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State Analysis and Optimal Control of Linear Time-Invariant Scaled Systems Using the Chebyshev Wavelets

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Abstract

A new approach to find the optimal solution of linear time-invariant scaled systems using the Chebyshev wavelets is proposed. The operational matrix of stretch is derived and together with the operational matrix of integration are used to change the system of state equations into a set of algebraic equations which can be solved using a digital computer. The approximated optimal solution with respect to a quadratic cost function is calculated by solving only these linear algebraic equations. The main feature of this article over other possible works is that with a relatively low number of terms we have accurate results. Numerical examples are given to support this claim.

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1. Introduction

The systems containing the terms with a stretched argument$s$,play important roles in describing the dynamics of current collection systems for electric locomotives [14] and in describing the particulate systems [15]. Therefore, spending some time to find the optimal solution of these systems is reasonable.
In the literature, some authors have tried block pulse series [3], Chebyshev polynomials [6], Shifted Legendre series[7], Laguerre series[8], Walsh series[16],Fourier series[17] methods to solve these systems.

Wavelets theory is a relatively new emerging area in mathematical research. It has been applied in a wide range of engineering disciplines; particularly, wavelets play an important role in establishing algebraic methods for the solution of problems described by differential equations, such as Numerical solutions of integral differential equations [2], analysis of time-varying [19, 5] or time-delay [12] systems, optimal control [9, 11, 18].

A method of constrained extremum is applied which consists of adjoining the constraint equations which are derived from the given dynamical system with the scale argument and the inequality constraints to the performance index by a set of undetermined Lagrange multipliers. As a result, the necessary conditions of optimality are derived as a system of algebraic equations in the unknown coefficients of $x(t)$,$u(t)$ and the Lagrange multipliers. These coefficients are determined in such a way that the necessary conditions for extremization are imposed.

For the first time in this paper, we have introduced an alternative numerical method to solve the linear quadratic optimal control problem using the Chebyshev wavelets. This method consists of reducing the optimal control problem to a set of algebraic equations by expanding the state vector $x(t)$ and the control vector $u(t)$ using the Chebyshev wavelet functions with unknown coefficients so-called reduced technique. As it is shown by the table, the approach gives more accurate results when compared with the existing approaches (polynomial series). Also the results of [8] are modified.

The paper is organized as follows: In Section 2, we describe the basic formulation of the Chebyshev wavelets and our proposed method required for our subsequent development. Section 3 is devoted to the formulation of the optimal control problem. In Sections $4-5$ the proposed method is used to approximate the optimal control problem. In Section 6, we report our numerical finding and demonstrate the accuracy of the proposed technique by considering some examples.

2. Properties of Chebyshev wavelets

2.1. The Chebyshev wavelets

Wavelets have been used by many researchers in many scientific and engineering fields. They constitute a family of functions constructed from the dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets:

$$\psi_{a,b} = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \quad a,b \in \mathbb{R}, a \neq 0$$

If we restrict the parameters a and b to discrete values as $a = {a_0}^{-k}$, $b = nb_0{a_0}^{-k}$ where $a_0 > 1$, $b_0 > 0$ and n, k are positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n} = |a|^{\frac{k}{2}}\psi({a_0}^k t - nb_0)$$

In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{k,n}$ forms an orthonormal basis. Chebyshev wavelets $\psi_{n,m}(t) = \psi(n,k,m,t)$ have four arguments: $m = 0,1,\ldots,M-1,\ n = 1,2,\ldots,2k-1,\ k = 0,1,2,\ldots$. The values of m are given in Eq.(1) and t is the normalized time. They are defined on the interval [0,1):

$$\psi(\mathrm{t})_{n,m} = \begin{cases} 2^{\frac{k}{2}}\tilde{T}_m(2^k t - 2n + 1) & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}} \\ 0 & otherwise \end{cases} \tag{1}$$

where

$$\tilde{T}_m(\mathrm{t}) = \begin{cases} \frac{1}{\sqrt{\pi}} & \mathrm{m} = 0 \\ \sqrt{\frac{2}{\pi}}\, T_m(t) & \mathrm{m} > 0 \end{cases} \tag{2}$$

In Eq.(2) the coefficients are used for orthonormality. Here $T_m(t)$ are Chebyshev polynomials of the first kind of degree m which are orthogonal with respect to the weight function $\mathrm{w(t)} = \frac{1}{\sqrt{1-t^2}}$, on [-1,1], and satisfy the following recursive formula:

$$T_0(t) = 1,\ T_1(t) = t,\ T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t) \quad m = 1,2,3, \tag{3}$$

