

Some Exact Values of Shannon Capacity for Evolving Systems

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Abstract

In the paper we describe the notion of Shannon capacity for evolving channels. Furthermore, using a computer search together with some theoretical results we establish some exact values of the measure.

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1 Introduction

There are many reasons to consider channels with a reliable transmission, i.e. a transmission with exactly zero probability of error [2]. In 1956, Claude Shannon introduced the notion of a capacity of a noisy channel for a reliable transmission [3]. In the paper we discuss a generalization of the problem, i.e. the Shannon capacity of evolving channels [1] and present some exact values of the Shannon capacity of reducing channels.

2 Preliminary notes

In the following, we will need the following definition

Definition 2.1 *A discrete channel $Q = (A_X, A_Y, M_{XY})$ consists of three parts: A_X and A_Y are input and output alphabets, respectively, M_{XY} is the transition matrix with $p(y|x)$ elements, which correspond to conditional probabilities.*

We say that a channel Q is noisy, if there are elements $y_1, y_2 \in A_Y$ and an element $x \in A_X$ such that $p(y_1|x)p(y_2|x) > 0$. Given a channel Q and element $x \in A_X$, we define

$$S_x = \{y \in A_Y : p(y|x) > 0\}. \quad (1)$$

S_x is the set of letters attainable on output, when there is x on input.

For a discrete channel Q , there exists the characteristic graph defined as follows

Definition 2.2 *The characteristic graph of a channel Q is a graph $G = (V, E)$ such that the vertex set $V = A_X$ and $\{x, y\} \in E(G)$ iff $S_x \cap S_y \neq \emptyset$.*

There are many operations acting on graphs. In the paper we will use the following operation

Definition 2.3 *Given two graphs G and H , the strong product $G \cdot H$ is defined as follows. The vertices of $G \cdot H$ are all pairs of the Cartesian product $V(G) \times V(H)$. There is an edge between (x, x') and (y, y') iff $\{x, y\} \in E(G)$ and $\{x', y'\} \in E(H)$ or $x = y$ and $\{x', y'\} \in E(H)$ or $x' = y'$ and $\{x, y\} \in E(G)$.*

We write G^n to denote $G \cdot G \cdot \dots \cdot G$, where G occurs n times.

Let G be a graph. A set of vertices S of G is said to be an independent set of vertices if they are pairwise nonadjacent. The independence number of G , denoted by $\alpha(G)$, is defined to be the size of a largest independent set of G . For m arbitrary graphs G_1, \dots, G_m , we have

$$\alpha(G_1 \cdot \dots \cdot G_m) \geq \alpha(G_1) \cdot \dots \cdot \alpha(G_m). \quad (2)$$

The notion of reliable capacity was introduced by Shannon [3]

Definition 2.4 *The Shannon capacity is defined as*

$$C_0 = \sup_n \sqrt[n]{\alpha(G^n)}. \quad (3)$$

We derived a generalization of this measure in [1]

Definition 2.5 *Given a sequence U_1, U_2, \dots of operations acting on graphs. The Shannon capacity of evolving channel is defined as*

$$C_0 = \sup_n \sqrt[n]{\alpha(\mathcal{U}_n(G))}. \quad (4)$$

where $\mathcal{U}_n(G) = U_n(U_{n-1}(\dots(U_1(G))))$.

Notice that U_1, U_2, \dots represent arbitrary operations on a channel, but sometimes the supremum does not exist, so the measure does not work. However, we get the Shannon capacity, if $\mathcal{U}_n(G) = G^n$.

Let Q be a noisy channel, G be its characteristic graph and $L = (v_1, \dots, v_l)$ be a sequence of vertices of G such that $v_i \neq v_j$, for $i, j = 1, \dots, l$ and $i \neq j$. Then we denote a sequence of reductions as

$$G^k[L] = G - v_1 \oplus (\{v_1\}, \emptyset) - v_2 \oplus (\{v_2\}, \emptyset) - \dots - v_k \oplus (\{v_k\}, \emptyset), \quad (5)$$

where $k \leq l \leq |V(G)|$ and the symbol $-$ means a vertex deletion and the symbol \oplus means a graph join.

In the next section, we will show some exact values for a special case of the Shannon capacity of evolving channel defined below

Definition 2.6 Given a graph G with n vertices and a permutation π of $\{1, \dots, n\}$. We define the Shannon capacity of reducing channels as

$$C_0^R = \max_{k \in [n]} \sqrt[k]{\alpha(G^k[\pi])}, \quad (6)$$

where $G^k[\pi] = G^0[\pi] \cdot G^1[\pi] \cdot G^2[\pi] \cdot \dots \cdot G^{k-1}[\pi]$ and $G^0[\pi] = G$.

