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By Using Fuzzy Rules for Applications to Risk Analysis in the Vesuvian Area

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Abstract

We experiment the use of the tool "Fuzzy SRA" in order to get the most relevant rule in a set of fuzzy rules related to a decision making problem in spatial analysis. The basic idea consists in the usage of the algebraic structure and the related string operations defined by A.Gisolfi and V.Loia (International Journal of Approximate Reasoning, 13, pp. 151-183, 1995) on a domain formed from fuzzy rules: the string attributes are the variables that appear in each logical clause of the antecedent rules. The rule to be adopted is represented in the final string from a triangular fuzzy number with the best central value. This rule will be the most "reliable" rule by considering the observable data defined as "facts". We test this approach by considering the problem of the eruption risk coming from the famous vulcan Vesuvio. We have considered a rule-set formed from 22 fuzzy rules used to evaluate the impact produced on the exposed values from an eruption risk in each of the 18 Municipal towns located in the so-called "Red Zone", which is the most exposed zone to the eruption risk, in accordance to the decisions of the Italian Civil Protection Agency. We have considered in the rules the following parameters: seismic vulnerability, urban environment, resident population, archaeological and architectural heritages. Here the "facts" coincide with the data measured from Italian Civil Protection Agency and ISTAT (Italian Statistical Institute), hence we determine the most correspondent rule to these facts.

Keywords: fuzzy rule, triangular fuzzy number

1 Introduction

In this work we use the tool Fuzzy SRA, defined in [4] and already tested in [2], [3], via a fuzzy rule-set employed to evaluate the impact produced on exposed values from the eruption risk of the Vesuvius, the famous Vulcan near Naples (Italy). This tool is applied on strings derived from the antecedents of the rules which give the most relevant rule as output, whose consequent represents the most reliable valuation of the impact produced. This valuation depends on the "facts" which are constituted from the data available, variable from zone to zone. In many planning problems, the decision-maker uses approximate reasoning on a set of empirical rules: Fuzzy SRA permits to determine for each area, with homogeneous values of the facts, the most reliable rule which the planner must take mainly in consideration .

From such approximate reasoning one deduces that it is possible to find the most reliable rule by resolving a problem of spatial reliability analysis, since the data taken in consideration (the facts) vary in the area under study. Like in [2], [3], we define an iso-reliable zone as a sub-area of the area under study in which the facts have the same value. In Section 2 we introduce the problem of the evaluation of the impact produced on exposed values from the risk of the Vesuvius eruption. In Section 3 we recall the algebraic structure of [4], in Section 4 we define the rule-set and Section 5 contains the final results. Concluding comments are in Section 6.

2 The impact produced from an eruption

On the base of the maximum attended event to be comparable with the eruption that happened in 1631, Italian Civil Protection Agency has elaborated in 1995 (and updated in the successive years) a national plan for the evacuation of the affected geographic area (cfr., the URL www.protezionecivile.it). This territory has been subdivided in three areas of risk:

- *Red Zone*, under risk of pyroclastic flows and enclosing 18 Municipal towns belonging to the District of Naples;
- *Blue Zone*, located at north of the Vesuvius, under risk of lahar;
- *Yellow Zone*, under risk of vulcanic ashes which could provoke the collapse of buildings.

The national plan provides the total evacuation of the "Red Zone" in case of various alarm phases determined from premonitory events of the eruption.

We have considered the 18 Municipal towns included in the "Red Zone". Each town has a variegated structure of urbanization and possesses many buildings and it has a more or less elevated value of exposition to the risk.

Many of these towns belong partially or totally to the Vesuvius National Park, and hence they are protected areas from an environmental point of view. Among these towns, there are Pompei and Ercolano, which are two very important archaeological sites of Roman age and architectural heritages exposed to the eruption risk. Figure 1 shows the 18 towns in the “Red Zone”.

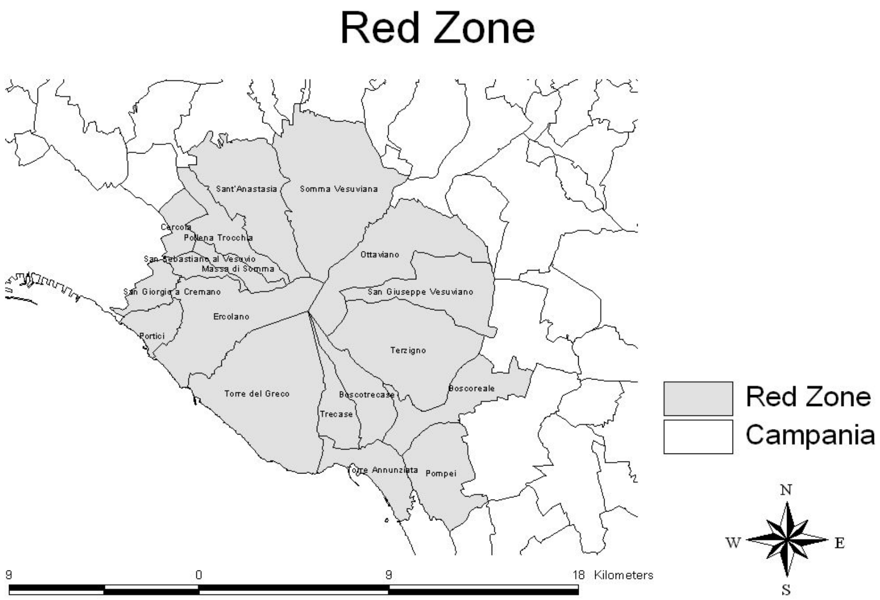

Fig. 1. The Red Zone

The most important exposed value to the eruption risk is represented from the resident population. The Regional Government (Regione Campania) has encouraged, under concession of various facilities (e.g., contributions for buying a house), the resident families since at least 5 years in the “Red Zone” to move in other areas considered safe from the eruption risk. In Table I we show the data related to the resident population in the towns of the “Red Zone” in 1998 and 2001, obtained from the ISTAT (Italian Statistical Institute).

