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# Optimal Policy for Stochastic Lot-Sizing Inventory Model with Deterioration and Partial Backlogging

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**Abstract**

A finite horizon inventory model for a single deteriorating product is studied. The system is under periodic review and there is a positive fixed order cost associated with any placed order. The demand in successive periods is independent and identically distributed. A constant fraction of any positive leftover stock is deteriorated at the end of each period. Any unsatisfied demand is partially backlogged and fulfilled immediately as a new order arrives. Previous research has proved that a (*s*, *S*) policy is optimal under complete backlogging and non-deterioration. This paper fulfills the vacancy of the current literature by considering the effects of deterioration and partial backlogging under stochastic demand. The conditions for optimality of (*s*, *S*) policy are successfully derived.

**Keywords**: Deteriorating Products, Periodic Review, Stochastic Inventory Control, Partial Backlogging

# 1 Introduction

The implicit assumption in conventional inventory models is that the stored products maintain the same utility forever, i.e., they can be stored for an infinite period of time without losing their value or characteristics. However, generally speaking, almost all products experience some sort of deterioration over time. Some products have very small deterioration rates, and henceforth the effect of such deterioration can be neglected. Some products may be subject to significant rates of deterioration. Fruits, vegetables, drugs, alcohol and radioactive materials are examples that can experience significant deterioration during storage. Therefore the effect of deterioration must be explicitly taken into account in developing inventory models for such products.

In general, deterioration is defined as decay, damage, spoilage, evaporation, obsolesce or loss of utility of an item such that it cannot be used for its original purpose. Some real-life phenomena also conform to the concept of deterioration, such as decay in radioactive elements, spoilage in food grain storage, breakages in glasswares and pilferages from on hand inventory, which are nearly proportional to the on hand inventory.

Although there are millions of literatures researching on inventory control issues, a very little portion of which are on the subject of deteriorating products. Ghare and Schrader (1963) were the first to start this type of research by developing a model with exponential deterioration of inventory. Covert and Philip (1973) and Philip (1974) studied models under the assumption that the time of deterioration of an item follows a Weibull distribution. Since then, there have been a number of researchers (e.g. Dave, 1981; Hariga, 1996; Wu, 2001; Deng, 2005; Tadj et al., 2006; Chung and Wee, 2008; Yang et al., 2009; Das et al., 2010) devoting to the inventory control of deteriorating products. Interested readers can refer to Raafat (1991), Shah and Shah (2000), Goyal and Giri (2001), and Li etc. (2010) for excellent reviews of such models..

Recent efforts of the deteriorating inventory research have been focused on considering the partial backlogging of the unsatisfied demand. The motivation is some reality issues since the case of complete backlogging is more likely only in a monopolistic market. In a non-monopolistic market, customers encountering shortages will respond differently. Some customers are willing to wait until the next replenishment, while others may be impatient and go elsewhere as waiting time increases. Therefore, partial backlogging is a necessary consideration for inventory management. Dye et al. (2006) derived a deteriorating inventory model in which the partial backlogging rate linearly depends on the total number of customers in the waiting line. Dye et al. (2007) studied the case that the partial backlogging rate is the reciprocal of a linear function of the waiting time up to the next replenishment. Chern et al. (2008) considered that the fraction of shortages backordered is a differentiable and decreasing function of time. Using the same partial backlogging ratio function as in Chern et al. (2008), Skouri et al. (2009) studied ramp type demand rate and Weibull distribution deterioration. Geetha and

Uthayakumar (2010) presented an EOQ model for deteriorating products with non-instantaneous deterioration, waiting-time-dependent partial backlogging and permissible delay in payments. Cheng et al. (2011) considered a deteriorating inventory model with trapezoidal type demand rate and a non-increasing exponential function of the waiting time up to the next replenishment.