We should note that in dealing with Chebyshev wavelets the weight function $w(x)$ has to be dilated and translated as:
$w_n(t) = w(2^k t - 2n + 1)$
Remark: The time interval [0,1) in Chebyshev wavelets can be extended to an arbitrary$[t_1, t_2)$ as follows [13]:

$$\psi(\mathrm{t})_{n,m} = \begin{cases} \frac{2^{\frac{k}{2}}}{\sqrt[2]{\Delta t}} \tilde{T}_m \left(\frac{2^k}{\Delta t} t - 2n + 1 \right) & t_1 + \Delta t \times \frac{n-1}{2^{k-1}} \le t < t_1 + \Delta t \times \frac{n}{2^{k-1}}, \\ 0 & otherwise \end{cases} \quad (4)$$

where $\Delta t = t_2 - t_1$.

2.2. Function approximation

A time function $f(t)$ that is square integrable on the time interval t $\in[0,t_f]$, may be expanded by Chebyshev wavelets as follows:

$$f(t) \cong \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) = C^T \Psi(t), \quad (5)$$

where C and $\Psi$are NM × 1 matrices given by

$$C = [c_{1,0}, c_{1,1}, \dots, c_{1,M-1}, c_{2,0}, \dots, c_{2,M-1}, \dots, c_{N,0}, \dots, c_{N,M-1}]^T, \quad (6)$$

$$\Psi(t) = \left[\psi_{1,0}(t), \psi_{1,1}(t), \dots, \psi_{1,M-1}(t), \psi_{2,0}(t), \dots, \psi_{2,M-1}(t), \dots, \psi_{N,0}(t), \dots, \psi_{N,M-1}(t)\right]^T \quad (7)$$

2.3. The Operational Matrix of Integration

We obtain the integrals of Chebyshev wavelets $\psi_i(t)$ which may be represented as:

$$\int_0^t \Psi(\tau)\, d\,\tau = P\Psi(t), \quad (8)$$

Where P is the NM×NM operational matrix of integration and is given in [1]

$$P = \frac{1}{2^k} \begin{bmatrix} \mathrm{F} & \mathrm{S} & \cdots & \mathrm{S} \\ 0 & \mathrm{F} & \cdots & \mathrm{S} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{F} \end{bmatrix}_{\boldsymbol{MN} \times MN}, \quad (9)$$

In Eq.(8) F and S are M×M matrices given by

$$
\mathrm{S}=\begin{bmatrix}
2 & 0 & \dots & 0\\
0 & 0 & \cdots & 0\\
\frac{-2\sqrt{2}}{3} & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots\\
\frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{\mathrm{m}+1}}{\mathrm{m}+1}-\frac{1-(-1)^{\mathrm{m}-1}}{\mathrm{m}-1}\right) & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots\\
\frac{\sqrt{2}}{2}\left(\frac{1-(-1)^{\mathrm{M}}}{\mathrm{M}}-\frac{1-(-1)^{\mathrm{M}-2}}{\mathrm{M}-2}\right) & 0 & \dots & 0
\end{bmatrix},
$$

$$
\mathrm{F}=\begin{bmatrix}
1 & \frac{1}{\sqrt[2]{2}} & 0 & \dots & \cdots & 0\\
-\frac{\sqrt[2]{2}}{4} & 0 & \frac{1}{4} & 0 & \dots & 0\\
-\frac{\sqrt[2]{2}}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & \ddots & 0\\
\vdots & 0 & \ddots & 0 & \ddots & 0\\
\frac{\sqrt{2}}{2}\left(\frac{(-1)^{\mathrm{m}-1}}{\mathrm{m}-1}-\frac{(-1)^{\mathrm{m}+1}}{\mathrm{m}+1}\right) & \vdots & 0 & \frac{-1}{2(\mathrm{m}-1)} & \ddots & \frac{1}{2(\mathrm{m}+1)}\\
\vdots & \vdots & \vdots & 0 & \ddots & 0\\
\frac{\sqrt{2}}{2}\left(\frac{(-1)^{\mathrm{M}-2}}{\mathrm{M}-2}-\frac{(-1)^{\mathrm{M}}}{\mathrm{M}}\right) & 0 & 0 & \dots & \frac{-1}{2(\mathrm{M}-2)} & 0
\end{bmatrix}
$$

2.4. The Operational Matrix of Stretch

The scaled matrix S may be defined for the Chebyshev wavelets having the following properties:

$$\Psi(st)=S\Psi(t) \tag{10}$$

For obtaining S, we assume $s=\frac{a}{b}$, in order to use equation (10) we should produce $\Psi(\frac{t}{b})$, then generate $\Psi(at)$, and finally by using $\Psi(st)=\Psi(\frac{at}{b})$, the scale operational matrix S can be obtained.