Table I shows a clear decreasing trend of the difference of population between the years 1998 and 2001. This trend is in contrast with the average increasing of 8% of population of the towns of the Region Campania during this period. Notwithstanding the evident dangerous situation and the facilities granted by the Regional government, the population is marginally increased from 1998 to 2001 in the towns of Pollena Trocchia, Somma Vesuviana, Terzigno and Torre Annunziata. In the national plan two important parameters are focused:

- the seismic vulnerability of the structural elements of the buildings in the urban centers,
- the vulnerability of the urban environment, that measures the ability of the

Table I. Resident population 1998 and 2001

Town	Population 1998	Population 2001
Boscoreale	29411	27381
Boscotrecase	11255	10638
Cercola	19356	18572
Ercolano	58254	54699
Massa di Somma	6036	5902
Ottaviano	23307	22549
Pollena Trocchia	13134	13326
Pompei	26143	25751
Portici	62106	58905
San Giorgio a Cremano	60357	52807
San Giuseppe Vesuviano	26838	23152
San Sebastiano al Vesuvio	10323	9851
Sant'Anastasia	28908	27472
Somma Vesuviana	32429	32838
Terzigno	15789	15831
Torre Annunziata	47659	48720
Torre del Greco	95665	90255
Trecase	9886	9179

urban centers to resist to a vulcanic eruption event. The analysis of such parameters allows to understand how much is vulnerable the area to the effects of an eruption.

Table II. Percentages seismic vulnerability

Town	**Low**	**Mean**	**High**
Boscoreale	37.5	37.5	25.0
Boscotrecase	12.5	37.5	50.0
Cercola	40.0	20.0	40.0
Ercolano	17.6	58.8	23.6
Massa di Somma	25.0	25.0	50.0
Ottaviano	28.6	28.6	42.8
Pollena Trocchia	28.6	42.8	28.6
Pompei	33.3	58.3	8.4
Portici	4.8	47.6	47.6
San Giorgio a Cremano	27.8	33.3	38.9
San Giuseppe Vesuviano	28.6	14.3	57.1
San Sebastiano al Vesuvio	14.3	57.1	28.6
Sant'Anastasia	50.0	25.0	25.0
Somma Vesuviana	20.0	60.0	20.0
Terzigno	16.7	50.0	33.3
Torre Annunziata	37.5	25.0	37.5
Torre del Greco	22.7	45.5	31.8
Trecase	33.4	33.3	33.3

The two parameters are obtained for all the towns of the “Red Zone” by using a methodology of surveying elaborated from Gruppo Nazionale dei Terremoti (the Earthquakes National Institute, cfr. the URL gndt.ingv.it) on the base of analogous experiences carried out from previous similar events. For the determination of the above mentioned parameters, the Municipal Govern of each town of the “Red Zone” has supplied a series of thematic spatial data in the cartographic scales 1:10000 or 1:5000. Table II contains the values in relative percentage of the index of seismic vulnerability of each town. The data are grouped in three classes of vulnerability: Low, Mean and High (of course, the sum of these three indexes is equal to 100%). We also take in account also the following parameters:

- the spatial extension of the Vesuvian National Park, which is an protected area from a paleontological and biological point of view,
- the presence of archaeological and architectural heritages, mainly the archaeological sites of Roman age at Pompei and Ercolano.

Table III gives the relative percentage of the index of vulnerability of urban environment for each town. The data are grouped in five classes (intervals) of vulnerability: [0,20[(very low), [20,40[(low), [40,60[(mean), [60,80] (high), [80,→[(very high).

Table III. Percentages vulnerability urban environment

Town	**[0,20[**	**[20,40]**	**[40,60[**	**[60,80[**	**[80,→[**
Boscoreale	37.5	--	--	62.5	--
Boscotrecase	37.5	--	25.0	25.0	12.5
Cercola	20.0	--	60.0	20.0	--
Ercolano	--	--	52.9	41.1	5.9
Ottaviano	28.6	--	71.4	--	--
Pollena Trocchia	14.3	14.3	71.4	--	--
Pompei	75.0	25.0	--	--	--
Portici	--	--	85.7	14.3	--
S. Giorgio a C.	22.2	--	44.4	33.4	--
San Giuseppe V.	42.8	--	28.6	28.6	--
S.Sebastiano al V.	14.3	--	85.7	--	--
Sant'Anastasia	50.0	--	25.0	25.0	--
Somma Vesuv.	20.0	--	60.0	20.0	--
Terzigno	16.7	8.3	75.0	--	--
Torre Annunziata	50.0	12.5	--	37.5	--
Torre del Greco	59.1	4.6	22.7	13.6	--
Trecase	33.4	33.3	33.3	--	--

In Table IV are reported the percentages of the area of each town included in the National Park area and the number of archaeological and architectonic assets present in each town.