One common feature of the above deteriorating inventory models is that customer demand is assumed to be deterministic. Hence, our paper fills the vacancy by studying the deteriorating inventory control with stochastic customer demand and partial backlogging. In our model, the planning horizon is finite and demand in each period is stochastic. The system is under periodic review, i.e. the inventory level is checked at the beginning of each period and a decision is made on how many to order. There is a fixed cost associated with any positive order. A constant fraction of the positive leftover stock will deteriorate. The excess demand will be partially backlogged at the end of the period. The objective is to determine the optimal ordering policy at the beginning of each period with minimum expected overall costs. There has been some research for this type of model if the effects of deterioration and partial backlogging are not present, which is usually addressed as stochastic lot-sizing model. It was Scarf (1960) who first established the optimal $(s, S)$ structure for the stochastic lot-sizing problem with independent and identically distributed demands in successive periods. This problem was re-studied by Porteus (2002) and a complete proof was provided. Schal (1976) generalized Scarf's result by finding some new conditions for the optimality of an $(s, S)$ policy without assuming particular demand distributions. Iyer etc. (1992) analyzed the deterministic $(s, S)$ inventory problem which is to determine parameters $s$ and $S$ such that implementing this $(s, S)$ policy results in the minimum possible total costs given a set of demands for $n$ periods. Sox (1997) considered the case in which the demand is random and the costs are non-stationary. Gallego etc. (2000) studied the finite ordering capacity and they showed that the optimal capacitated policy has an $(s, S)$-like structure. Sobel and Zhang (2001) considered that the demands arrive simultaneously from a deterministic source and a random source. They proved that a modified $(s, S)$ policy is optimal assuming that the stochastic demand is satisfied immediately if there is sufficient stock on hand. Dellaert and Melo (2003) considered a stochastic manufacturing system with only partial knowledge on future demand because customers tend to order in advance of their actual needs. More recently, Ozer and Wei (2004) considered a capacitated production system faced by a manufacturer who has the ability to obtain advance demand information. Bensoussan etc. (2006) considered the effect of information delay between the current time and the time of the most recent inventory level known to the inventory manager. The optimal ordering policy is respectively base stock policy when there is no fixed order cost and $(s, S)$ policy when fixed order cost exists.

The rest of this paper is organized as follows. Problem description and notations are given in Section 2. Section 3 and 4 present the model and solutions. Section 5 concludes the paper.

# 2 Problem Description and Notations

## 2.1 Problem Description

The problem concerned here can be described as follows. It is a single product, single location problem. The system will be run for $N$ periods. The product has a random life and will deteriorate over time. The customer demand in each period is stochastic. At the beginning of each period, one needs to decide if it is necessary to place an order, and if so, how much to order. A fixed order cost is incurred whenever an order is placed. There is a per-unit cost associated with each order too. Any on-hand inventory at the end of a period can be used in the next period. Any unsatisfied demand can be partially backlogged until fulfilled, or lost. The order decisions are made such that the total expected long-run cost is minimized.

It is assumed that

(1) All demands are independent and identically distributed.
(2) A constant fraction of the positive leftover stock will deteriorate.
(3) The excess demand will be partially backlogged at the end of the period

A penalty cost will be incurred for any backlogging and lost sale amount. The system is under periodic review, i.e. the inventory level is checked at the beginning of each period and a decision is made on how many to order. Porteus (2002) has shown that a ($s$, $S$) ordering policy is optimal under complete backlogging and non-deterioration. Our objective is to identify under what conditions the ($s$, $S$) policy still holds when deterioration and partial backlogging are explicitly taken into account.

## 2.2 Notations

(1) $c$ – unit purchasing cost ($/unit)
(2) $h$ – unit holding cost, charged against positive ending inventory ($/unit)
(3) $b$ – unit backlogging cost, charged against shortages backlogged at the end of a period ($/unit)
(4) $p$ – penalty cost of a lost sale including lost profit ($/unit)
(5) $K$ – fixed ordering cost
(6) $\alpha$–one period discount factor
(7) $D$ – generic random variable representing demand, which is i.i.d over each period
(8) $\Phi$ – one-period demand distribution
(9) $\phi$ – demand density distribution
(10) $x$ – inventory level before ordering (the state of the system)
(11) $y$ – inventory level after ordering (the decision variable)
(12) $\theta$ – constant fraction of positive leftover stock at the end of the period that is deteriorated

