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Risk-Criterion for a Single Robot with External Safety Device

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Abstract

We introduce a robot-safety device system characterized by cold stand-by and by an admissible risky state. The system is attended by a single repairman and all the underlying repair time distributions are arbitrary. We obtain an explicit formula for the point availability of the robot-safety device system via an integral equation of the convolution-type. In order to decide whether the risky state is admissible, we introduce a risk-criterion based on the notion of rare events. The criterion is always satisfied in the case of fast repair. As an example, we consider the case of Coxian repair and we display a computer-plotted graph of the point availability obtained by numerical inversion of the corresponding Laplace-transform.

Keywords: robot, safety device, cold stand-by, point availability, risk-criterion, Coxian repair, computer-plotted graph

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1 Introduction

Innovations in the field of microelectronics and micromechanics have enhanced the involvement of "smart" robots in all kind of advanced technical applications, e.g. Brandin [3]. Unfortunately, no robot is completely reliable. Therefore, up-to-date robots are often connected with a safety device, e.g. Dhillon and Yang [6], Vanderperre and Makhanov [12]. Such a device prevents possible damage, caused by a robot failure or by hazardous man-machine interactions, in the robot's neighbouring environment. The usual "bugbears" are software failures, e.g. Gaskill and Went [8], common-cause failures, e.g. Dhillon and Yang [5], human errors, e.g. Dhillon and Liu [7] and physical failures, e.g. Birolini [2]. Moreover, the random behaviour of the entire system (robot, safety device, repair facility) could jeopardize some prescribed safety requirements if we allow, in particular, the robot to operate during the repair time of the failed safety unit.

Such a "risky" state is called *admissible* if the associated event: "The robot is operating and the safety device is under repair", constitutes a rare event. Therefore, an appropriate statistical analysis of robot-safety device systems is indispensable to support the system designer in problems of risk acceptance and safety assessments.

Apart from various robot-safety device systems introduced in the previous Literature, e.g. Dhillon and Yang [6], Vanderperre and Makhanov [12,13,14] we introduce a modified system (henceforth called the **L**-system) endowed with the following characteristics:

- The robot and the safety device are attended by a single repairman.
- The repair discipline is first-in-first-out and any repair is perfect.
- The robot is allowed to operate when the safety unit is under repair, provided that a suitable risk-criterion is satisfied.
- The safety device need not to operate when the robot is under repair. Therefore, upon failure of the robot, the *operative* safety device is shut-off and put in cold stand-by until the repair of the robot has been completed.

The notion of "cold" stand-by signifies that the safety unit is failure-free in stand-by.

In order to describe the random behaviour of the **L**-system, we employ a stochastic process endowed with state probability functions satisfying integral equations of the convolution-type. Next, we define a risk-function associated with the risky state of the robot. In order to decide whether the risky state is admissible, we introduce an associated risk-criterion compatible with the notion of rare events. The criterion is always satisfied in the case of "fast" repair. As an example, we consider the case of Coxian repair and we display the graph of system's point availability obtained by numerical inversion of the corresponding Laplace-transform.

2 Formulation

Consider the **L**-system satisfying the following assumptions:

- The *operative* safety device has a constant failure rate λ_s, a zero failure rate in stand-by (the so-called cold stand-by) and a general repair time distribution $R_s(\cdot)$, $R_s(0) = 0$. Let f_s be the random variable corresponding to λ_s. The repair time is denoted by r_s.

- The robot has a constant failure rate λ and a general repair time distribution $R(\cdot)$, $R(0) = 0$. The failure-free time and the repair time are respectively denoted by f and r.

- The random variables f, f_s, r, r_s are supposed to be statistically independent.

In order to describe the random behaviour of the **L**-system, we employ a stochastic process $\{N_t,\, t \geq 0\}$ with discrete state space $\{A,\, B,\, C,\, D\} \subset [0, \infty)$, characterized by the following set of mutual exclusive events:

$\{N_t = A\}$: "The robot and the safety device are jointly operative at time t."

State A is called the safe state.

$\{N_t = B\}$: "The robot is operative but the safety device is under progressive repair at time t."

State B is the so-called risky state. We tacitly assume that state B is admissible. See Ch. 4 for further details.

$\{N_t = C\}$: "The robot is under progressive repair and the safety device is in (cold) stand-by at time t."

$\{N_t = D\}$: "The safety device is under progressive repair and the robot is waiting for repair at time t."

Note that a transition from state D into state A is only possible via state C. We assume that the **L**-system starts operating in state A at some time origine $t = 0$, i.e. let $N_0 = A$ with probability one.

