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Relation of Finite Mellin Integral Transform with Laplace and Fourier Transforms

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Abstract

The aim of this paper is to derive the relation between the Finite Mellin integral transform with the Laplace transform by using the double Laplace and Fourier – Finite Mellin integral. Properties like linearity property, scaling roperty, power property and f(ax)g(by) are also derived. The shifting and inversion theorems for Laplace-Finite Mellin integral ransform and Fourier-Finite Mellin integral transforms are also discussed.

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1: Introduction

The Laplace transform and Fourier Transform are widely used for solving differential and integral equations. In physics and engineering it is used for analysis of linear time -invariant systems such as electrical circuits , harmonic oscillators , optical devices and mechanical systems. In this analysis , the Laplace transform is often interpreted as transformation from the time-domain in which inputs and outputs functions of time to the frequency-domain, where the same inputs and outputs are functions of complex annular frequency in radians per unit time. Given a simple mathematical or functional description of an input or output to a system. The Laplace transform provides an alternative functional as description that often simplifies the process of analyzing the behavior of the system or in synthesizing a view system based on a set of specifications. Fourier Transform is often use in signal processing.
The theory of integral has presented a direct and systematic technique for the resolution of certain type of classical boundary and initial value problems .To be successful the transform must be adopted to the form of the differential operator- to be eliminated as well as to the range of interest and the associated boundary conditions. There are numerous cases for which no suitable transform exists.
Here we consider Laplace Finite Mellin integral transform to the removal of the polar operators that occur when Laplace operator is expressed in either spherical or plane polar coordinates

The Double Laplace Transform can be used to find the Laplace –Finite Mellin Integral Transform in the range $[0,\infty]$.and [0,a]. Fourier Laplace transform is used to find the Fourier- Finite Mellin integral transform in the range $[0,\infty]$.and [0,a].

2: Preliminary Results

2.1: Relation Of The Finite Mellin Integral Transform With Laplace Transform

The Laplace transform of the function f(x) of x is denoted by L[f(x),r] and defined as L[f(x),s]= $\int_0^\infty e^{-sx} f(x)dx,$

whenever this integral is exists for r>0 is the parameter

The inverse of the Laplace transform is denoted by $L^{-1}[f(x),r] = f(x)$ and defined as $L^{-1}[f(x),s] = f(x) = \frac{1}{2\pi i}\int_{c1-i\infty}^{c1+i\infty} e^{sx} L[f(x),r]dr$

The Mellin integral transform of the function of f(y) of y is denoted by M[f(y),s] and is defined as M[f(y),r]= $\int_0^\infty y^{r-1} f(y)dy$

whenever this integral is exists for s>0 parameter.

The inversion of the Mellin integral transform is denoted by $M^{-1}[f(y),r)] = f(y)$ and defined as $M^{-1}[f(y),r)] = f(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-r} M[f(y),s]ds$

The Finite Mellin integral transform for f(y) of y is denoted by $M\ [f(y(,r]$ and defined as $M\ [f(y(,r] = \int_0^a a^{-r} y^{r-1} f(y)dy$, whenever this integral is exists for r>0 ,parameter by using x= $-\log(\frac{y}{a})$ in the Laplace transform.

Its inverse is denoted by $f(y) = M_f^{-1}[f(y),r]$ and defined as

$$f(y) = M_f^{-1}[f(y),r] = \frac{1`}{2\pi i}\int_{c-ia}^{c+ia} y^{-r} M_f[f(y),r]dr$$

for r>0, whenever this integral is exists.

The double Laplace transform is denoted by $L_2[f(x,z),s,r]$ and defined as

$$L_2[f(x,z),s,0,\infty,r,0,\infty] = \int_0^\infty \int_0^\infty e^{-(sx+rz)} f(x,z)dxdz$$

whenever this double integral is exists for r>0 and s>0 are parameters.

Substitute z= -log($\frac{y}{a}$) then y=ae^{-z} , dz= -$\frac{dy}{y}$,if z=0 then y=a and if z=∞ then y=0

$$L_2[f(x,z),s,0,\infty,r,0,\infty]=\int_0^\infty\int_0^\infty e^{-(sx+rz)}f(x,z)dxdz$$

$$=\int_0^\infty\int_0^\infty e^{-sx}e^{-rz}f(x,z)dxdz$$

$$=\int_0^\infty\int_0^\infty e^{sx}(e^{-z})^r f(x,z)dxdz$$

$$=\int_0^\infty\int_a^0 e^{-sx}(\frac{y}{a})^r f(x,-\log(\frac{y}{a}))dx(\frac{-dy}{a})$$

$$L_2[f(x,z),s,0,\infty,r,0,\infty]=\int_0^\infty\int_0^a a^{-r}e^{-sx}y^{r-1}f(x,y)dxdy$$