Table IV. Percentages and number of archaeological/architectural heritages

Town	Percentage of town area	Number of arch./archit. heritages
Boscoreale	2.3	4
Boscotrecase	76.5	0
Cercola	0.0	0
Ercolano	13.0	53
Massa di Somma	71.5	0
Ottaviano	51.7	2
Pollena Trocchia	37.9	0
Pompei	0.0	215
Portici	0.0	10
San Giorgio a Cremano	0.0	2
San Giuseppe Vesuviano	0.1	0
San Sebastiano al Vesuvio	18.1	0
Sant'Anastasia	31.8	3
Somma Vesuviana	36.6	7
Terzigno	51.2	0
Torre Annunziata	0.0	7
Torre del Greco	0.4	9
Trecase	68.9	0

3 Definition of the algebraic structure

The algebraic structure defined in [4] refers to triangular fuzzy numbers. The motivation for this choice stands mainly in the simplicity of the calculations while the use of other types of fuzzy numbers (e.g., trapezoidal or Gaussian) should increase noticeably the computational complexity. In accordance to [1] and [5] we recall that a triangular fuzzy number (TFN) is a map $\mu : R \rightarrow [0,1]$, R being the reals, such that $\mu(x)=0$ if $x \le a$, $\mu(x)=(x-a)/(M-a)$ if $a \le x \le M$, $\mu(x)=(b-x)/(b-M)$ if $M \le x \le b$, $\mu(x)=0$ if $b \le x$, where a and b are the extremes of the interval $[a,b] = \text{supp } \mu = \{x \in R : \mu(x) \ne 0\}$. In the sequel the above TFN is represented by [a,M,b] and let $\mu'=[a',M',b']$ be another TFN. We recall that the addition of μ and μ' is the TFN $(\mu+\mu')$ defined as $\mu+\mu' = [a+a',M+M',b+b']$ and that the multiplication ''$\cdot$'' of μ by $k \in R$ is the TFN $(k\cdot\mu)$ defined as $k\cdot\mu = [ka, kM, kb]$. Let U be the universe of discourse and $\{\alpha_1, \alpha_2,\ldots, \alpha_n\}$ be an ordered n-tuple of linguistic labels, each composed from one or more linguistic modifiers and a variable (for instance, "α_1 = False", "α_2 = More or Less Good",..., "α_i = Good", "α_{i+1} = Very Good",..., "α_n = Completely Good"), each represented by a suitable TFN which is denoted, without loss of generality, also with α_i , i = 1,2,…,n. A fuzzy attribute is a map A: $U \rightarrow \{\alpha_1,\alpha_2,\ldots,\alpha_n\}$, represented by a string of the following type:

$$A = [a_n]^{\alpha_n}[a_2]^{\alpha_{n-1}}...[a_1]^{\alpha_1} \tag{1}$$

where $a_i=A^{-1}(\alpha_i)$ is a subset of U, called class. If $A^{-1}(\alpha_i)= \emptyset$, then we write a_i=[-]. Let B be the following fuzzy attribute represented by the following string:

$$B = [b_m]^{\beta_m}[b_2]^{\beta_{m-1}}...[b_1]^{\beta_1} \tag{2}$$

where the used symbols have a similar meaning to the above ones. Let $n \ge m$ and following [6], we put:

$$C = A\Delta B = [c_{m+n-1}]^{\gamma_{m+n-1}}[c_{m+n-2}]^{\gamma_{m+n-2}}...[c_1]^{\gamma_1} \tag{3}$$

being the subsets $\{c_i\}$ defined with the following formulas for i=1,…,m+n-1:

$$c_i = \begin{cases} \vee_{j=1}^{i} (a_{i-j+1} \wedge b_j) & \text{if } 1 \le i \le m-1 \\ \vee_{j=1}^{m} (a_{i-j+1} \wedge b_j) & \text{if } m \le i \le n-1 \\ \vee_{j=i-n+1}^{m} (a_{i-j+1} \wedge b_j) & \text{if } n \le i \le m+n-1 \end{cases} \tag{4}$$

The c_i' s can be found by using the simple rule of the usual arithmetical multiplication [6]. The TFN's γ_i, for i=1,…,m+n-1, are given by

$$\gamma_i = \begin{cases} \frac{1}{k1+k2} \cdot \sum_{j=1}^{i} d2_j \cdot d1_{i-j+1} \cdot (k1 \cdot \alpha_{i-j+1} + k2 \cdot \beta_j) & \text{if } 1 \le i \le m-1 \\ \frac{1}{k1+k2} \cdot \sum_{j=1}^{m} d2_j \cdot d1_{i-j+1} \cdot (k1 \cdot \alpha_{i-j+1} + k2 \cdot \beta_j) & \text{if } m \le i \le n-1 \\ \frac{1}{k1+k2} \cdot \sum_{j=i-n+1}^{m} d2_j \cdot d1_{i-j+1} \cdot (k1 \cdot \alpha_{i-j+1} + k2 \cdot \beta_j) & \text{if } n \le i \le m+n-1 \end{cases} \quad (5)$$

where the coefficients d_i are defined, for i=1,…,m+n-1, as

$$d_i = \begin{cases} \sum_{j=1}^{i} d2_j \, d1_{i-j+1} & \text{if } 1 \le i \le m-1 \\ \sum_{j=1}^{m} d2_j \, d1_{i-j+1} & \text{if } m \le i \le n-1 \\ \sum_{j=i-n+1}^{m} d2_j \, d1_{i-j+1} & \text{if } n \le i \le m+n-1 \end{cases} \quad (6)$$

The integer $d1_i$ (resp., $d2_i$) is the number of subsets $\{a_i\}$ (resp., $\{b_i\}$) of the string A (resp., B) involved in the operation of union performed from (4), whereas the integer k1 (resp., k2) stands for the total number of subsets $\{a_i\}$ of A (resp., $\{b_i\}$ of B) involved in the operation of intersection performed from (4).

The following example from [4] clarifies the above concepts and definitions. Let {A,B,A'} be the set of three parameters (fuzzy attributes), U={O1,O2} be two iso-reliable zones and $\alpha_4 = \beta_4$ = Cv, $\alpha_3 = \beta_3$ = V, $\alpha_2 = \beta_2$ = Mv, $\alpha_1 = \beta_1$ = F be four TFNs with the linguistic labels, in decreasing order in accordance to their meaning, shown in Table V.