2.4.1 The $\Psi\left(\frac{t}{b}\right)$ function

By considering Eq.(1) over $[0,1]$, the function $\Psi\left(\frac{t}{b}\right)$ over $[0,b]$ is given by:

$$\Psi\left(\frac{1}{b}t\right) = S_b \underbrace{\left[\Psi^{\top}(t)\Psi^{\top}(t-1) \ \ldots \ \Psi^{\top}\big(t-(b-1)\big)\right]^{\top}}_{b\ times} \tag{11}$$

let

$$\Psi(t) = [\psi_0{}^{\top}, \psi_1{}^{\top}, \ldots, \psi_{2^k-1}{}^{\top}]^{\top} \tag{12}$$

where

$$\psi_n{}^{\top} = [\psi_{n,0}, \psi_{n,1}, \ldots, \psi_{n,M-1}]$$

From Eq.(1) and Eq.(3) we know $\Psi_{n,m}(t)$ is defined over $\left[\frac{n}{2^k}, \frac{n+1}{2^k}\right]$, so $\Psi_0(t)$ is defined over $\left[0, \frac{1}{2^k}\right]$ and $\Psi(\frac{t}{b})$ can be defined over $\left[0, \frac{b}{2^k}\right]$. Thus by using equation (7), equation (11) can be written as:

$$\psi_0\left(\frac{1}{b}t\right) = S\,[\psi_0{}^{\top}(t), \psi_1{}^{\top}(t), \ldots, \psi_{b-1}{}^{\top}(t)]^{\top} \tag{13}$$

where

$$S_{M\times bM} = \begin{pmatrix} S_{0_{M\times M}} & S_{1_{M\times M}} & \cdots & S_{(b-1)_{M\times M}} \end{pmatrix} \tag{14}$$

Thus by considering the interval $\left[0, \frac{b}{2^k}\right]$, we can restate equation (13) for one entry as:

$$\psi_0\left(\frac{1}{b}\left(t+\frac{i}{2^k}\right)\right) = S_i\psi_0(t) \quad i = 0, 1, \ldots, b-1 \tag{15}$$

In other words

$$L\left(\frac{1}{b}(t+i)\right) = S_i L(t) \tag{16}$$

We know that

$$L_0\left(\frac{1}{b}(t+i)\right) = 1$$

Thus the first row of matrix S_i can be expressed as:

$$S_{i0} \equiv [1, 0, 0, \ldots, 0]$$

also we have

$$L_1\left(\frac{1}{b}(t+i)\right)=1-\frac{2t}{b}-\frac{2i}{b}=1-\frac{(1+2\mathrm{i})}{b}+\frac{1}{b}L_1(t)$$

Similarly for the second row of matrix $S_i$we have:

$$S_{i1}\equiv[1-\frac{(1+2\mathrm{i})}{\mathrm{b}},\frac{1}{b},0,\dots,0]$$

Then by continuing the procedure, for the jth row of matrix S_i we obtain:

$$L_j\left(\frac{1}{b}(t+i)\right)=\Big\{2\left(1-\frac{1+2\mathrm{i}}{\mathrm{b}}\right)S_{i,j-1}-S_{i,j-2}+\left[0,\frac{2}{b}S_{i,j-1}(1),0,\dots,0\right]+\left[0,0,\frac{1}{b}S_{i,j-1}(2:M-1)\right]+\left[\frac{1}{b}S_{i,j-1}(2:M),0\right]\Big\}L(t)\tag{17}$$

where $S_{i,j-1}(1)$ is the first element in the $S_{i,j-1}$ row and $S_{i,j-1}(2:M)$ are a row from the second of column of $S_{i,j-1}$ to $(M-1)$th column of $S_{i,j-1}$.