This is the relation between Finite Mellin integral and Laplace transform for f(x, y) with parameters r>0 and s>0 in the range [0,0; ∞, a] and is denoted as $L_f M[f(x,y),r,s]$

$$L_f M[f(x,y),s,0,\infty,r,0,a]=\int_0^\infty\int_0^a a^{-r}e^{-sx}y^{r-1}f(x,y)dxdy \qquad (1)$$

where $0 \prec x \le \infty$ **and** $0 \le y \le a$

2.2: Relation Of The Finite Mellin Integral Transform With Fourier Transform

The Fourier transform is denoted by F{f(x),s]=L[s] and can be defined as

$$F[f(x),s]=L[s]=\int_0^\infty e^{-isx}f(x)dx,$$

whenever this integral is exists. for parameter s>0
The inverse of the Fourier Transform is denoted by $F^{-1}[f(x),s]=f(x)$ and defined as

$$F^{-1}[f(x),s]=f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{isx}F[f(x),s]ds$$

whenever parameter s>0

The Mellin Integral Transform in the following way

$$M[f(x),r]=\int_0^\infty x^{r-1}f(x)dx \quad ,\alpha<\mathrm{Re}(s)<\beta$$

where α and β are real numbers deterring the maximum range of values of Re(z) such that the integral converges.

The inversion of the Mellin integral transform is denoted by $M^{-1}[f(y),s)]=f(y)$

and defined as $M^{-1}[f(y),s)]=f(y)=\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} y^{-r}M[f(y),s]ds$

The Finite Mellin integral transform for f(y) of y is denoted by $M\ [f(y(,r]$

and defined as $M\ [f(y(,r]=\displaystyle\int_0^a a^{-r}y^{r-1}f(y)dy$,whenever this integral is exists for r>0 ,parameter by using x= $-\log(\frac{y}{a})$ in the Laplace transform.

Its inverse is denoted by $f(y)=M_f^{-1}[f(y),r]$ and defined as

$$f(y)=M_f^{-1}[f(y),r]=\frac{1`}{2\pi i}\int_{c-ia}^{c+ia} y^{-r}M_f[f(y),r]dr$$

for r>0, whenever this integral is exists.

The Fourier Laplace transform $FL[f(x,z),s,r]$ can be defined as

$$FL[f(x,z),s,0,\infty,r,0,\infty]=\int_0^\infty\int_0^\infty f(x,z)e^{-(isx+rz)}dxdz$$

whenever this integral is exists. for parameter s>0 ,r>0

substitute z= -log(y/a) ,y= ae^{-z} ,dz= $-\dfrac{dy}{y}$,

If z=0 then y= a and if z= ∞ then y=0

$$FL[f(x,z),s,0,\infty,r,0,\infty]=\int_0^\infty\int_a^0 f(x,-\log(\frac{y}{a}))e^{-isx}e^{r\log(\frac{y}{a})}dx(\frac{-dy}{y})$$

$$=\int_0^\infty\int_a^0 f(x,-\log(\frac{y}{a}))e^{-isx}e^{\log(\frac{y}{a})^r}dx(\frac{-dy}{y})$$

$$=\int_0^\infty\int_a^0 f(x,-\log(\frac{y}{a}))e^{-six}(\frac{y}{a})^r dx(\frac{-dy}{y})$$

$$FL[f(x,z),s,0,\infty,r,0,\infty]=\int_0^\infty\int_0^a a^{-r}f(x,y)e^{-isx}y^{r-1}dxdy$$

This is the **Fourier Finite Mellin Integral Transform in [0, inf],[0,a],** where a is greater than zero (a>0) .
This integral is denoted by $F_f M[f(x,y),s,r]$.in $[0,0;\infty,a]$

$$F_f M[f(x,y),s,0,\infty,r,0,a]=\int_0^\infty\int_0^a a^{-r}f(x,y)e^{-isx}y^{r-1}dxdy \qquad (2)$$

where $0\prec x\le\infty$ **and** $0\le y\le a$

3: LEMMAS (Laplace-Finite Mellin integral Transform-LFMIT)

3.1:Lemma.1

3.1.1: The LFMIT is

$$L_f M\ [f(x,y),s,0,\infty,r,0,a]=\int_0^\infty\int_0^a a^{-r}f(x,y)e^{-sx}y^{r-1}dxdy \text{ ,then}$$

3.1.1: $L_f M\ [\alpha\ f(x,y)+\beta\ g(x,y),s,0,\infty,r,0,a]$

$$=\alpha\ L_f M\ [f(x,y),s,0,\infty,r,0,a]+\beta\ L_f M\ [g(x,y),s,0,\infty,r,0,a]$$

3.1.2: $L_f M\ [f(x,y^n),s,0,\infty,r,0,a]=\frac{1}{n}L_f M[f(x,z),s,0,\infty,\frac{r}{n},0,a^n]$

Proof

3.1.1: If $L_f M\ [f(x,y),s,0,\infty,r,0,a]=\int_0^\infty\int_0^a a^{-r}f(x,y)e^{-sx}y^{r-1}dxdy$,then

$$L_f M\ [\alpha\ f(x,y)+\beta\ g(x,y),s,0,\infty,r,0,a]=\int_0^\infty\int_0^a a^{-r}[\alpha f(x,y)+\beta g(x,y)]e^{-sx}y^{r-1}dxdy$$

$$=\int_0^\infty \int_0^a a^{-r}\alpha f(x,y)e^{-st}z^{r-1}dtdz+\int_0^\infty \int_0^a a^{-r}\beta g(x,y)e^{-st}z^{r-1}dtdz$$

$$=\alpha \int_0^\infty \int_0^a a^{-r} f(x,y)e^{-sx}y^{r-1}dxdy+\beta \int_0^\infty \int_0^a a^{-r} g(x,y)e^{-sx}y^{r-1}dxdy$$