Table V. Examples of linguistic labels (TFN)

Label	**Description**	**a**	**M**	**b**
Cv	Optimum Reliability	0.80	0.90	1.00
V	Good Reliability	0.65	0.75	0.80
Mv	Sufficient Reliability	0.55	0.60	0.65
F	Mediocre Reliability	0.45	0.50	0.55
Sc	Scanty Reliability	0.35	0.40	0.45

Suppose to have the following strings:

$$A = [O1]^{Cv}\ [-]^{V}\ [-]^{Mv}\ [O2]^{F},$$

$$B = [-]^{Cv}\ [O1,O2]^{V}\ [-]^{Mv}\ [-]^{F},$$

$$A' = [O2]^{Cv}\ [-]^{V}\ [-]^{Mv}\ [O1]^{F}.$$

This means that in the zone O1 (resp., O2) the parameter A (resp. A') has a optimum reliable measure but not A' (resp. A) which has a mediocre reliable measure, whereas the parameter B has good reliable measure in both zones. Now we find the subsets c_i (here m=n=4):

$$\begin{array}{r} [O1]\ [-]\ [-]\ [O2] \\ [-]\ [O1,O2]\ [-]\ [-] \\ \hline [-]\ [-]\ [-]\ [-] \end{array}$$

$$[-]\ [-]\ [-]\ [-]$$

$$\begin{array}{l} \quad\quad\quad [O1]\ [-]\ [-]\ [O2] \\ \quad [-]\ [-]\ [-]\ [-] \\ \hline [-]\ [O1]\ [-]\ [-]\ [O2]\ [-]\ [-] \end{array}$$

In the calculation of the TFN's γ_i, we put for brevity:

$$a_{i,j} = d2_j \cdot d1_{i-j+1} \cdot (k1 \cdot \alpha_{i-j+1} + k2 \cdot \beta_j)$$

We observe that k1= k2=1, $d2_i = d1_i = 1$ for every i = 1,…,4 and thus we have:

$a_{1,1}$ = 1·1·(1·[0.45,0.50,0.55]+1·[0.45,0.50,0.55]) = [0.90,1.00,1.10],
$a_{2,1}$ = 1·1·(1·[0.55,0.60,0.65]+1·[0.45,0.50,0.55]) = [1.00,1.10,1.20],
$a_{3,1}$ = 1·1·(1·[0.65,0.70,0.80]+1·[0.45,0.50,0.55]) = [1.10,1.20,1.35],
$a_{4,1}$ = 1·1·(1·[0.80,1.00,1.20]+1·[0.45,0.50,0.55]) = [1.25,1.50,1.75],
$a_{1,2}$ = 1·1·(1·[0.45,0.50,0.55]+1·[0.55,0.60,0.65]) = [1.00,1.10,1.20],
$a_{2,2}$ = 1·1·(1·[0.55,0.60,0.65]+1·[0.55,0.60,0.65]) = [1.10,1.20,1.30],
$a_{3,2}$ = 1·1·(1·[0.65,0.70,0.80]+1·[0.55,0.60,0.65]) = [1.20,1.30,1.45],
$a_{4,2}$ = 1·1·(1·[0.80,1.00,1.20]+1·[0.55,0.60,0.65]) = [1.35,1.60,1.85],
$a_{1,3}$ = 1·1·(1·[0.45,0.50,0.55]+1·[0.65,0.70,0.80]) = [1.10,1.20,1.35],
$a_{2,3}$ = 1·1·(1·[0.55,0.60,0.65]+1·[0.65,0.70,0.80]) = [1.20,1.30,1.45],
$a_{3,3}$ = 1·1·(1·[0.65,0.70,0.80]+1·[0.65,0.70,0.80]) = [1.30,1.40,1.60],
$a_{4,3}$ = 1·1·(1·[0.80,1.00,1.20]+1·[0.65,0.70,0.80]) = [1.45,1.70,2.00],

$a_{1,4}$ = 1·1·(1·[0.45,0.50,0.55]+1·[0.80,1.00,1.20]) = [1.25,1.50,1.75],

$a_{2,4}$ = 1·1·(1·[0.55,0.60,0.65]+1·[0.80,1.00,1.20]) = [1.35,1.60,1.85],
$a_{3,4}$ = 1·1·(1·[0.65,0.70,0.80]+1·[0.80,1.00,1.20]) = [1.45,1.70,2.00],
$a_{4,4}$ = 1·1·(1·[0.80,1.00,1.20]+1·[0.80,1.00,1.20]) = [1.60,2.00,2.40].

Furthermore we have that:

$d_1=d1_1\cdot d2_1=1\cdot 1=1$,
$d_2= d1_1\cdot d2_2+d1_2\cdot d2_1=1\cdot 1+1\cdot 1=2$,
$d_3=d1_3\cdot d2_1+d1_2\cdot d2_2+d1_1\cdot d2_3=1\cdot 1+1\cdot 1+1\cdot 1=3$,
$d_4=d1_4\cdot d2_1+d1_3\cdot d2_2+d1_2\cdot d2_3+d1_1\cdot d2_4=1\cdot 1+1\cdot 1+1\cdot 1+1\cdot 1=4$,
$d_5=d1_4\cdot d2_2+d1_3\cdot d2_3+d1_2\cdot d2_4=1\cdot 1+1\cdot 1+1\cdot 1=3$,
$d_6=d1_4\cdot d2_3+d1_3\cdot d2_4=1\cdot 1+1\cdot 1=2$,
$d_7=d1_4\cdot d2_4=1\cdot 1=1$

and hence we deduce the following TFN's

γ_1= (1/2)·$a_{1,1}$ = [0.45,0.50,0.55],
γ_2= (1/4)·($a_{2,1}$+$a_{1,2}$)=[0.50,0.55,0.60],
γ_3= (1/6)·($a_{3,1}$+$a_{2,2}$+$a_{1,3}$)=[0.55,0.90,1.00],
γ_4=(1/8)·($a_{4,1}$+$a_{3,2}$+$a_{2,3}$+$a_{1,4}$)=[0.61,0.70,0.80],
γ_5= (1/6)·($a_{4,2}$+$a_{3,3}$+$a_{2,4}$)=[0.66,0.76,0.88],
γ_6= (1/4)·($a_{4,3}$+$a_{3,4}$)=[0.72,0.85,1.00],
γ_7= (1/2)·$a_{4,4}$=[0.80,1.00,1.20].