The state probability functions are defined by $p_K(t) := Pr\{N_t = K\}$, $K = A,\, B,\, C,\, D$ where $p_A(t) + p_B(t) + p_C(t) + p_D(t) = 1$, $p_A(0) = 1$.

We recall that the robot and the safety device are only jointly available in the safe state A. Therefore, the point availability of the robot-safety device system, denoted by $\mathcal{A}(t)$, is given by $p_A(t)$.

The long-run availability, denoted by $\mathcal{A}(\infty)$, is defined as the limit of $\mathcal{A}(t)$ if t tends to ∞. Observe that

$$\mathcal{A}(\infty) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathcal{A}(t)dt,$$

provided that the precious limit exists.

2.1 Notations

- The notation $N << 1$ signifies that the number N is considerably smaller than 1.
- The expectation operator, e.g. Brémaud [4, Appendix A, page 268] is denoted by $\mathbf{E}$. Note that, for instance,

$$\mathbf{E}e^{-zr} = \int_0^\infty e^{-zx}\, dR(x),\ \mathrm{Re}\, z \geq 0.$$

- The notation $X \sim Y$ means that the random variables X and Y are equal in distribution.
- Let $F(t)$ be any probability distribution on $[0, \infty)$. The n-fold convolution of F is denoted by $F^{n*}(t)$. For $n = 0$, $F^{0*}(t)$ represents the Heaviside unit-step function with the unit-jump at $t = 0$, i.e.

$$F^{0*}(t) := \begin{cases} 1, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

- Let $p(t)$, $t \geq 0$ be a locally integrable and bounded function on $[0, \infty)$. The Laplace-transform of $p(t)$ is denoted by the corresponding character marked with a star, i.e.

$$p^\star(z) := \int_0^\infty e^{-zt}p(t)dt,\ \mathrm{Re}\, z > 0.$$

3 Point availability

The following theorem 3.1 shows that $\mathcal{A}(t)$ is an infinite mixture of compound Lebesgue-Stieltjes convolutions.

Theorem 3.1
The point availability of the robot-safety device system satisfies the integral equation

$$\mathcal{A}(t) = e^{-(\lambda+\lambda_s)t} + \int_0^t \mathcal{A}(t-x)d\varphi(x) \tag{3.1}$$

where

$$\varphi(t) := \int_0^t \left(1 - e^{-(\lambda+\lambda_s)(t-u)}\right) d\psi(u),$$

$$\psi(t) := \frac{\lambda}{\lambda+\lambda_s} R(t) + \frac{\lambda_s}{\lambda+\lambda_s} \left\{ \int_0^t e^{-\lambda u} dR_s(u) + \int_0^t \int_0^{t-v} (1 - e^{-\lambda u}) dR_s(u) dR(v) \right\}.$$

The unique solution of Eq. (3.1) is given by

$$\mathcal{A}(t) = \int_{0-}^t e^{-(\lambda+\lambda_s)(t-x)} d \sum_{n=0}^{\infty} \varphi^{n*}(x). \tag{3.2}$$

Furthermore,

$$\mathcal{A}(\infty) = \frac{1}{1 + \lambda_s \mathbf{E} r_s + (\lambda + \lambda_s(1 - \mathbf{E} e^{-\lambda r_s}))\mathbf{E} r}. \tag{3.3}$$

Proof

First note that state A is *regenerative* for the process $\{N_t\}$, i.e. the Markov property of the exponential distribution ensures that state A is a renewal state of the $\mathbf{L}$-system. Furthermore, observe that a busy period of the repairman can either start via a robot failure with $Pr\{f < f_s\}$ or via a failure of the safety device with $Pr\{f_s < f\}$. Note that $Pr\{f = f_s\} = 0$ and that $Pr\{f < f_s\} = \lambda/(\lambda + \lambda_s)$, whereas $Pr\{f_s < f\} = \lambda_s/(\lambda + \lambda_s)$. Let θ_a, respectively θ_b, be a busy period starting with a robot failure, respectively with a failure of the safety device. The random variable

$$\theta := \begin{cases} \theta_a, & \text{if } f < f_s, \\ \theta_b, & \text{if } f_s < f \end{cases}$$

is called the *global* busy period of the repairman.