(13) $\beta$ – constant fraction of unsatisfied demand during a period that is backlogged
(14) $N$ – the length of the planning horizon
(15) $x^+ = \max(x, 0)$
(16) $x^- = \min(x, 0)$

# 3 Model

## 3.1 Formulation

Let us first examine the one-period problem. Expected one-period holding, backlogging, shortage and deteriorating cost function of level $y$ of inventory after ordering is

$$\begin{aligned} L(y) &= E\left[h(y-D)^+ + c\theta(y-D)^+ + b\beta(D-y)^+ + p(1-\beta)(D-y)^+\right] \\ &= E\left[(h+c\theta)(y-D)^+ + (b\beta + p - p\beta)(D-y)^+\right] \\ &= El(y-D) \end{aligned}$$

where $l(x) = (h+c\theta)x^+ + (b\beta + p - p\beta)(-x)^+$.

Let $G_t(y) = cy + L(y) + \alpha\int_0^\infty f_{t+1}\left[y - D - \theta(y-D)^+ - (1-\beta)(y-D)^-\right]\phi(D)dD$,

then the optimality equations (OE) will be

$$f_t(x) = -cx + \min\left\{G_t(x), \min_{y\geq x}\left[K + G_t(y)\right]\right\}.$$

That is, there is no fixed order cost associated with zero order and a fixed order cost $K$ will be incurred if we order something. One has to make a choice on placing an order or not. If the inventory level at the end of period $N$ is $x$, then the terminal cost $v_T(x)$ is incurred. Letting $G_t^*(x) = \min\left\{G_t(x), \min_{y\geq x}\left[K + G_t(y)\right]\right\}$, then the OE can be rewritten as $f_t(x) = -cx + G_t^*(x)$.

## 3.2 (s, S) Policy

Since there is a fixed ordering cost incurred for any non-zero order, then the ordering cost function is concave, which is different from Periodic-Review Stochastic Lot-Sizing models. If there is no deterioration and backlogging is complete, a ($s$, $S$) policy will be optimal in each period. This policy means whenever the inventory is below some amount $s$, we will place an order to bring the inventory level up to $S$ (where $s \leq S$). The order quantity is greater than or equal to ($S$-$s$). If the inventory level is above $s$, we will not place an order. This ensures that the fixed ordering cost only occurs for a certain amount (i.e. ≥ $S$-$s$). The order will not be placed if the amount is too small.

### 3.3 K-Convex Functions

(Porteus, 2002) A function $f$: **R**→**R** (a real valued function of a single real variable) is $K$-convex if $K\geq 0$, and for each $x \leq y$, $0 \leq \theta \leq 1$, and $\bar{\theta} = 1-\theta$:

$$f(\theta x + \bar{\theta} y) \leq \theta f(x) + \bar{\theta}[K + f(y)].$$

The following Lemma (Porteus, 2002) provides some important properties of $K$-convex function.

Lemma 1

(a) If $f$ is $K$-convex and $a$ is a positive scalar, then $af$ is $k$-convex for all $k \geq aK$.

(b) The sum of a $K$-convex function and a $k$-convex function is $(K+k)$-convex.

(c) If $f$ is $K$-convex, $x < y$, and $f(x) = K + f(y)$, then $f(z) \leq K + f(y)$ for all $z \in [x, y]$.

## 4 Optimality of (*s*, *S*) Policy

In this section, the proof is presented about under what conditions the ($s$, $S$) policy is still optimal in each period.

Lemma 2 If $f_{t+1}$ is a continuous $K$-convex function and $\beta = 1-\theta$, then the following hold:

(a) $G_t$ is a continuous $K$-convex function.

(b) A ($s$, $S$) policy is optimal in period $t$.

(c) $G_t^*$ is a continuous $K$-convex function.

(d) $f_t$ is a continuous $K$-convex function.