Then we get the following resulting fuzzy attribute:

$$A \Delta B = [-]^{\gamma_7}[O1]^{\gamma_6}[-]^{\gamma_5}[-]^{\gamma_4}[O2]^{\gamma_3}[-]^{\gamma_2}[-]^{\gamma_1}$$

By considering (A Δ B) Δ A', we have that k1 = 2 and the new $d1_i$, for i = 1,…,7, are $d1_1$ = 1, $d1_2$ = 2, $d1_3$ = 3, $d1_4$ = 4, $d1_5$ = 3, $d1_6$ = 2, $d1_7$ = 1 while it is k2 = 1 and $d2_1$ = 1, $d2_2$ = 1, $d2_3$ = 1, $d2_4$ = 1. Since now n=7 and m = 4, (AΔB)ΔA' has 10 classes to which 10 TFN's are associated. It is possible to show that the associativity holds, that is (AΔB)ΔA' = A'Δ(AΔB) but we omit this fact for brevity .

4 The rule-set in Fuzzy SRA

We take in consideration an empirical fuzzy rule-set used in a specific domain. It is represented from a set of rules in which the antecendents of connected logical clauses by the logical operator AND:

R_k: IF (b_1 is $B_{1,k}$) AND (b_2 is $B_{2,k}$) AND…AND (b_n is $B_{n,,k}$) THEN (a is A_k) (7)

where $B_{1,k,}$ $B_{2,k}$,…, $B_{n,,k}$ A_k are linguistic labels of fuzzy sets and $b_1 \ldots b_n$ are attributes in the rule. The single rules are objects in the universe U of the rules and the parameters $b_1 \ldots b_n$ in the antecedent of the rules are the attributes forming the strings of the algebraic structure. The following short example is illustrative: we consider a set of 3 rules with 2 attributes b_1 and b_2 in the antecedent B_1:

$$\begin{aligned} &R_1 : \text{IF } (b_1 \text{ is } B_1) \text{ AND } (b_2 \text{ is } B_1) \text{ THEN } (a \text{ is } A_1);\\ &R_2 : \text{IF } (b_1 \text{ is } B_2) \text{ AND } (b_2 \text{ is } B_2) \text{ THEN } (a \text{ is } A_2);\\ &R_3 : \text{IF } (b_1 \text{ is } B_2) \text{ AND } (b_2 \text{ is } B_3) \text{ THEN } (a \text{ is } A_3). \end{aligned} \quad (8)$$

For the rule set (8), the two strings are the following:

$$\begin{aligned} &b_1 :\ [R_1]^{B_1} [R_2, R_3]^{B_2} [-]^{B_3}\\ &b_2 :\ [R_1]^{B_1} [R_2]^{B_2} [R_3]^{B_3} \end{aligned} \quad (9)$$

We use Fuzzy SRA to get from (8) the rule that fits better the facts, by applying recursively the operation Δ as described in Section III. To obtain the final string we assume the consequent of such rule as final result. In order to achieve this purpose, the antecedents of the rules in (8) are rewritten by assigning the linguistic label B_k to the attribute b_i and by considering the difference between the linguistic label assigned to the corresponding fact b_i^* and that one assigned to the attribute in the rule. In order to be more precise, we consider the facts as

$$\begin{aligned} &b_1^* : \ b_1 \ \text{is} \ B_3\\ &b_2^* : \ b_2 \ \text{is} \ B_2 \end{aligned}$$

Let I: B→ {1,2,…, k,…} be the function that assigns the integer k to the linguistic label B_k, $I(B_k)=k$. Let B_h (resp. B_k) be the linguistic labels assigned to the attribute b_i in the fact b_i^* (resp. in the antecedent of the rule). Then we substitute in the antecedent of the rule, for the attribute b_i, the linguistic label B_k with the linguistic labels B_m with $m=abs(I(B_k)-I(B_h))+1$. With these modifications, we have for the antecedent of each rule:

$$\begin{aligned} &R1: (b_1 \text{ is } B3) \text{ AND } (b_2 \text{ is } B2)\\ &R2: (b_1 \text{ is } B2) \text{ AND } (b_2 \text{ is } B1)\\ &R3: (b_1 \text{ is } B2) \text{ AND } (b_2 \text{ is } B2) \end{aligned} \quad (10)$$

from which we obtain the following strings of the algebraic structure:

$$\begin{aligned} b_1 &: \ [-]^{B_1} [R_2, R_3]^{B_2} [R_1]^{B_3} \\ b_2 &: \ [R_2]^{B_1} [R_1, R_3]^{B_2} [-]^{B_3} \end{aligned} \tag{11}$$

We assign as weight for each attribute the relative frequence in which the attribute appears in the antecedent of the rules of the fuzzy rule-set. If N_a is the number in the fuzzy rule-set and N_i is the number of logical clauses in which appears the attribute A_i, we assign to the attribute A_i the weight $W_i = N_i/N_a$. Further we apply an amplification factor S to the weight by obtaining a final weight $\mu_i = S \cdot W_i + 1$; we put S = 9 and so the final weight μ_i is a value between 1 and 10. So we assign a minimal weight 1 to the attribute which appears with very low frequency in the antecedent of the rules. In our example, we have $N_1 = N_2 = 3$, $W_1 = W_2 = 1$, $\mu_1 = \mu_2 = 10$ for the attributes B_1 and B_2. In this case they appear in each rule of the fuzzy rule-set and the final weight for each attribute assumes the maximum value 10. The determination of the final string allows to know which is (or are) the rule(s) whose antecedent is more adherent to the facts; the linguistic label (of the fuzzy set of the class to which the rules belong to) determines qualitatively the importance of such rules: the central value of this fuzzy set is a valuation of how much is "reliable" the choice of those rules.