$=\alpha$ $L_f M$ [f(x,y),s,0,∞,r,0,a]+ β $L_f M$ [g(x,y),s ,0,∞,r,0,a]

$L_f M$ [α f(x,y)+ β g(x,y),s,0,∞,r,0,a]

$=\alpha$ $L_f M$ [f(x,y),s,0,∞,r,0,a]+ β L_m [g(x,y),s,0,∞,r,0,a] (3)

3.1.2: The Laplace-Finite Mellin integral transform is

$L_f M$ [f(x, y),s,0,∞,r,0,a]= $\int_0^\infty \int_0^a a^{-r} f(x,y)e^{-sx}y^{r-1}dxdy$,then

$L_f M$ [f(x, y^n),s,0,∞,r,0,a]= $\int_0^\infty \int_0^a a^{-r} f(x,y^n)e^{-sx}y^{r-1}dxdy$

substitute $y^n = z$, $y = z^{\frac{1}{n}}$,dy= $\frac{1}{n}z^{\frac{1}{n}-1}dz$, If y=0 then z=0 and if y=a then z= a^n

$L_f M$ [f(x , y^n),s,0,∞,r,0,a] = $\int_0^\infty \int_0^{a^n} a^{-r} f(x,y)e^{-st}(x^{\frac{1}{n}})^{r-1}dx\frac{1}{n}z^{\frac{1}{n}-1}dz$

$$=\frac{1}{n}\int_0^\infty \int_0^{a^n} a^{-r} f(x,z)e^{-sx}z^{\frac{r}{n}-1}dxdz$$

$$=\frac{1}{n}L_f M[f(x,z),s,0,\infty,\frac{r}{n},0,a^n]$$

$L_f M$ [f(x , y^n),s,0,∞,r,0,a] $=\frac{1}{n}L_f M[f(x,z),s,0,\infty,\frac{r}{n},0,a^n]$ (4)

3.2:Lemma-2

The LFMIT is

$L_f M$ [f(t ,z),s,0,∞,r,0,a]= $\int_0^\infty \int_0^a a^{-r} f(t,z)e^{-st}z^{r-1}dtdz$,then

3.2.1 $L_f M$ [f(cx , dy),s,0,∞,r,0,a] $=\frac{1}{cd^r}$ $L_f M$ [f(t,z),s/c,0,∞,r,0,ad]

3.2.2: $L_f M$ [f(cx)g(dy),s,0,∞,r,0,a] $=\frac{1}{cd^p}$ $L_f M$ [f*x),s/c]M[g(y),r,0,ad]

Proof

3.2.1: The Laplace-Finite Mellin integral transform is

$$L_f M\ [f(x,y),s,0,\infty,r,0,a]=\int_0^\infty\int_0^a a^{-r} f(x,y)e^{-sx}y^{r-1}dxdy \text{ ,then}$$

$$L_f M\ [f(cx,dy),s,0,\infty,r,0,a]=\int_0^\infty\int_0^a a^{-r} f(cx,dy)e^{-sx}y^{r-1}dxdy$$

Substitute cx=t and dy=z then z=t/c and y=z/d , dx=dt/c and dy=dz/d
If x=y=0 then t=z=0 and if x=∞ then t=∞ and if y=a then z=ad

$$L_f M\ [f(cx,dy),s,0,\infty,r,0,a]=\int_0^\infty\int_0^{ad} a^{-r} f(t,z)e^{-\frac{st}{c}}(\frac{z}{d})^{r-1}\frac{dtdz}{cd}$$

$$=\frac{1}{cd^r}\int_0^\infty\int_0^{ad} a^{-r} f(t,z)e^{-\frac{st}{c}}z^{r-1}dtdz$$

$$L_f M\ [f(cx,dy),s,0,\infty,r,0,a]=\frac{1}{cd^r}\ L_f M\ [f(t,z),s/c,0,\infty,r,0,ad] \qquad (5)$$

3.2.2: The Laplace-Finite Mellin integral transform is

$$L_f M\ [f(x,y),s,0,\infty,r,0,a]=\int_0^\infty\int_0^a a^{-r} f(x,y)e^{-sx}y^{r-1}dxdy \text{ ,then}$$

$$L_f M\ [f(cx)g(dy),s,0,\infty,r,0,a]=\int_0^\infty\int_0^a a^{-r} f(ax)d(dy)e^{-sx}y^{r-1}dxdy$$

$$=\int_0^\infty e^{-st} f(cx)dx\int_0^a a^{-r} g(dy)y^{r-1}dy$$

substitute cx=t and dy=z then x=t/c and y=z/d ,dx=dt/c and dy=dz/d
If x=y=0 then t=z=0 and if x=∞ then t=∞ and if y=a then z=ad

$$=\int_0^\infty e^{-\frac{st}{c}} f(t)\frac{dt}{c}\int_0^{ad} a^{-r} g(z)(\frac{z}{d})^{r-1}\frac{dz}{d}$$

$$=\frac{1}{cd^r}\int_0^\infty e^{-\frac{st}{c}} f(t)dt\int_0^{ad} a^{-r} g(z)z^{r-1}dz$$

$$=\frac{1}{cd^r}\text{ L[f(t),s/c]M[g(z),r,0,ad]}$$

$$L_f M\ [f(cx)g(dy),s,0,\infty,r,0,a]=\frac{1}{cd^r}\ L_f M\ [f(t),s/c]M[g(z),r,0,ad] \quad (6)$$