Proof (a) From section 3.1, we know that

$$G_t(y) = cy + L(y) + \alpha \int_0^\infty f_{t+1}\left[y - D - \theta(y-D)^+ - (1-\beta)(y-D)^-\right]\phi(D)dD.$$

Define $g(y) = \alpha \int_0^\infty f_{t+1}\left[y - D - \theta(y-D)^+ - (1-\beta)(y-D)^-\right]\phi(D)dD$, then

$$G_t(y) = cy + L(y) + g(y).$$

If we can show that for each $y_1 \leq y_2$, $0 \leq \lambda \leq 1$, and $\bar{\lambda} = 1-\lambda$, the following holds:

$$g(\lambda y_1 + \bar{\lambda} y_2) \leq \lambda g(y_1) + \bar{\lambda}[K + g(y_2)].$$

Then according to Lemma 1 (a), $g(y)$ is a $\alpha K$-convex function. Since $cy$ and $L(y)$ are convex, then according to Lemma 1 (b), it can be shown that $G_t(y)$ is $k$-convex. The following is to show that $g(\lambda y_1 + \bar{\lambda} y_2) \leq \lambda g(y_1) + \bar{\lambda}[K + g(y_2)]$.

$$
\begin{aligned}
& g(\lambda y_1 + \bar{\lambda} y_2) \\
&= \int_0^{\infty} f_{t+1}\left[\lambda y_1 + \bar{\lambda} y_2 - D - \theta(\lambda y_1 + \bar{\lambda} y_2 - D)^+ - (1-\beta)(\lambda y_1 + \bar{\lambda} y_2 - D)^-\right]\phi(D)dD \\
&= \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD \\
&+ \int_{y_1}^{y_2} f_{t+1}\left[\lambda y_1 + \bar{\lambda} y_2 - D - \theta(\lambda y_1 + \bar{\lambda} y_2 - D)^+ - (1-\beta)(\lambda y_1 + \bar{\lambda} y_2 - D)^-\right]\phi(D)dD \\
&+ \int_{y_2}^{\infty} f_{t+1}\left[\beta(\lambda y_1 + \bar{\lambda} y_2 - D)^-\right]\phi(D)dD \\
&= \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD \\
&+ \int_{y_1}^{\lambda y_1 + \bar{\lambda} y_2} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD + \int_{\lambda y_1 + \bar{\lambda} y_2}^{y_2} f_{t+1}\left[\beta(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD \\
&+ \int_{y_2}^{\infty} f_{t+1}\left[\beta(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD \\
&\le \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD \\
&+ \int_{y_1}^{\lambda y_1 + \bar{\lambda} y_2} \left[\lambda f_{t+1}\left[(1-\theta)(y_1 - D)\right] + \bar{\lambda} f_{t+1}\left[(1-\theta)(y_2 - D)\right] + \bar{\lambda} k\right]\phi(D)dD \\
&+ \int_{\lambda y_1 + \bar{\lambda} y_2}^{y_2} \left[\lambda f_{t+1}\left[\beta(y_1 - D)\right] + \bar{\lambda} f_{t+1}\left[\beta(y_2 - D)\right] + \bar{\lambda} k\right]\phi(D)dD \\
&+ \int_{y_2}^{\infty} f_{t+1}\left[\beta(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD \\
&= \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD \\
&+ \int_{y_1}^{\lambda y_1 + \bar{\lambda} y_2} \left[\lambda f_{t+1}\left[\beta(y_1 - D)\right] + \bar{\lambda} f_{t+1}\left[(1-\theta)(y_2 - D)\right] + \bar{\lambda} k\right]\phi(D)dD \\
&+ \int_{\lambda y_1 + \bar{\lambda} y_2}^{y_2} \left[\lambda f_{t+1}\left[\beta(y_1 - D)\right] + \bar{\lambda} f_{t+1}\left[(1-\theta)(y_2 - D)\right] + \bar{\lambda} k\right]\phi(D)dD \\
&+ \int_{y_2}^{\infty} f_{t+1}\left[\beta(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD \\
&\le \lambda g(y_1) + \bar{\lambda}\left[K + g(y_2)\right]
\end{aligned}
$$

Proof of (a) is completed.