Since the attributes here used in the example are constituted from mutually independent variables in the logical clauses of antecedents of the rules, then there are not "concomitance effects". These effects must be corrected in accordance to the suggestions given in [4].

5 Final results

In order to deduce an indicator of the impact produced from the eruption risk, we take a rule-set of 6 parameters considered as the most meaningful ones and below listed. They are the following:

b_1 = (Percentage vulnerability urban environment $\in$ [80,→]) +
0.7·(Percentage vulnerability urban environment $\in$ [60,80[) +
0.5·(Percentage vulnerability urban environment $\in$ [40,60[),

b_2 = (Percentage seismic vulnerability High)+
0.5·(Percentage seismic vulnerability Mean),

b_3 = Resident population at 2001 (ISTAT),

b_4 = Difference between resident population at 2001 and 1998,
b_5 = Percentage of area town included in the National Park,

b_6 = Number of archaeological/architectural heritages.

The empirical rules are determined from the following linguistic labels (TFNs) of Table VI.

Table VI. Linguistic labels used in Fuzzy SRA

Label	**Description**	**a**	**M**	**b**
B_1	Good	0.9	1	1
B_2	Discrete	0.7	0.8	0.9
B_3	Sufficient	0.55	0.6	0.7
B_4	Scanty	0.2	0.4	0.55
B_5	Bad	0	0	0.2

The empirical rules are described in the sequel:

- R_1 : IF ((b_1 is B_1) OR (b_1 is B_2)) AND (b_2 is B_1) AND (b_6 is B_1) THEN (a is B_1);
- R_2 : IF ((b_1 is B_1) OR (b_1 is B_2)) AND (b_3 is B_1) AND (b_6 is B_1) THEN (a is B_1);
- R_3 : IF (b_1 is B_1) AND (b_2 is B_2) AND (b_3 is B_1) AND (b_6 is B_1) THEN (a is B_2);
- R_4 : IF (b_1 is B_2) AND (b_2 is B_1) AND (b_3 is B_1) AND (b_6 is B_1) THEN (a is B_2);
- R_5 : IF (b_1 is B_2) AND (b_2 is B_2) AND (b_3 is B_1) AND (b_4 is B_1) THEN (a is B_3);
- R_6 : IF (b_1 is B_2) AND (b_2 is B_2) AND (b_3 is B_1) AND (b_5 is B_1) THEN (a is B_3);
- R_7 : IF (b_1 is B_3) AND (b_5 is B_3) AND (b_6 is B_3) THEN (a is B_4);
- R_8 : IF ((b_1 is B_1) OR ((b_1 is B_2)) AND ((b_6 is B_4) OR (b_6 is B_5)) THEN (a is B_4);
- R_9 : IF (b_1 is B_4) AND (b_2 is B_4) AND (b_3 is B_2) THEN (a is B_4);
- R_{10}: IF (b_1 is B_4) AND (b_4 is B_5) AND (b_6 is B_4) THEN (a is B_5);
- R_{11}: IF (b_1 is B_4) AND (b_3 is B_5) AND (b_4 is B_5) AND (b_5 is B_5) THEN (a is B_5);
- R_{12}: IF ((b_1 is B_4) OR ((b_1 is B_5)) AND (b_6 is B_5) THEN (a is B_5);
- R_{13}: IF ((b_2 is B_4) OR ((b_2 is B_5)) AND (b_6 is B_5) THEN (a is B_5);
- R_{14}: IF (b_1 is B_5) AND (b_3 is B_5) THEN (a is B_5);
- R_{15}: IF (b_2 is B_5) AND (b_3 is B_5) THEN (a is B_5),

where the parameter “a” in the consequent represents an indicator of the impact produced from the eruption over the values exposed to the risk. By decomposing the rules containing the OR operator, we have the following rule-set :

- R_{1A} : IF (b_1 is B_1) AND (b_2 is B_1) AND (b_6 is B_1) THEN (a is B_1);
- R_{1B} : IF (b_1 is B_2) AND (b_2 is B_1) AND (b_6 is B_1) THEN (a is B_1);
- R_{2A} : IF (b_1 is B_1) AND (b_3 is B_1) AND (b_6 is B_1) THEN (a is B_1);