For more details of the section, see [1].

3 Some exact values

It is interesting that we could facilitate the problem of calculating the Shannon capacity of reducing channels.

Lemma 3.1 Given graphs H and $K_n - e$, i.e. complete graph without an edge, then

$$\alpha((K_n - e) \cdot H) = 2\alpha(H). \quad (7)$$

Proof. Given a graph H with $V(H) = \{1', 2', \dots, l'\}$ and an edge $e = \{v_1, v_2\}$ of K_n , with $V(K_n) = \{1, 2, \dots, n\}$, there are two copies H_1, H_2 of H in $(K_n - e) \cdot H$ on vertex sets $\{(v_1, i') : i' \in V(H)\}$ and $\{(v_2, i') : i' \in V(H)\}$, respectively, which are not linked. Hence, $2\alpha(H) \leq \alpha((K_n - e) \cdot H)$. On the other hand, let S be a largest independent set of $(K_n - e) \cdot H$. If $S \subset V(H_1) \cup V(H_2)$, then $|S| \leq 2\alpha(H)$. Otherwise, for an vertex $(s_1, s_2) \in S$, which is outside $V(H_1) \cup V(H_2)$, we can replace (s_1, s_2) to vertices (v_1, s_2) and (v_2, s_2) , without breaking the independent set condition. Therefore, $\alpha((K_n - e) \cdot H) \leq 2\alpha(H)$.

The reduction Shannon capacity has the following property

Theorem 3.2 Given a graph G and a permutation $\pi = (p_1, \dots, p_n)$ of all vertices of G . If $G^i[\pi], \dots, G^n[\pi]$ are complete graphs, then:

$$C_0^R = \begin{cases} i + 1, & \text{if } i = 0, 1 \\ \max_{k \in [i-1]} \sqrt[k]{\alpha(G^k[\pi])}, & \text{if } 2 \leq i < n \end{cases} \quad (8)$$

Proof. It is easy to see that the theorem is true for $i = 0$. If $G^i[\pi], \dots, G^n[\pi]$ are complete graphs for some $i \in \{1, \dots, n-1\}$, then $G^{i-1}[\pi]$ is a complete graph without an edge. Using the well known formula $\alpha(H \cdot K_j) = \alpha(H)$, where j is an integer number, we conclude that for $k \geq i$

$$\alpha(G^k[\pi]) = \alpha(G^i[\pi]). \quad (9)$$

For $i = 1$, we have $\alpha(G^1[\pi]) = \alpha(K_n - e) = 2$. For $2 \leq i < n$, from Lemma 3.1

$$\alpha(G^i[\pi]) = \alpha(G^{i-1}[\pi] \cdot G^{i-1}[\pi]) = 2\alpha(G^{i-1}[\pi]). \quad (10)$$

In addition, from (2), we get $\alpha(G^{i-1}[\pi]) \geq \alpha(G^0[\pi]) \cdot \dots \cdot \alpha(G^{i-2}[\pi])$. Because $G^0[\pi] \subset \dots \subset G^n[\pi]$, hence $\alpha(G^0[\pi]) \geq \dots \geq \alpha(G^{i-2}[\pi]) \geq \alpha(G^{i-1}[\pi]) = 2$. So, we get $\alpha(G^{i-1}[\pi]) \geq 2^{i-1}$ and therefore

$$\sqrt[i-1]{\alpha(G^{i-1}[\pi])} \geq \sqrt[i]{2\alpha(G^{i-1}[\pi])}. \quad (11)$$

From these considerations, we get the thesis.

Using preceding results and a computer search we have established among others that for all permutations

- $C_0^R = 1$, for K_n , $n > 0$,
- $C_0^R = 2$, for $2P_2$, $K_3 \cup K_1$, P_4 , C_4 , $K_4 - e$, W_5 , $K_5 - e$,
- $C_0^R = 3$, for S_4 , $K_2 \cup E_2$,
- $C_0^R = 4$, for E_4 .

These computations brought us to $\alpha(G^k[\pi]) = \alpha(G^0[\pi]) \cdot \dots \cdot \alpha(G^{k-1}[\pi])$, for all permutations π and graphs on n vertices, where $1 \leq k \leq n \leq 4$. Additionally, we find that $C_0^R = \alpha(G)$, for all graphs on $n \leq 4$.

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