Clearly,

$$Pr\{\theta \le t\} = \frac{\lambda}{\lambda+\lambda_s} Pr\{\theta_a \le t\} + \frac{\lambda_s}{\lambda+\lambda_s} Pr\{\theta_b \le t\}.$$

But the assumption of cold stand-by implies that $\theta_a \sim r$, whereas

$$\theta_b \sim \begin{cases} r_s, & \text{if } f > r_s, \\ r_s + r, & \text{if } f < r_s. \end{cases}$$

Consequently,

$$Pr\{\theta \le t\} = \frac{\lambda}{\lambda+\lambda_s}R(t)+\frac{\lambda_s}{\lambda+\lambda_s}\left\{\int_0^t e^{-\lambda u}dR_s(u)+\int_0^t\int_0^{t-v}(1-e^{-\lambda u})dR_s(u)dR(v)\right\} \equiv \psi(t).$$

Observe that the *idle* time of the repairman, denoted by i_R, has distribution

$$Pr\{i_R \le t\} = Pr\{\min(f,\ f_s) \le t\}. \tag{3.4}$$

Consequently, the *return* time from state A to state A (via states B, C or D), denoted by r_{AA}, has distribution

$$Pr\{r_{AA} \le t\} = Pr\{\theta + i_R \le t\}. \tag{3.5}$$

Therefore, by Renewal Theory, e.g. Serfoso [10, page 45-46],

$$\mathcal{A}(t) = Pr\{\min(f,\ f_s) > t\} + \int_0^t \mathcal{A}(t-u)dPr\{r_{AA} \le t\}. \tag{3.6}$$

But $Pr\{\min(f,\ f_s) > t\} = e^{-(\lambda+\lambda_s)t}$, whereas $Pr\{r_{AA} \le t\} = \varphi(t)$.

Hence, Eq. (3.1) follows from Eqs. (3.4), (3.5) and (3.6). The unique solution of Eq. (3.1), given by Eq. (3.2), can be obtained by iteration or via the Laplace-transform. Such solution procedures are well-known (e.g. Feller [9]) and therefore omitted. In order to obtain $\mathcal{A}(\infty)$, we remark that $\varphi(t)$ is non-lattice, whereas $e^{-(\lambda+\lambda_s)t}$ is *directly* Riemann integrable. Whence, by the key renewal theorem, e.g. Serfoso [10, page 47],

$$\mathcal{A}(\infty) = \frac{\mathbf{E}i_R}{\mathbf{E}i_R + \mathbf{E}\theta}. \tag{3.7}$$

On the other hand,

$$\mathbf{E}i_R = \frac{1}{\lambda+\lambda_s}, \tag{3.8}$$

whereas

$$\mathbf{E}e^{-z\theta} = \frac{\lambda}{\lambda+\lambda_s}\mathbf{E}e^{-zr} + \frac{\lambda_s}{\lambda+\lambda_s}\left(\mathbf{E}e^{-z(r+r_s)} + \mathbf{E}e^{-(z+\lambda)r_s}(1-\mathbf{E}e^{-zr})\right).$$

The relation

$$\mathbf{E}\theta = -\frac{\partial}{\partial z}\mathbf{E}e^{-z\theta}|_{z=0}$$

reveals that

$$\mathbf{E}\theta = \frac{\lambda}{\lambda+\lambda_s}\mathbf{E}r + \frac{\lambda_s}{\lambda+\lambda_s}\left\{\mathbf{E}(r+r_s) - \mathbf{E}e^{-\lambda r_s}\mathbf{E}r\right\}. \tag{3.9}$$

Hence, $\mathcal{A}(\infty)$ follows from Eqs. (3.7), (3.8) and (3.9).

Remarks 3.1

- The long-run availability of the robot-safety device system fails to satisfy the so-called distribution-free property, i.e. $\mathcal{A}(\infty)$ depends on the canonical structure of R_s by means of $\mathbf{E}e^{-\lambda r_s}$. The latter phenomenon seems to be a rule rather than an exception, e.g. Vanderperre [14, page 70].
- The Laplace-transform is often quite useful to evaluate the original function by numerical inversion. For direct reference, we state the following result:

$$\mathcal{A}^\star(z) = \frac{1}{z + \lambda(1 - \mathbf{E}e^{-zr}) + \lambda_s[1 - (\mathbf{E}e^{-zr}\mathbf{E}e^{-zr_s} + \mathbf{E}e^{-(\lambda+z)r_s}(1 - \mathbf{E}e^{-zr}))]}. \tag{3.10}$$

4 Risk-criterion

An obvious risk-function associated with the risky state B is given by $p_B(t)$, henceforth denoted by $\mathfrak{R}(t)$. The following theorem 4.1 shows that $\mathfrak{R}(t)$ is closely related to $\mathcal{A}(t)$. The quantity $\mathfrak{R}(\infty)$ is called the long-run risk.