- R_{2B} : IF (b_1 is B_2) AND (b_3 is B_1) AND (b_6 is B_1) THEN (a is B_1);
- R_3 : IF (b_1 is B_1) AND (b_2 is B_2) AND (b_3 is B_1) AND (b_6 is B_1) THEN (a is B_2);
- R_4 : IF (b_1 is B_2) AND (b_2 is B_1) AND (b_3 is B_1) AND (b_6 is B_1) THEN (a is B_2);
- R_5 : IF (b_1 is B_2) AND (b_2 is B_2) AND (b_3 is B_1) AND (b_4 is B_1)THEN (a is B_3);
- R_6: IF (b_1 is B_2) AND (b_2 is B_2) AND (b_3 is B_1) AND (b_5 is B_1) THEN (a is B_3);
- R_7 : IF (b_1 is B_3) AND (b_5 is B_3) AND (b_6 is B_3) THEN (a is B_4);
- R_{8A} : IF (b_1 is B_1) AND (b_6 is B_4) THEN (a is B_4);
- R_{8B} : IF (b_1 is B_1) AND (b_6 is B_5) THEN (a is B_4);
- R_{8C} : IF (b_1 is B_2) AND (b_6 is B_4) THEN (a is B_4);
- R_{8D} : IF (b_1 is B_2) AND (b_6 is B_5) THEN (a is B_4);
- R_9 : IF (b_1 is B_4) AND (b_2 is B_4) AND (b_3 is B_2) THEN (a is B_4);
- R_{10} : IF (b_1 is B_4) AND (b_4 is B_5) AND (b_6 is B_4) THEN (a is B_5);
- R_{11}: IF (b_1 is B_4) AND (b_3 is B_5) AND (b_4 is B_5) AND (b_5 is B_5) THEN (a is B_5);
- R_{12A} : IF (b_1 is B_4) AND (b_6 is B_5) THEN (a is B_5);
- R_{12B} : IF (b_1 is B_5) AND (b_6 is B_5) THEN (a is B_5);
- R_{13A} : IF (b_2 is B_4) AND (b_6 is B_5) THEN (a is B_5);
- R_{13B} : IF (b_2 is B_5) AND (b_6 is B_5) THEN (a is B_5);
- R_{14} : IF (b_1 is B_5) AND (b_3 is B_5) THEN (a is B_5);
- R_{15} : IF (b_2 is B_5) AND (b_3 is B_5) THEN (a is B_5).

Table VII contains the values of the weights obtained via S = 9. The parameter b_1 is the most relevant one because it appears in 19 of the above 22 fuzzy rules. The tool finds the winning rule by comparing each antecedent with the facts.

Table VII. Values of the weights of the attributes

Attribute	$\mathbf{N_i}$	$\mathbf{W_i}$	μ_i
b_1	19	0.86	10
b_2	10	0.45	6
b_3	5	0.23	3
b_4	3	0.14	2
b_5	3	0.14	2
b_6	16	0.73	8

In order to assign to each fact the corresponding linguistic label, we have used the empirical classification of Table VIII, in which the domain of each attribute b_i (i = 1,…,6) is decomposed in intervals, to which the linguistic label B_j belongs (j

= 1,…, 5).

Table VIII. Linguistic labels in Fuzzy SRA

Label	B_1	B_2	B_3	B_4	B_5
b1	[0,20[	[20,40[	[40,60[	[60,80[	[80,100]
b2	[0,20[	[20,40[	[40,60[	[60,80[	[80,100]
b3	[0,2e4[	[2e4,4e4[	[4e4,6e4[	[6e4,8e4[	≥ 8e4
b4	< -0.1	[-0.1,-0.07[	[-0.07,-0.03[	[-0.03,0[	≥0
b5	[0,20[	[20,40[	[40,60[	[60,80[	[80,100]
b6	[0,3[	[3,10[	[10,20[	[20,50[	≥50

In Tables IX÷XIV the facts are reported for each attribute and town with the related values.

Table IX. Attribute b_1 and facts

Town	**Value**	**Fuzzy Set**
Boscoreale	43.75	B_3
Boscotrecase	42.50	B_3
Cercola	44.00	B_3
Ercolano	61.12	B_4
Massa di Somma	12.50	B_1
Ottaviano	35.70	B_2
Pollena Trocchia	35.70	B_2
Pompei	0	B_1
Portici	52.86	B_3
San Giorgio a Cremano	45.58	B_3
San Giuseppe Vesuviano	34.32	B_3
San Sebastiano al Vesuvio	42.85	B_3
Sant'Anastasia	30.00	B_2
Somma Vesuviana	44.00	B_3
Terzigno	37.50	B_2
Torre Annunziata	26.25	B_2
Torre del Greco	20.87	B_2
Trecase	16.65	B_1

Table X. Attribute b_2 and facts

Town	**Value**	**Fuzzy Set**
Boscoreale	43.75	B_3
Boscotrecase	68.75	B_4
Cercola	50.00	B_3
Ercolano	53.00	B_3
Massa di Somma	59.02	B_3
Ottaviano	57.10	B_3
Pollena Trocchia	50.00	B_3
Pompei	37.55	B_2
Portici	71.40	B_4
San Giorgio a Cremano	55.55	B_3
San Giuseppe Vesuviano	64.25	B_4
San Sebastiano al Vesuvio	57.15	B_3
Sant'Anastasia	37.50	B_2
Somma Vesuviana	50.00	B_3
Terzigno	58.30	B_3
Torre Annunziata	50.00	B_3
Torre del Greco	54.55	B_3
Trecase	49.95	B_3

Table XI. Attribute b_3 and Facts

Town	**Value**	**Fuzzy Set**
Boscoreale	27381	B_2
Boscotrecase	10638	B_1
Cercola	18572	B_1
Ercolano	54699	B_3
Massa di Somma	5902	B_1
Ottaviano	22549	B_2
Pollena Trocchia	3326	B_1
Pompei	26143	B_2
Portici	58905	B_3
San Giorgio a Cremano	52807	B_3
San Giuseppe Vesuviano	23152	B_2
San Sebastiano al Vesuvio	9851	B_1
Sant'Anastasia	27475	B_2
Somma Vesuviana	32838	B_2
Terzigno	15831	B_1
Torre Annunziata	48720	B_3
Torre del Greco	90255	B_5
Trecase	9179	B_1

Table XII. Attribute b_4 and Facts

Town	**Value**	**Fuzzy Set**
Boscoreale	-0.069	B_3
Boscotrecase	-0.055	B_3
Cercola	-0.041	B_3
Ercolano	-0.061	B_3
Massa di Somma	-0.022	B_4
Ottaviano	-0.033	B_3
Pollena Trocchia	0.015	B_5
Pompei	-0.015	B_4
Portici	-0.051	B_3
San Giorgio a Cremano	-0.046	B_3
San Giuseppe Vesuviano	-0.137	B_1
San Sebastiano al Vesuvio	-0.046	B_3
Sant'Anastasia	-0.050	B_3
Somma Vesuviana	0.013	B_5
Terzigno	0.003	B_5
Torre Annunziata	0.022	B_5
Torre del Greco	-0.057	B_3
Trecase	-0.072	B_2