4: LEMMAS (Fourier-Finite Mellin Integral Transform-FFMIT)

4.1: Lemma

Let f(x ,y), x ,y$\in \Re_+$,

$$F_f M[f(x,y),s,r]=\int_0^\infty \int_0^a a^{-r} f(x,y)e^{-isx} y^{r-1} dxdy \text{ , then}$$

4.1.1: $F_f M[\alpha f(x,y)+\beta g(x,y),s,r]=\alpha\ F_f M[f(x,y),s,r]+\beta\ F_f M[g(x,y),s,r]$

4.1.2: $F_f M[f(x,y^n),s,r]=\frac{1}{n}\ F_f M[f(x,y),s,\frac{r}{n}]$

Proof

4.1.1 : Let f(x ,y) and g(x ,y) , x ,y$\in \Re_+$, and α and β are constants then

$$F_f M[f(x,y),s,r]=\int_0^\infty \int_0^a a^{-r} f(x,y)e^{-isx} y^{r-1} dxdy \quad \text{becomes}$$

$$F_f M[\alpha f(x,y)+\beta g(x,y),s,r] \int_0^\infty \int_0^a a^{-r}[\alpha f(x,y)+\beta g(x,y)]e^{-isx} y^{r-1} dxdy$$

$$=\int_0^\infty \int_0^a a^{-r}\alpha g(x,y)e^{-isx} y^{r-1} dxdy+\int_0^\infty \int_0^a a^{-r}\beta g(x,y)e^{-isx} y^{r-1} dxdy$$

$$=\alpha \int_0^\infty \int_0^a a^{-r}\alpha f(x,y)e^{-isx} y^{r-1} dxdy+\beta \int_0^\infty \int_0^a a^{-r} g(x,y)e^{-isx} y^{r-1} dxdy$$

$$=\alpha\ F_f M[f(x,y),s,r]+\beta\ F_f M[g(x,y),s,r]$$

$$F_f M[\alpha f(x,y)+\beta g(x,y),s,r]=\alpha\ F_f M[f(x,y),s,r]+\beta\ F_f M[g(x,y),s,r] \quad (7)$$

4.1.2: Let f(x , y) and g(x ,y) , x ,y$\in \Re_+$, and α and β are constants then

$$F_f M[f(x,y),s,r]=\int_0^\infty \int_0^a a^{-r} f(x,y)e^{-isx} y^{r-1} dxdy \quad \text{becomes}$$

$$F_f M[f(x,y^n),s,r]=\int_0^\infty \int_0^a a^{-r} f(x,y^n)e^{-isx} y^{r-1} dxdy$$

Substitute $y^n = z$, $y = z^{\frac{1}{n}}$, $dy = \frac{1}{n} z^{\frac{1}{n}-1} dz$

If y=0 then z=0 and if y=a then z=a^n

$$F_f M[f(x,y^n),s,r]=\int_0^\infty \int_0^{a^n} a^{-r} f(x,z)e^{-isx}(z^{\frac{1}{n}})^{r-1}dx\frac{1}{n}z^{\frac{1}{n}-1}dz$$

$$=\frac{1}{n}\int_0^\infty \int_0^{a^n} a^{-r} f(x,z)e^{-isx}z^{\frac{r}{n}-1}dxdz$$

$$=\frac{1}{n} F_f M[f(x,y),s,\frac{r}{n}]$$

$$F_f M[f(x,y^n),p,q]=\frac{1}{n} F_f M[f(x,y),s,\frac{r}{n}] \qquad (8)$$

4.2: Lemma

Let **f** and g be two real valued functions, and

$$F_f M[f(x,y),s,r]=\int_0^\infty \int_0^{a} a^{-r} f(x,y)e^{-isx}y^{r-1}dxdy \quad , \text{ then}$$

4.2.1: $F_f M[f(cx,dy),s,r] = \frac{1}{cd^r} F_f M[f(t,z),\frac{s}{c},r]$

4.2.1: $F_f M[f(ax)g(by),s,r]=\frac{1}{ab^r} F_f M[f(u),\frac{s}{a}]M[g(v),r,0,\frac{b}{a}]$