Theorem 4.1
The risk-function is given by

$$\mathfrak{R}(t) = \lambda_s \int_0^t \mathcal{A}(t-x)e^{-\lambda x}(1 - R_s(x))dx.$$

In addition,

$$\mathfrak{R}(\infty) = \lambda_s\, \mathcal{A}(\infty)\frac{1 - \mathbf{E}e^{-\lambda r_s}}{\lambda}.$$

Proof
Note that a transition of the process $\{N_t\}$ into state B can only occur via state A. Hence,

$$\begin{aligned}\mathfrak{R}(t) &= \int_0^t \mathcal{A}(x)Pr\{r_s > t-x,\ f > t-x,\ f_s \in dx | f_s > x\} \\ &= \int_0^t \mathcal{A}(x)e^{-\lambda(t-x)}(1 - R_s(t-x))Pr\{f_s \in dx | f_s > x\}.\end{aligned}$$

However, a straightforward application of the law $Pr\{\varepsilon_1 \cap \varepsilon_2\} = Pr\{\varepsilon_1 \,|\, \varepsilon_2\}\, Pr\{\varepsilon_2\}$ and the Markov property of the exponential distribution entails that $Pr\{f_s \in dx \,|\, f_s > x\} = \lambda_s\, dx$.

Hence,

$$\begin{aligned}\mathfrak{R}(t) &= \int_0^t \mathcal{A}(x)e^{-\lambda(t-x)}(1 - R_s(t-x))\lambda_s\, dx \\ &= \lambda_s \int_0^t \mathcal{A}(t-x)e^{-\lambda x}(1 - R_s(x))dx.\end{aligned}$$

Moreover, the existence of $\mathcal{A}(\infty)$ also implies the existence of $\mathfrak{R}(\infty)$. Since $\mathcal{A}(t) \leq 1$, we obtain by bounded convergence

$$\mathfrak{R}(\infty) = \lambda_s \, \mathcal{A}(\infty) \int_0^\infty e^{-\lambda x}(1 - R_s(x))dx = \lambda_s \mathcal{A}(\infty) \frac{1 - \mathbf{E}e^{-\lambda r_s}}{\lambda}.$$

Remarks 4.1

In order to obtain a suitable risk-criterion, compatible with the notion of rare events, we employ the supremum norm

$$||\mathfrak{R}(t)|| := \sup \{\mathfrak{R}(t), \, t \in \, [0, \infty]\} \, .$$

Finally, we propose the following risk-criterion: *State B is admissible if* $||\mathfrak{R}(t)|| << 1$.

5 Fast repair

Next, we consider the case of fast repair, i.e. $\lambda_s \, \mathbf{E} r_s << 1$. Observe that the notion of "fast" repair does not necessarily imply a small average repair time of the failed safety device. The mean $\mathbf{E} r_s$ is only supposed to be considerably smaller than the average lifetime $1/\lambda_s$ of the operative safety device. A condition that is usually satisfied in practice!

Note that by Theorem 4.1, $||\mathfrak{R}(t)|| < \lambda_s \mathbf{E} r_s$. Thus, our risk-criterion is always satisfied in the case of fast repair. Moreover, our proposed criterion provides a simple (λ, R)-insensitive test to verify whether the risky state is admissible.

6 Application example

We recall that the point availability of the robot-safety device (see theorem 3.1) is given by

$$\mathcal{A}(t) = \int_{0-}^t e^{-(\lambda + \lambda_s)(t-x)} d \sum_{n=0}^\infty \varphi^{n*}(x).$$

Unfortunately, an explicit (exact) evaluation of $\mathcal{A}(t)$ in terms of *finite* linear combinations of elementary algebraic and/or transcendental functions is in general excluded. Therefore, in order to present computational results, we assume that our repair time distributions have rational Laplace-Stieltjes transforms (so-called Coxian distributions) and we compute $\mathcal{A}(t)$ via the Laplace-transform. Note that the family of Coxian random variables is surprisingly large. For instance, consider the hyper-exponential distribution with a negative weight,

$$H(t) := p_1(1 - e^{-\lambda_1 t}) + p_2(1 - e^{-\lambda_2 t}),\ t \geq 0$$

where $p_1 > 0$, $p_2 < 0$, $p_1 + p_2 = 1$, $\lambda_1 p_1 + \lambda_2 p_2 = 0$, $0 < \lambda_1 < \lambda_2$.

Observe that we allow p_2 to be negative! However, since H is supposed to be a probability distribution on $[0, \infty)$, we must have $p_1 > 0$. Note that $(1 - H(t))^{-1}$ is log-convex. Hence, H has an increasing hazard rate. Consequently, H belongs to an important sub-family of Coxian distributions with interesting engineering applications. For instance, H is suitable to model repair times.