To find $S_{i,j}$ by using equation (17) we have:

$$S_{i,j}\equiv\Big\{2\left(1-\frac{1+2\mathrm{i}}{\mathrm{b}}\right)S_{i,j-1}-S_{i,j-2}+\left[0,\frac{2}{b}S_{i,j-1}(1),0,\dots,0\right]+\left[0,0,\frac{1}{b}S_{i,j-1}(2:M-1)\right]+\left[\frac{1}{b}S_{i,j-1}(2:M),0\right]\Big\}\tag{18}$$

Thus we approximated the $\Psi\left(\frac{t}{b}\right)$as follows:

$$\psi_0\left(\frac{1}{b}t\right)=S\,[\psi_0{}^{\top}(t),\psi_1{}^{\top}(t),\dots,\psi_{b-1}{}^{\top}(t)]^{\top}$$

with the generalization of this method for finding matrix S_b we have:

$$\Psi\left(\frac{1}{b}t\right)=S_b\underbrace{\left[\Psi^{\top}(t)\Psi^{\top}(t-1)\ \dots\ \Psi^{\top}\big(t-(b-1)\big)\right]}_{b\ times}{}^{\top}$$

The final equation for S_b is:

$$S_b=\left(I_{2^k}\otimes S\right)\tag{19}$$

2.4.2 The $\Psi(\mathbf{at})$ function

To produce $\Psi(at)$**,** we can restate Eq.(12) as:

$$\Psi\left(\frac{1}{a}t\right)=S_a\underbrace{\left[\Psi^{\top}(t)\Psi^{\top}(t-1)\ \dots\ \Psi^{\top}\big(t-(a-1)\big)\right]}_{a\ times}{}^{\top}\tag{20}$$

we can also write

$$S^{\top}{}_a\Psi\left(\frac{1}{a}t\right)=(S^{\top}{}_aS_a)\underbrace{[\Psi^{\top}(t)\Psi^{\top}(t-T)\ \dots\ \Psi^{\top}(t-(a-1)T)]}_{a\ times}{}^{\top}$$

Similarly we have

$$(S^{\top}{}_a S_a)^{-1} S^{\top}{}_a \Psi\left(\frac{1}{a}t\right) = \underbrace{\left[\Psi^{\top}(t)\Psi^{\top}(t-T) \ \ \dots\ \Psi^{\top}(t-(a-1)T)\right]}_{a\ times}^{\top} \tag{21}$$

therefore

$$\Psi(t) = K(1{:}\,2^k M, :)\Psi\left(\frac{1}{a}t\right)$$

where

$K = (S^{\top}{}_a S_a)^{-1} S^{\top}{}_a$

Note that $K(1{:}\,2^k M, :)$represents the first row to the $2^k M$th row of the matrix K.
Thus we obtain $\Psi(at)$given by

$$\Psi(at) = \bar{S}_a \Psi(t) \tag{22}$$

where

$\bar{S}_a$=$K(1{:}\,2^k M, :)$

2.4.3 The $\mathbf{\Psi\left(\frac{a}{b}t\right)}$function

By considering 2.4.1 and 2.4.2, we can obtain $\Psi\left(\frac{a}{b}t\right)$ as follows:

$$\Psi\left(\frac{a}{b}t\right) = \bar{S}_b (S_a^{\top} S_a)^{-1} S_a^{\top} \underbrace{\left[\Psi^{\top}(t)\Psi^{\top}(t-1) \ \ \dots\ \Psi^{\top}\big(t-(a-1)\big)\right]}_{a\ times}^{\top} \tag{23}$$

where

$\bar{S}_b = S_b(:, 1{:}\,a2^k M)$

Note that $a2^k M$ represents the first column to the $a2^k M$thcolumn of thematrix S_b.

Thus the scale matrix S in equation (11) by using equation (23) is:

$$S = \bar{S}_b \bar{S}_a \tag{24}$$

3. Problem Statement

Consider linear time-invariant scaled systems characterized by

$$\dot{x}(t) = Ax(t) + Bx(st) + Cu(t), \quad x\,(0)\ \ specified \tag{25}$$

where the state variable $x(t)$ and the input variable $u(t)$ are b-vector and r-vector respectively. The time-invariant coefficient matrices A, B and $C$are matrices of appropriate dimensions.

The problem is to find the optimal control $u(t)$ and the corresponding state trajectory $x(t)$, satisfying Eq. (25) while minimizing the quadratic performance index

$$J = \frac{1}{2}\int_0^{t_f}[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]\,dt, \tag{26}$$

using the Lagrange approach. In Eq. (27), Q (t) is a positive semi-definite $b \times b$ matrix and $R(t)$ is a positive-definite $r \times r$ matrix.