(b) Please refer to (Porteus, 2002).

(c) Please refer to (Porteus, 2002).

(d) Please refer to (Porteus, 2002).

The proof of Lemma 2 is completed.

Lemma 3 If $f_{t+1}$ is a continuous decreasing *K*-convex function, $\beta > 1-\theta$ and $\Phi(x_i) < \dfrac{b\beta + p - c - p\beta}{h + c\theta + b\beta + p - p\beta}$ ($i = 1,2,\ldots N$) where $x_i$ is the inventory level before ordering for each period, then the following hold.

(a) $G_t$ is a continuous *K*-convex function.

(b) A (*s*, *S*) policy is optimal in period *t*.

(c) $G_t^*$ is a continuous *K*-convex function.

(d) $f_t$ is a continuous decreasing *K*-convex function.

Proof (a) As of Lemma 2, the objective is to show that for each $y_1 \le y_2$, $0 \le \lambda \le 1$, and $\bar\lambda = 1-\lambda$, it is true that $g(\lambda y_1 + \bar\lambda y_2) \le \lambda g(y_1) + \bar\lambda\left[K + g(y_2)\right]$.

$$g(\lambda y_1 + \bar\lambda y_2)$$

$$= \int_0^\infty f_{t+1}\left[\lambda y_1 + \bar\lambda y_2 - D - \theta(\lambda y_1 + \bar\lambda y_2 - D)^+ - (1-\beta)(\lambda y_1 + \bar\lambda y_2 - D)^-\right]\phi(D)dD$$

$$= \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar\lambda y_2 - D)\right]\phi(D)dD$$

$$+ \int_{y_1}^{y_2} f_{t+1}\left[\lambda y_1 + \bar\lambda y_2 - D - \theta(\lambda y_1 + \bar\lambda y_2 - D)^+ - (1-\beta)(\lambda y_1 + \bar\lambda y_2 - D)^-\right]\phi(D)dD$$

$$+ \int_{y_2}^\infty f_{t+1}\left[\beta(\lambda y_1 + \bar\lambda y_2 - D)\right]\phi(D)dD$$

$$= \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar\lambda y_2 - D)\right]\phi(D)dD$$

$$+ \int_{y_1}^{\lambda y_1 + \bar\lambda y_2} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar\lambda y_2 - D)\right]\phi(D)dD + \int_{\lambda y_1 + \bar\lambda y_2}^{y_2} f_{t+1}\left[\beta(\lambda y_1 + \bar\lambda y_2 - D)\right]\phi(D)dD$$

$$+ \int_{y_2}^\infty f_{t+1}\left[\beta(\lambda y_1 + \bar\lambda y_2 - D)\right]\phi(D)dD$$

$$\le \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar\lambda y_2 - D)\right]\phi(D)dD$$

$$+ \int_{y_1}^{\lambda y_1 + \bar\lambda y_2} \left[\lambda f_{t+1}\left[(1-\theta)(y_1 - D)\right] + \bar\lambda f_{t+1}\left[(1-\theta)(y_2 - D)\right] + \bar\lambda k\right]\phi(D)dD$$

$$+\int_{\lambda y_1+\bar{\lambda}y_2}^{y_2}\left[\lambda f_{t+1}[\beta(y_1-D)]+\bar{\lambda}f_{t+1}[\beta(y_2-D)]+\bar{\lambda}k\right]\phi(D)dD$$

$$+\int_{y_2}^{\infty} f_{t+1}\left[\beta(\lambda y_1+\bar{\lambda}y_2-D)\right]\phi(D)dD$$

$$\lambda g(y_1)+\bar{\lambda}[K+g(y_2)]$$

$$=\lambda\int_0^{y_1} f_{t+1}[(1-\theta)(y_1-D)]\phi(D)dD+\lambda\int_{y_1}^{\lambda y_1+\bar{\lambda}y_2} f_{t+1}[\beta(y_1-D)]\phi(D)dD$$

