi = c_1$ and $\phi = (c_2 / \mathrm{k}^2)\, u + \alpha$ where c_1 and c_2 are arbitrary constant and α is an arbitrary solution of the Helmholtz equation.

Proof: Using lemma 1, the equation (1) gives $\beta = Q$, form the equation (5) we found $\xi = c_1$ and from equation (2) we get $\beta_x = c_1 / \mathrm{k}^2$. Thus most general infinitesimal symmetry of the one-dimensional Helmholtz equation in a semi-infinite medium has coefficient function of the form $\xi = c_1$ and $\phi = (c_2 / \mathrm{k}^2)\, u + \alpha$ where c_1 and c_2 are arbitrary constant and α is an arbitrary solution of (2.1)

Lemma 3: The Lie algebras of infinitesimal symmetries of the Helmholtz equation in a semi-infinite medium is spanned by the two vector fields $v_1 = \partial_x$, $v_2 = (1/\ k^2)\ u\partial_u$ and the infinite-dimensional sub-algebra $v_\alpha = \alpha\partial_u$ where α is an arbitrary solution of the Helmholtz equation.

Proof: The proof is evident by using lemma 1 and lemma 2.

Theorem 1: The symmetry Lie algebra ϑ of the Helmholtz equation in a semi-infinite medium is spanned by the set of vector field v_1, v_2 and v_α.

Proof: Using lemma 3, the commutation relation betweens these vector fields are given by the following see Table 2

Table 2: The Commutation - Relation Table

	v_1	v_2	v_α
v_1	0	0	v_{α_x}
v_2	0	0	$-\left(1/k^2\right)v_\alpha$
v_α	$-v_{\alpha_x}$	$\left(1/k^2\right)v_\alpha$	0

by using commutation-relation table it is easy to derive ϑ is a Lie algebra with Lie bracket operation.

Lemma 4: The one-parameter groups G_i (i =1,2, α) generated by the v_i are given as follows G_1: $(x + \varepsilon, u)$, G_2: $(x, e^{\left(\varepsilon/k^2\right)}u)$, G_α: $(x, u + \varepsilon\alpha)$ where each G_i is a symmetry group.

Proof: The one parameter group generated by v_i is given by $\exp(\varepsilon v_i)(x, u) = \left(\tilde{x},\tilde{u}\right)$, so by using lemma 3 it is obvious.

Theorem 2: The solution of Helmholtz equation by using its different symmetry groups are given by $u^{(1)} = f(x-\varepsilon)$, $u^{(2)} = e^{\left(\varepsilon/k^2\right)}f(x)$, $u^{(\alpha)} = f(x) + \varepsilon\alpha$ where $u = f(x)$ be an assume solution to the Helmholtz equation, ε is any real number and α any other solution to the Helmholtz equation.

Proof: Putting the value of x and u in solution $u = f(x)$ and using $(x, u) = \left(\tilde{x},\tilde{u}\right)$ for each G_i and using lemma 4 we get the above function which are the solution to the Helmholtz equation

3 Conclusion

In our investigation the symmetry group G_2 and G_α reflects the linearity of the Helmholtz equation. The group G_1 is space translation symmetry group. At the end the most general solution that we can obtain from a given solution $u = f(x)$, by group transformations is in the form given below

$$u = e^{(\varepsilon_2/k^2)} f(x - \varepsilon_1) + \alpha \tag{3.1}$$

where ε_1, and ε_2 are real constant and α be an arbitrary solution to the one-dimensional Helmholtz equation for the harmonic wave excitation in a semi-infinite medium. The most general solution (3.1) gives us all possible most general infinitesimal symmetries of Helmholtz equation (2.1).

4 Special Case

If we take k = 1 then equation 3.1 reduces to $u = e^{\varepsilon_2} f(x - \varepsilon_1) + \alpha$ (4.1)

where ε_1, and ε_2 are real constant and α be an arbitrary solution to the one-dimensional Helmholtz equation for the harmonic wave excitation in a semi-infinite medium.

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