4. Approximation of the System Dynamics

Integrating Eq.(25) from 0 to t yields

$$\int_0^t \dot{x}(\tau)\,d\tau = A\int_0^t x(\tau)\,d\tau + B\int_0^t x(s\tau)\,d\tau + C\int_0^t u(\tau)\,d\tau \tag{27}$$

By expanding $x(0)$, $x(\tau)$,$x(s\tau)$, $u(\tau)$ in terms of Chebyshev wavelet functions and using Eqs.((8),(10)) we have

$$\int_0^t \dot{x}(\tau)\,d\tau = x(t) - x(0) = X^T\Psi(t) - {X_0}^T\Psi(t) \tag{28}$$

$$\int_0^t x(\tau)\,d\tau = \int_0^t X^T\Psi(\tau)\,d\tau = X^T\int_0^t \Psi(\tau)\,d\tau = X^T P\Psi(t) \tag{29}$$

$$\int_0^t u(\tau)\,d\tau = \int_0^t U^T\Psi(\tau)\,d\tau = U^T\int_0^t \Psi(\tau)\,d\tau = U^T P\Psi(t) \tag{30}$$

$$\int_0^t x(s\tau)\,d\tau = \int_0^t X^T\Psi(s\tau)\,d\tau = X^T\int_0^t S\Psi(\tau)\,d\tau = X^T SP\Psi(t) \tag{31}$$

By substituting Eqs. (28) and (31) into Eq. (27) we get

$$X^T\Psi(\mathrm{t}) - {X_0}^T\Psi(\mathrm{t}) = AX^T P\Psi(\mathrm{t}) + BX^T SP\Psi(\mathrm{t}) + CU^T P\Psi(\mathrm{t}) \tag{32}$$

or

$$X^T - {X_0}^T = AX^T P + BX^T SP + CU^T P \tag{33}$$

And by using the properties of Kronecker product [10]we obtain

$$\left[I_{\mathrm{rn}} - {P^T}_{\mathrm{r\times r}} \otimes A_{n\times n} - (S_{\mathrm{r\times r}}P_{\mathrm{r\times r}})^T \otimes B_{n\times n}\right]\bar{X} = \overline{X_0} + [{P^T}_{\mathrm{r\times r}} \otimes C_{n\times m}]\bar{U},$$

Where

$r = mk$

$$X(t) = X^T \Psi(t) \quad , \qquad \bar{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

$$U(t) = U^T \Psi(t), \qquad \bar{U} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix}$$

Finally we obtain

$$\bar{X} = {A_s}^{-1}[\overline{X_0} + [P^T{}_{\mathrm{r}\times\mathrm{r}} \otimes C_{n\times m}]\bar{U}\,] \tag{34}$$

Where

$$A_s = [\boldsymbol{I} - P^T{}_{\mathrm{r}\times\mathrm{r}} \otimes B_{n\times n} - (S_{\mathrm{r}\times\mathrm{r}} P_{\mathrm{r}\times\mathrm{r}})^T \otimes B_{n\times n}]$$

and $\boldsymbol{I}$ is the rn-dimensional identity matrix.

5. An optimal control problem

5.1.The Lagrange approach

5.1.1. The performance index approximation

Given the generalized time-invariant scaled systems described by Eq. (25), we want to find the optimal control that minimizes the cost functional

$$J = \frac{1}{2}\int_{t_0}^{T} [x^T(t)Qx(t) + u^T(t)Ru(t)]dt, \tag{35}$$

where the matrices Q and R areconstant positive semi-definite and positive definite, respectively. Eq. (35) can be computed more efficiently by writing J as

$$J = \frac{1}{2}[\bar{X}^T \bar{Q} \bar{X} + \bar{U}^T \bar{R} \bar{U}] \tag{36}$$

where

$$\bar{Q} = W \otimes Q \quad , \quad \bar{R} = W \otimes R$$

$$\int_0^t \Psi(t)\, \Psi^T(t) dt = W.$$

5.1.2. Solution of the optimization problem

The optimal control problem has been reduced to a parameter optimization problem which can be stated as follows. Find X and U so that J is minimized subject to the constraint given in Eq. (34).