$$+\lambda\int_{\lambda y_1+\bar{\lambda}y_2}^{y_2} f_{t+1}[\beta(y_1-D)]\phi(D)dD+\lambda\int_{y_2}^{\infty} f_{t+1}[\beta(y_1-D)]\phi(D)dD$$

$$+\bar{\lambda}\int_0^{y_1} f_{t+1}[(1-\theta)(y_2-D)]\phi(D)dD+\bar{\lambda}\int_{y_1}^{\lambda y_1+\bar{\lambda}y_2} f_{t+1}[(1-\theta)(y_2-D)]\phi(D)dD$$

$$+\bar{\lambda}\int_{\lambda y_1+\bar{\lambda}y_2}^{y_2} f_{t+1}[(1-\theta)(y_2-D)]\phi(D)dD+\bar{\lambda}\int_{y_2}^{\infty} f_{t+1}[\beta(y_2-D)]\phi(D)dD$$

$$+\int_0^{y_1}\bar{\lambda}k\phi(D)dD+\int_{y_1}^{\lambda y_1+\bar{\lambda}y_2}\bar{\lambda}k\phi(D)dD+\int_{\lambda y_1+\bar{\lambda}y_2}^{y_2}\bar{\lambda}k\phi(D)dD+\int_{y_2}^{\infty}\bar{\lambda}k\phi(D)dD$$

$$=\int_0^{y_1}\left[\lambda f_{t+1}[(1-\theta)(y_1-D)]+\bar{\lambda}f_{t+1}[(1-\theta)(y_2-D)]+\bar{\lambda}k\right]\phi(D)dD$$

$$+\int_{y_1}^{\lambda y_1+\bar{\lambda}y_2}\left[\lambda f_{t+1}[\beta(y_1-D)]+\bar{\lambda}f_{t+1}[(1-\theta)(y_2-D)]+\bar{\lambda}k\right]\phi(D)dD$$

$$+\int_{\lambda y_1+\bar{\lambda}y_2}^{y_2}\left[\lambda f_{t+1}[\beta(y_1-D)]+\bar{\lambda}f_{t+1}[(1-\theta)(y_2-D)]+\bar{\lambda}k\right]\phi(D)dD$$

$$+\int_{y_2}^{\infty}\left[\lambda f_{t+1}[\beta(y_1-D)]+\bar{\lambda}f_{t+1}[\beta(y_2-D)]+\bar{\lambda}k\right]\phi(D)dD$$

$$\geq\int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1+\bar{\lambda}y_2-D)\right]\phi(D)dD$$

$$+\int_{y_1}^{\lambda y_1+\bar{\lambda}y_2}\left[\lambda f_{t+1}[\beta(y_1-D)]+\bar{\lambda}f_{t+1}[(1-\theta)(y_2-D)]+\bar{\lambda}k\right]\phi(D)dD$$

$$+\int_{\lambda y_1+\bar{\lambda}y_2}^{y_2}\left[\lambda f_{t+1}[\beta(y_1-D)]+\bar{\lambda}f_{t+1}[(1-\theta)(y_2-D)]+\bar{\lambda}k\right]\phi(D)dD$$

$$+\int_{y_2}^{\infty} f_{t+1}\left[\beta(\lambda y_1+\bar{\lambda}y_2-D)\right]\phi(D)dD$$

Since $\beta > 1-\theta$ and $f_{t+1}$ is a decreasing function according to the assumption, the following must hold:

$$\int_{y_1}^{\lambda y_1+\bar{\lambda}y_2} f_{t+1}\left[\beta(y_1-D)\right]\phi(D)dD \geq \int_{y_1}^{\lambda y_1+\bar{\lambda}y_2} f_{t+1}\left[(1-\theta)(y_1-D)\right]\phi(D)dD$$

and $$\int_{\lambda y_1+\bar{\lambda}y_2}^{y_2} f_{t+1}\left[(1-\theta)(y_2-D)\right]\phi(D)dD \geq \int_{\lambda y_1+\bar{\lambda}y_2}^{y_2} f_{t+1}\left[\beta(y_2-D)\right]\phi(D)dD.$$

Then $g(\lambda y_1+\bar{\lambda}y_2) \leq \lambda g(y_1)+\bar{\lambda}\left[K+g(y_2)\right]$, i.e. $g(y)$ is a K-convex function.