Proof (4.2.1) Let **f** and g be two real valued functions , then

$$F_f M[f(x,y),s,r]=\int_0^\infty \int_0^{a} a^{-r} f(x,y)e^{-isx}y^{r-1}dxdy \quad , \text{becomes}$$

$$F_f M[f(cx,dy),s,r]=\int_0^\infty \int_0^{a} a^{-r} f(x,y)e^{-isx}y^{r-1}dxdy$$

substitute If cx=t ,x=t/c , dx=dt/c and dy=z ,y=z/d ,dy=dz/d if x=y=0 then t=z=0 and if x=∞ then t=∞ and if y=a then z=ad

$$F_f M[f(cx,dy),s,r]=\int_0^\infty \int_0^{ad} a^{-r} f(t,z)e^{-ist/c}(\frac{z}{d})^{r-1}\frac{dtdz}{cd}$$

$$=\frac{1}{cd^r}\int_0^\infty \int_0^{ad} a^{-r} f(t,z)e^{-ist/c}z^{r-1}dtdz$$

$$=\frac{1}{cd^r} F_f M[f(t,z),\frac{s}{c},r]$$

$$F_f M[f(cx,dy),p,q] = \frac{1}{cd^r} F_f M[f(t,z),\frac{s}{c},r] \qquad (9)$$

4.2.2: Let **f** and g be two real valued functions , then

$$F_f M[f(x,y),s,r] = \int_0^\infty \int_0^a a^{-r} f(x,y) e^{-isx} y^{r-1} dxdy \text{ , becomes}$$

$$F_f M[f(ax)g(by),s,r] = \int_0^\infty \int_0^a a^{-r} f(ax)g(by) e^{-isx} y^{r-1} dxdy$$

$$= \int_0^\infty f(ax) e^{-isx} dx \int_0^a a^{-r} g(by) y^{r-1} dy$$

If ax=u ,x=u/a , dx=du/a and by=v , y=v/b ,dy=dv/b

If x=y=0 then u=v=0 and x=∞ then u=∞ and if y=a then v=ab

$$F_f M[f(ax)g(by),s,r] = \int_0^\infty f(u) e^{-isu/a} \frac{du}{a} \int_0^{\frac{b}{a}} a^r g(y) (\frac{v}{b})^{r-1} \frac{dv}{b}$$

$$= \frac{1}{ab^r} \int_0^\infty f(u) e^{-isu/a} du \int_0^{\frac{b}{a}} a^r g(y) v^{r-1} dv$$

$$= \frac{1}{ab^r} \; F_f M[f(u), \frac{s}{a}] M[g(v), r, 0, \frac{b}{a}]$$

$$F_f M[f(ax)g(by),s,r] = \frac{1}{ab^r} \; F_f M[f(u), \frac{s}{a}] M[g(v), r, 0, \frac{b}{a}] \qquad (10)$$

5. SHIFTING THEOREMS

5.1 Shifting Theorem For Laplace-Finite-Finite Mellin Integral Transform

The shifting theorem for Laplace- Finite Mellin integral transform is

$$L_f M \; [f(x,y),s,0,\infty,r,0,a] = \int_0^\infty \int_0^a a^{-r} f(x,y) e^{-sx} y^{r-1} dxdy \text{ , then}$$

$$L_f M \; [e^{-bx} y^n f(x,y),s,0,\infty,r,0,a] = L_f M \; [f(x,y),s+a,0,\infty,r+n,0,a]$$

Proof

If $L_f M$ [f(x,y),s,0,∞,r,0,a]= $\int_0^\infty \int_0^a a^{-r} f(x,y) e^{-sx} y^{r-1} dxdy$,then

$$L_f M \; [e^{-bx} y^n f(x,y),s,0,\infty,r,0,a] = \int_0^\infty \int_0^a a^{-r} e^{-bx} y^n f(x,y) e^{-sx} y^{r-1} dxdy$$

$$=\int_0^\infty \int_0^a a^{-r} f(x,y) e^{-(s+a)x} y^{r+n-1} dxdy$$

$$L_f M\ [e^{-bx} y^n f(x,y), s, 0, \infty, r, 0, a] = L_f M\ [f(x,y), s+a, 0, \infty, r+n, 0, a] \quad (11)$$

5.2 Shifting Theorem For The Fourier-Finite Mellin Integral Transform

Let f(x ,y) be the two variable function in x and y , then the LFMIT is

$$F_f M[f(x,y), s, r] = \int_0^\infty \int_0^{\frac{1}{a}} a^r f(x,y) e^{-isx} y^{r-1} dxdy \ , \text{ then}$$

$$F_f M[f(x,y), s, r] = F_f M[f(x,y), s+a, r+n]$$

Proof

If $F_f M[f(x,y), s, r] = \int_0^\infty \int_0^{\frac{1}{a}} a^r f(x,y) e^{-isx} y^{r-1} dxdy$, then

$$F_f M[e^{-iax} y^n f(x,y), s, r] = \int_0^\infty \int_0^{\frac{1}{a}} a^r e^{-iax} y^n f(x,y) e^{-isx} y^{r-1} dxdy$$

$$= \int_0^\infty \int_0^{\frac{1}{a}} a^r f(x,y) e^{-i(s+a)x} y^{r+n-1} dxdy$$

$$F_f M[e^{-iax} y^n f(x,y), s, r] = F_f M[f(x,y), s+a, r+n] \quad (12)$$

6: Inversion Theorems

6.1: Inversion Theorem For The Laplace-Finite Mellin Integral Transform

If the Laplace transform of the function f(x) is defined as

$$L[f(x), r, 0, \infty] = \int_0^\infty e^{=rx} f(x) dx \text{ , whenever this integral is exist,}$$

s>0,is the parameter ,then its inverse transform is

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} L[f(x), r, 0, \infty] ds \ .$$