$$L = \frac{1}{2}[\bar{X}^T\bar{Q}\bar{X} + \bar{U}^T\bar{R}\bar{U}] + \lambda^T\left\{\left[I_{\mathrm{rn}} - (P_{\mathrm{r\times r}}S)^T \otimes A_{n\times n}\right]\bar{X} - \bar{X}_0 - [P^T{}_{\mathrm{r\times r}} \otimes B_{n\times m}]\bar{U}\right\}, \tag{37}$$

where the vector λ represents the unknown Lagrange multipliers, then the necessary conditions for stationary are given by

$$\frac{\partial L}{\partial x} = 0, \tag{38}$$

$$\frac{\partial L}{\partial u} = 0, \tag{39}$$

$$\frac{\partial L}{\partial \lambda} = 0. \tag{40}$$

6. Numerical examples

6.1. Example 1

Consider the time-invariant system with a stretch 2 described by [16]

$$\dot{x} = -x(2t), \quad x(0) = 1 \tag{41}$$

Then by choosing K = 4,M = 15 we have the Chebyshev wavelet solutions$x(t)$.Table 1 shows the comparison between the solutions of this example, obtained by both the Lanczos $\boldsymbol{\tau}$ method [4] and the Laguerre series method [8].From Table 1 it is clear that the Chebyshev wavelet solutions converge faster than the Laguerre series solution and also the errors of the Chebyshev wavelet solutions are much less than that of Laguerre series solution.

6.2. Example 2

Consider the scaled system with a stretch $\frac{1}{2}$ described by

$$\dot{x} = x(0.5t) + 4u(t) \tag{42}$$

$$x(0) = 1 \tag{43}$$

with the cost functional

$$J = \frac{1}{2}\int_0^T [x^2(t) + u^2(t)]dt \tag{44}$$

The problem is to find the optimal control $u(t)$ which minimizes Eq. (44) subject to Eqs. (43) and (42).The approximated values of x, u and the results of the cost function are shown in Tables2 and 3, respectively.

7. Conclusion

A simple and effective algorithm based on Chebyshev wavelets for obtaining operational matrix of stretch has been presented. Then theoretical elegance of the Chebyshev wavelet approach has been used to find the optimal solution of linear time-invariant scaled systems subject to a quadratic cost function. It has also been shown that the key idea is to transform the time-invariant functions including a stretch into wavelet functions using the operational matrices of integral and stretch. Some illustrative examples demonstrate that only a small number of terms are required to obtain accurate approximations. Hence the present method is extremely convenient for computer programming.

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Table 1:A comparison between the Chebyshev wavelets and Laguerre series solution of Example 1.

Time	Lanczos $\boldsymbol{\tau}$ method [17]	Laguerre series for M=40 [19]	Chebyshev wavelets for M=15, k=4
0.0	1.000000	0.999741	1.000000
0.5	0.658273	0.658294	0.658276
1.0	0.475058	0.474977	0.475020
1.5	No result	0.360199	0.360655
2.0	0.28257	0.283142	0.282575
2.5	No result	0.227676	0.226550
3.0	0.18583	0.185863	0.185110

Table 2:A comparison between the Chebyshev wavelets and the Legendre series approximations of x, u ofexample 2 using the Lagrange approach.

t	$x(t)$		$u(t)$	
	Chebyshev wavelets M=10, K=4	Legendre series M=10	Chebyshev wavelets M=7, N=4	Legendre series M=10
0.0	0.9998949795822	0.9890616782370	-1.3938312038283	-1.374004651366
0.5	0.1366591248892	0.1362303747910	-0.2089137911844	-0.208875054179
1.0	0.0274153302628	0.0270684379297	-0.0550498227868	-0.052509434535
1.5	0.0077454934927	0.0079516205838	-0.0189796405242	-0.019221145812
2.0	0.0029468917679	0.0034918983906	-0.0084465274238	-0.009730719982
2.5	0.0013209984812	0.0001060464474	-0.0040222778117	-.0007383773401
3.0	0.0006358942286	0.0023701015419	-0.0021381039881	-0.005995237759
3.5	0.0004242466779	-0.0021263525545	-0.0011541997066	0.0032761051024
4.0	0.0029365777549	-0.0096643482969	-0.0000059483052	0.0174262025031

Table 3:The values of the cost functions using the Chebyshev wavelets and the Legendre series of example 2.

t	Chebyshev wavelets K=4 ,M=10	Legendre series M=10	Chebyshev wavelets K=4 ,M=15	Legendre series M=15
4	0.1739515745797	0.1739588567336	0.1739500743866	0.1739517273872
8	0.1739469815391	0.1743308883144	0.1739508732497	0.1739596296828
16	0.1739939638621	0.1744282127363	0.1739458286408	0.1740236675092

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