The proof of (a) is completed.

(b) Please refer to (Porteus, 2002).

(c) Please refer to (Porteus, 2002).

(d) According to section 3.1, we have:

$$f_t(x) = -cx + \min\left\{G_t(x), \min_{y\geq x}\left[K+G_t(y)\right]\right\},$$

where $G_t(x) = cx + L(x) + \partial\int_0^\infty f_{t+1}\left[y-D-\theta(y-D)^+ -(1-\beta)(y-D)^-\right]\phi(D)dD$ and

$$L(x) = E\left[h(x-D)^+ + c\theta(x-D)^+ + b\beta(D-x)^+ + p(1-\beta)(D-x)^+\right]$$
$$= \int_0^x\left[(h+c\theta)(x-D)\right]\Phi(D)dD + \int_x^\infty\left[(b\beta+p-p\beta)(D-x)\right]\Phi(D)dD.$$

To prove $G_t(x)$ is a decreasing function, we only need to prove $\left[cx+L(x)\right]$ is decreasing because $f_{t+1}(x)$ is given to be a decreasing function.

The first derivative of $\left[cx+L(x)\right]$ is given by:

$$\left[cx+L(x)\right]' = c + h\Phi(x) + c\theta\Phi(x) - b\beta + b\beta\Phi(x) - p + p\Phi(x) + p\beta - p\beta\Phi(x)$$
$$= c - b\beta - p + p\beta + (h + c\theta + b\beta + p - p\beta)\Phi(x),$$

since $\Phi(y_1) < \dfrac{b\beta+p-c-p\beta}{h+c\theta+b\beta+p-p\beta}$, then $\left[cx+L(x)\right]' < 0$. So $\left[cx+L(x)\right]$ is a decreasing function, and therefore, $G_t$ is a decreasing function.

Since $\min_{y\geq x}\left[K+G_t(y)\right] = K + G_t(\bar{y})$, which is a constant, we obtain that $f_t(x) = -cx + \min\left\{G_t(x), \min_{y\geq x}\left[K+G_t(y)\right]\right\}$ is a decreasing function.

From part (*c*) we know that $G_t^*$ is k-convex, then $f_t(x)$ is the summation of a convex function and a *k*-convex function, therefore, *k*-convex itself.

The proof of Lemma 3 is completed.

Based on Lemma 2 and Lemma 3, there exists an optimal (*s*, *S*) policy for each period of the finite planning horizon if either of the following two conditions holds.

(1) $v_T$ is a continuous $K$-convex function and $\beta = 1-\theta$.

(2) $v_T$ is a continuous decreasing $K$-convex function, $\beta > 1-\theta$ and $\Phi(x_i) < \dfrac{b\beta + p - c - p\beta}{h + c\theta + b\beta + p - p\beta}$ ($i = 1,2,\ldots N$) where $x_i$ is the inventory level before ordering for each period.

## 5. Summary

A finite horizon inventory model for a single product is considered in this paper. The system is under periodic review and there is a fixed order cost associated with any non-zero order. The demand in successive periods is independent and identically distributed. A constant fraction of any positive leftover stock is deteriorated at the end of each period. Any unsatisfied demand is partially backlogged and fulfilled immediately as a new order arrives. This paper identified under what conditions the (*s*, *S*) policy is still optimal. One drawback is that the two conditions derived are both rigid to some level and the explicit form of (*s*, *S*) is very difficult to obtain. This can be overcome by either formulating the problem as a mixed-integer programming model with a service-level constraint or a stochastic programming model with discretized customer demand, which will be discussed in our future research.

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