The Mellin integral transform of f(y) in 0 to a is defined as

$$M[f(y),s,0,a]=\int_0^a a^{-s}y^{p-1}f(y)dy$$,whenever this integral is exist

and p>0 is the parameter .,then its inversion formula is

$$f(y)=\frac{1}{2\pi i}\int_{c-ia}^{c+ia} y^{-s}M[f(y),s,0,a]ds,$$

for the Laplace- Finite Mellin transform is

$$L_f M\ [f(x,y),s,0,\infty,r,0,a]=\int_0^\infty\int_0^a a^{-r}f(x,y)e^{-sx}y^{r-1}dxdy,$$

and its inversion formula is

$$f(x,y)=\frac{1}{(2\pi i)^2}\int_{c2-i\infty}^{c2+i\infty}\int_{c1-ia}^{c1+ia}\frac{e^{sx}y^{-r}}{sp}L_f M[f(x,y),s,0,\infty;r,0,a]drds$$

Pr**oof**

Assume that $L_f M$ [f(x ,y),s,0,∞;r,0,a] is a regular function in the strips Re(s) $|<$ r ('r' to be real number) of the s-plane and that 0<c<ν_1, c_1-i∞ ≤s ≤c_1+ i∞ where c_1 is constant and $|$Re(p) $|<$ q ('q' to be real number) of the p-plane and that 0<c<ν_2,c_2-i∞ ≤p ≤c_2+ i∞ where c_2 is constant ,

If the Laplace- Finite Mellin integral transform is

$$L_f M\ [f(x,y),s,0,\infty r,s,0,a]=\int_0^\infty\int_0^a a^{-r}f(x,y)e^{-sx}y^{r-1}dxdy,$$

then

$$L_f M\ [f(x,y),s,0,\infty,r,0,a]$$

$$=\int_0^\infty\int_0^a a^{-r}e^{-sx}y^{r-1}\left[\frac{1}{(2\pi i)^2}\int_{c2-iN_2}^{c2+iN_2}\int_{c1-iN_1}^{c1+iN_1}e^{sx}y^{-r}Lm[f(x,y),s,0,\infty;r,0,a]drds\right]dxdy$$

$$=1\int_0^\infty\int_0^a a^{-r}e^{-sx}y^{r-1}dxdy$$

$$=\frac{1}{(2\pi i)^2}\int_{c2-iN_2}^{c2+iN_2}\int_{c1-iN_1}^{c1+iN_1}e^{sx}y^{-r}Lm[f(x,y),s,0,\infty;r,0,a]drds\int_0^\infty e^{-sx}dx\int_0^a a^{-r}y^{r-1}dy$$

$$=\frac{1}{(2\pi i)^2}\int_{c2-iN_2}^{c2+iN_2}\int_{c1-iN_1}^{c1+iN_1}e^{sx}y^{-r}Lm[f(x,y),s,0,\infty;r,0,a]drds\ a^{-r}\left[\frac{e^{-sx}}{-s}\right]_0^\infty\left[\frac{y^r}{p}\right]_0^a$$

$$=\frac{1}{(2\pi i)^2}\int_{c2-iN_2}^{c2+iN_2}\int_{c1-iN_1}^{c1+iN_1} e^{sx}y^{-r}Lm[f(x,y),s,0,\infty;r,0,a]drds\,\frac{1}{sp},$$

$$=\frac{1}{(2\pi i)^2}\int_{c2-iN_1}^{c2+iN_1}\int_{c1-iN_2}^{c1+iN_2} \frac{e^{sx}y^{-r}}{sp}L_f M[f(x,y),s,0,\infty;r,0,a]drds$$

As $N_1 \to \infty$ and $N_2 \to a$

This is the inversion of the Laplace-Finite Mellin integral transform and it is denoted as f(x,y), then

$$f(x,y)=\frac{1}{(2\pi i)^2}\int_{c2-\infty a}^{c2+i\infty}\int_{c1-ia}^{c1+ia} \frac{e^{sx}y^{-r}}{sp}L_f M[f(x,y),s,0,\infty;r,0,a]drds \qquad (13)$$

6.2: Inversion Theorem For The Fourier-Finite Mellin Integral Transform

Let f(x ,y) be the two variable function in x and y , then the LFMIT

$$F_f M[f(x,y),s,r]=\int_0^\infty\int_0^{\frac{1}{a}} a^r f(x,y)e^{-isx}y^{r-1}dxdy \quad ,$$

Then the inversion result is

$$f(x,y)=F_f M^{-1}[f(x,y),s,r]=\frac{1}{(2\pi i)^2}\int_{c1-i\infty}^{c1+i\infty}\int_{c2-ia}^{c2+ia}\frac{1}{isr}e^{isx}y^{-r}F_f M[f(x,y),s,r]dsdr$$