Table XIII. Attribute b_5 and Facts

Town	**Value**	**Fuzzy Set**
Boscoreale	2.3	B_1
Boscotrecase	76.5	B_4
Cercola	0.0	B_1
Ercolano	13.0	B_1
Massa di Somma	71.5	B_4
Ottaviano	51.7	B_3
Pollena Trocchia	37.9	B_2
Pompei	5	B_1
Portici	0.0	B_1
San Giorgio a Cremano	0.0	B_1
San Giuseppe Vesuviano	0.0	B_1
San Sebastiano al Vesuvio	18.1	B_1
Sant'Anastasia	31.8	B_2
Somma Vesuviana	36.6	B_2
Terzigno	51.2	B_3
Torre Annunziata	0.0	B_1
Torre del Greco	0.4	B_1
Trecase	0.0	B_1

Table XIV. Attribute b_6 and Facts

Town	**Value**	**Fuzzy Set**
Boscoreale	4	B_2
Boscotrecase	0	B_1
Cercola	0	B_1
Ercolano	53	B_5
Massa di Somma	0	B_1
Ottaviano	2	B_3
Pollena Trocchia	0	B_1
Pompei	215	B_5
Portici	10	B_3
San Giorgio a Cremano	2	B_1
San Giuseppe Vesuviano	0	B_1
San Sebastiano al Vesuvio	0	B_1
Sant'Anastasia	3	B_2
Somma Vesuviana	7	B_2
Terzigno	0	B_1
Torre Annunziata	7	B_2
Torre del Greco	9	B_2
Trecase	0	B_1

Table XV contains the final results: the mean value of the fuzzy set to which is close the most reliable rule and the linguistic label of the consequent of the first of such rules which appears on the left of the final string.

Table XV. Final results

Town	**M**	**Consequent**
Boscoreale	0.812	B3
Boscotrecase	0.834	B3
Cercola	0.789	B3
Ercolano	0.832	B3
Massa di Somma	0.864	B3
Ottaviano	0.821	B3
Pollena Trocchia	0.815	B4
Pompei	0.867	B4
Portici	0.819	B4
San Giorgio a Cremano	0.852	B4
San Giuseppe Vesuviano	0.843	B4
San Sebastiano al Vesuvio	0.787	B4
Sant'Anastasia	0.790	B4
Somma Vesuviana	0.851	B4
Terzigno	0.823	B4
Torre Annunziata	0.817	B3
Torre del Greco	0.827	B4
Trecase	0.910	B2

A thematic map containing the final valuations of Table XV with the related linguistic labels is given in Figure 2. Strictly speaking, “Trecase” is the unique town with a discrete valuation of impact since it has a low seismic vulnerability and does not contain architectural and archaeological heritages. Ten towns have a sufficient valuation of impact. In seven towns, like, for instance, “Portici” and “San Sebastiano al Vesuvio”, we have a scanty valuation of impact because of a very high value of the parameter concerning the seismic vulnerability.

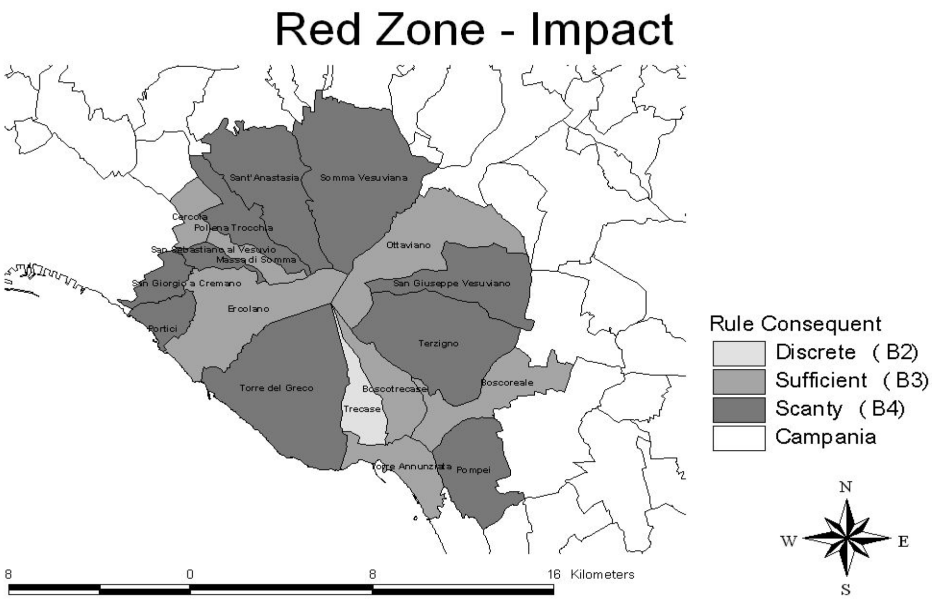

Fig. 2. Thematic map of impact deduced from Table XV

6 Conclusion

The results prove the validity of the usage of the tool Fuzzy SRA [2],[3],[4]. We have experimented this tool on the 18 towns included in the “Red Zone”, which is the most exposed zone to the eruption risk of the vulcan Vesuvio. Finally we have determined a thematic map of impact by considering the parameters of seismic vulnerability, urban environment, resident population, the number of archaeological and architectural heritages and the percentage of area town included in the related National Park which surrounds the volcan. These parameters are studied and quantified in a rule-set of 22 fuzzy rules. The most reliable rule is the final rule which is determined via the algebraic structure inside Fuzzy SRA and it gives the conclusive thematic map of impact.

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