Another important (better-known) Coxian distribution with an increasing hazard rate, is the so-called Erlang-K distribution

$$E_K(t) := 1 - e^{-t} \sum_{k=0}^{K-1} \frac{t^k}{k!},\ t \geq 0,\ K = 1, 2, \ldots$$

As a numerical example, let $R = H$, $R_s = E_2$, $\lambda_1 = 1$, $\lambda_2 = 2$. Note that $p_1 = 2$, $p_2 = -1$,
$\mathbf{E}e^{-zr} = 2/(1+z)(2+z)$, $\mathbf{E}r = 3/2$, $\mathbf{E}e^{-zr_s} = 1/(1+z)^2$, $\mathbf{E}r_s = 2$ and $\mathbf{E}e^{-(\lambda+z)r_s} = 1/(1+\lambda+z)^2$. Finally, let $\lambda = 1/3$ and $\lambda_s = 0.01$.

Inserting these data into Eq. (3.10) reveals that

$$\mathcal{A}^\star(z) = \frac{N(z)}{zD(z)},\ \text{Re } z > 0$$

where $N(z) := 300(1+z)^3(2+z)(4+3z)^2$ and

$$D(z) := 14,655 + 67,147z + 127,442z^2 + 128,224z^3 + 72,207z^4 + 21,627z^5 + 2,700z^6.$$

Note that $D(z) = 2,700\ \Pi_{k=1}^6 (z - z_k)$, where

$$\begin{aligned}
z_1 &= -1.56187;\ z_2 = -1.19194;\ z_3 = -1.65687 + 0.50255i; \\
z_4 &= -1.65687 - 0.50255i;\ z_5 = -0.97122 + 0.17116i; \\
z_6 &= -0.97122 - 0.17116i.
\end{aligned}$$

Applying Cauchy's residue theorem, e.g. Apostol [1, page 460], to the inversion formula

$$\mathcal{A}(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{-iT+\delta}^{iT+\delta} e^{zt} \mathcal{A}^\star(z) dz,\ t > 0,\ \delta > 0$$

sustained by Maple, e.g. Shingareva & Lizarraga [11] and using the identity

$$(u + iv)e^{(a+ib)t} + (u - iv)e^{(a-ib)t} = 2e^{at}(u \cos bt - v \sin bt)$$

reveals that

$$\begin{aligned}\mathcal{A}(t) \;=\;& 0.65506 - 0.07101e^{-1.56187t} + 0.00708e^{-1.19194t} + \\ & 0.03523e^{-0.97122t}\cos 0.17116t + 0.02137e^{-0.97122t}\sin 0.17116t + \\ & 0.37363e^{-1.65687t}\cos 0.50255t + 0.40554e^{-1.65687t}\sin 0.50255t.\end{aligned}$$

Figure 1 displays the graph of $\mathcal{A}(t)$, $0 \le t \le 10$.

Case $\lambda = 1/3$, $\lambda_s = 0.01$. Note that $\mathcal{A}(\infty) = 0.65506$.

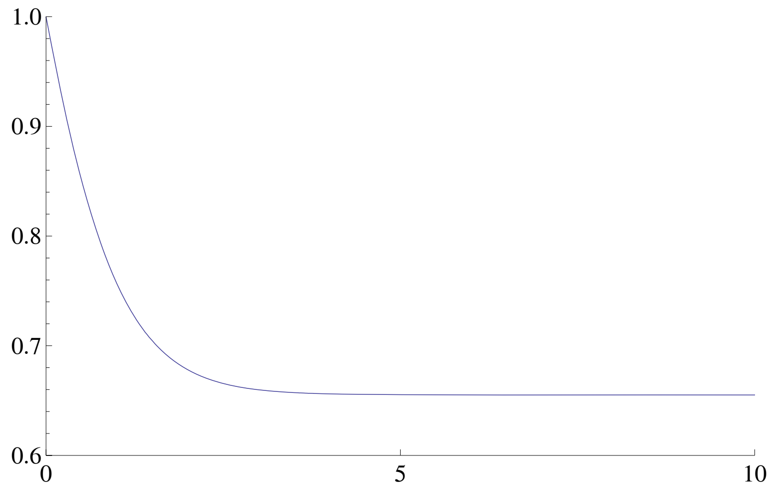

Fig. 1: Graph of $\mathcal{A}(t)$, $\lambda = 1/3$, $\lambda_s = 0.01$

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