Proof

Assume that $F_f M[f(x,y),s,r]$ is a regular function in the strips

$|\text{Re}(s)| <$ r ('r' be a real number) of the s-plane and that 0<c< ν_1,

c_1-i∞ ≤s ≤ c_1+ i∞ where c_1 is constant and $|\text{Re}(p)| <$ q

('q' be a real number) of the p-plane and that 0<c< ν_2, c_2-i∞ ≤p ≤ c_2+ i∞

where c_2 is constant

Let f(x ,y) be the two variable function in x and y , then the FFMIT is

$$F_f M[f(x,y),s,r]=\int_0^\infty\int_0^a a^{-r} f(x,y)e^{-isx}y^{r-1}dxdy \quad ,\text{then}$$

$$L_f M[f(x,y),s,r]=\int_0^\infty\int_0^a a^{-r}[\frac{1}{(2\pi i)^2}\int_{c1-iN_1}^{c1+iN_1}\int_{c2-iN_2}^{c2+iN_2} e^{isx}y^{-r}F_f M[f(x,y),s,r]dsdr]e^{-isx}y^{r-1}dxdy$$

$$=\frac{1}{(2\pi i)^2}\int_{c1-iN_1}^{c1+iN_1}\int_{c2-iN_2}^{c2+iN_2} e^{isx}y^{-r}F_f M[f(x,y),s,r]dsdr[\int_0^\infty\int_0^{\frac{1}{a}a} a^r e^{-isx}y^{r-1}dxdy]$$

$$=\frac{1}{(2\pi i)^2}\int_{c1-iN_1}^{c1+iN_1}\int_{c2-iN_2}^{c2+iN_2} e^{isx}y^{-r}F_fM[f(x,y),s,r]dsdra^{-r}[\int_0^{\infty}e^{-ist}dx\int_0^{a} y^{r-1}dy]$$

$$=\frac{1}{(2\pi i)^2}\int_{c1-iN_1}^{c1+iN_1}\int_{c2-iN_2}^{c2+iN_2} e^{isx}y^{-r}F_fM[f(x,y),s,r]dsdra^{r}[\frac{e^{-ist}}{-is}]_0^{\infty}[\frac{y^r}{r}]_0^a$$

$$=\frac{1}{(2\pi i)^2}\int_{c1-iN_1}^{c1+iN_1}\int_{c2-iN_2}^{c2+iN_2} e^{isx}y^{-r}F_fM[f(x,y),s,r]dsdra^{-r}[\frac{e^{-\infty}+e^0}{-s}][\frac{(a)^r}{r}]$$

$$=\frac{1}{(2\pi i)^2}\int_{c1-iN_1}^{c1+iN_1}\int_{c2-iN_2}^{c2+iN_2} e^{sx}y^{-r}F_fM[f(x,y),s,r]dsdra^{r}[\frac{-e^{-\infty}-e^0}{-is}][\frac{(a)^r}{r}]$$

$$=\frac{1}{(2\pi i)^2}\int_{c1-iN_1}^{c1+iN_1}\int_{c2-iN_2}^{c2+iN_2} \frac{1}{isr}e^{isx}y^{-r}F_fM[f(x,y),s,r]dsdr$$

This is the inversion of the Laplace Fractional Mellin integral transform in $[0,0]$ to $[\infty,\frac{1}{a}]$ as $N_1\to\infty$ and $N_2\to a$ is denoted by f(x,y)=$L_fM^{-1}[f(x,y),s,r]$

$$\text{f(x ,y)}=F_fM^{-1}[f(x,y),s,r]=\frac{1}{(2\pi i)^2}\int_{c1-i\infty}^{c1+i\infty}\int_{c2-ia}^{c2+ia}\frac{1}{isr}e^{isx}y^{-r}F_fM[f(x,y),s,r]dsdr \qquad (14)$$

7: TABLES

7.1:Laplace Transform And Finite Mellin Integral Transform

1	**Laplace Transform**	**Finite Mellin Integral Transform**
	The Laplace Transform of f(x) is denoted by L[f(x),s] and defined as L[f(x),s]= $\int_0^\infty e^{-sx} f(x)dx,$ whenever this integral is exists for s>0 is the parameter	The Finite Mellin integral transform of the function of f(y) of y is denoted by M[f(y),r] and is defined as M[f(y),r]= $\int_0^a a^{-r} y^{r-1} f(y)dy$ whenever this integral is exists for r>0 is the parameter
2	**Inverse Laplace Transform**	**Inverse Finite Mellin Integral Transform**
	The inverse of the Laplace transform is denoted by $L^{-1}[f(x),s] = f(x)$ and defined as $L^{-1}[f(x),s] = f(x) =$ $\frac{1}{2\pi i}\int_{c1-i\infty}^{c1+\infty} e^{sx} L[f(x),r]dr$	The inversion of the Mellin integral transform is denoted by $M^{-1}[f(y),r)] = f(y)$ and defined as $M^{-1}[f(y),r)] = f(y) =$ $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-r} M[f(y)],s]ds$

7.2: Fourier Transform And Finite Mellin integral Transform

1	**Fourier Transform**	**Finite Mellin Integral Transform**
	The Fourier transform is denoted by F{f(x),s]=L[s] and defined as $F[f(x),s]=L[s]=\int_0^{\infty} e^{-isx} f(x)dx$, whenever this integral is exists. for parameter s>0	The Finite Mellin integral transform of the function of f(y) of y is denoted by M[f(y),r] and is defined as M[f(y),r]=$\int_0^a a^{-r} y^{r-1} f(y)dy$ whenever this integral is exists for r>0 is the parameter
2	**Inverse Fourier Transform**	**Inverse Finite Mellin Integral Transform**
	The inverse of the Fourier Transform is denoted by $F^{-1}[f(x),s]=f(x)$ and defined as $F^{-1}[f(x),s]=f(x)=$ $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{isx} F[f(x),s]ds$ whenever parameter s>0	The inversion of the Mellin integral transform is denoted by $M^{-1}[f(y),r)]=f(y)$ and defined as $M^{-1}[f(y),r)]=f(y)=$ $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-r} M[f(y)],s]ds$

7.3: Relation between Finite Mellin Integral Transform With Laplace And Fourier Transforms

1	**The Laplace- Finite Mellin Integral Transform**
	$L_f M[f(x,y),s,0,\infty,r,0,a]=\int_0^{\infty}\int_0^{a} a^{-r} e^{-sx} y^{r-1} f(x,y)dxdy$
2	**The Fourier- Finite Mellin Integral Transform**
	$F_f M[f(x,y),s,0,\infty,r,0,a]=\int_0^{\infty}\int_0^{a} a^{-r} f(x,y)e^{-isx} y^{r-1}dxdy$

7.4: Relation between Laplace Transform and Finite Mellin Integral Transform

	Laplace –Finite Mellin Transform	Fourier-Finite Mellin Transform
1	Lemma::3.1.1 L $_f M$ [α f(x,y)+β g(x,y),s,0,∞, r,0,a] =α L $_f M$ [f(x,y),s,0,∞,r,0,a]+β L $_m$ [g(x,y),s,0,∞,r,0,a]	Lemma:4.1.1 $F_f M[\alpha f(x,y)+\beta g(x,y),s,r]$ $=\alpha\ F_f M[f(x,y),s,r]+\beta\ F_f M[g(x,y),s,r]$
2	Lemma:3.1.2 L $_f M$ [f(x ,y^n),s,0,∞,r,0,a] $=\frac{1}{n} L_f M[f(x,z),s,0,\infty,\frac{r}{n},0,a^n]$	Lemma:4.1.2 $F_f M[f(x,y^n),p,q]=\frac{1}{n}\ F_f M[f(x,y),s,\frac{r}{n}]$
3	Lemmq:3.2.1 L $_f M$ [f(cx, dy),s,0,∞,r,0,a] $=\frac{1}{cd^r}$ L $_f M$ [f(t,z),s/c,0,∞,r,0,ad]	Lemma:4.2.1 $F_f M[f(cx,dy),p,q] = \frac{1}{cd^r}\ F_f M[f(t,z),\frac{s}{c},r]$
4	Lemma:3.2.2 L $_f M$ [f(cx)g(dy),s,0,∞,r,0,a] $=\frac{1}{cd^r}$ L $_f M$ [f(t),s/c]M[g(z),r,0,ad]	Lemma:4.2.2 $F_f M[f(ax)g(by),s,r]=\frac{1}{ab^r}$ $F_f M[f(u),\frac{s}{a}]M[g(v),r,0,\frac{b}{a}]$

7.5: Shifting Theorem For Laplace and Fourier Transforms With Finite Mellin Integral Transform

1	**The Shifting Theorem of Laplace- Finite Mellin Integral Transform**
	$L_f M$ $[e^{-bx} y^n$ f(x,y),s,0,∞,r,0,a]= $L_f M$ [f(x,y),s+a,0,∞,r+n,0,a]
2	**The Shifting Theorem of Fourier- Finite Mellin Integral Transform**
	$F_f M[e^{-iax} y^n f(x,y), s, r] = F_f M[f(x,y), s+a, r+n]$

7.6: Inversion Theorems For Laplace And Fourier Transforms With Finite Mellin Integral Transform

1	**The Inverse of Laplace- Finite Mellin Integral Transform**
	F(x,y)=$\frac{1}{(2\pi i)^2} \int_{c2-\infty a}^{c2+i\infty} \int_{c1-ia}^{c1+ia} \frac{e^{sx} y^{-r}}{sp} L_f M[f(x,y), s, 0, \infty; r, 0, a] drds$
2	**The Inverse of Fourier- Finite Mellin Integral Transform**
	f(x ,y)=$F_f M^{-1}[f(x,y), s, r] = \frac{1}{(2\pi i)^2} \int_{c1-i\infty}^{c1+i\infty} \int_{c2-ia}^{c2+ia} \frac{1}{isr} e^{isx} y^{-r} F_f M[f(x,y), s, r] dsdr$

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