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Shape and Topological Optimization for Electromagnetism Problems

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Abstract

This paper presents a topological shape optimization problem technique for electromagnetic problems using the topological sensitivity analysis and topological derivative. The objective function that represents the design objective is expressed in terms of magnetic field. The adjoint method is used to optimize the distribution of magnetics fields. Some numerical results that demonstrated the validity of the proposed approach are presented.

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1 Introduction

Shape and topological optimization is the most flexible optimization method that can simultaneously deal with geometrical and topological distribution. This method involves defining a fixed design domain such that it is larger than the resulting domain. In the fixed domain, an arbitrary configuration can be express, putting hole, allowing large change in the geometrical and topological design during the optimization process. Shape and topological optimization were originally developed for structures design and recently adapted for many other areas of design by taking other branches of science into consideration, such as electromagnetic.

The aim of the topological sensitivity analysis is to obtain an asymptotic expansion of shape functional with respect to the creation of a small hole inside the domain. The principe is the following. One consider a cost function $j(\Omega) = J(\Omega, u_\Omega)$ where u_Ω is solution to a partial differential equation defined in the domain $\Omega \subset \mathbb{R}^d$, $d = 2\ or d = 3$, a point $x_0 \in \Omega$ and a fixed domain $B \subset \mathbb{R}^d$, containing the origin. One searches for an asymptotic expansion of $j(\Omega \backslash \overline{(x_0 + \epsilon B)}$ when ϵ tends to zeros. In most cases, it reads in the form

$$j(\Omega \backslash \overline{(x_0 + \epsilon B)} - j(\Omega) = f(\epsilon)g(x_0) + o(f(\epsilon)). \tag{1}$$

Here $f(\epsilon)$ is an explicit positive function going to zeros with ϵ and function g, the topological gradient or topological derivative, is in general easy to compute. Expression (1) is called the asymptotic analysis. Hence to minimize the criterion J, one must to create holes at some points $\bar{x}$ where the topological gradient is negative. For more details about this approach we refer the reader to S. Garreau, P. Guillaume and Masmoudi [14], M. Masmoudi [21], A.A Novotny and al [24], J. Sokolowski and A.Zochowski [27]. For all these work perturbing the domain consist to insert hole.

B. Samet and al [26] obtain an asymptotic expansion of a functional with respect to the creation of a small hole in the domain. Such expansion is obtained for Helmholtz equation with Dirichlet condition on the boundary of circular hole. They presented some applications to waveguide problem.

In this paper our study consists to determine the geometrical and topological distribution of the magnetic fields in a considered space.

The electromagnetism takes care with describing phenomena governed by electric and magnetic fields which are connected. The study of phenomena results from interaction of the electric current and magnetic fields. Useful simulations are made on precise examples by taking account various pulsations in the case of the atomic physics.

This paper is organized as follows. In section 2 the modeling of electromagnetism phenomena is presented. In the third section a topological sensitivity framework using an adaptation of the adjoint method is introduced and we compute the expression of the topological sensitivity by using the approaches of Masmoudi [21] and Nazarov and Sokolowski [23]. In the last section, we conclude by some numerical simulations concerning the geometrical and topological distribution of the magnetic fields $\overrightarrow{B} = \overrightarrow{rot}\overrightarrow{A}$, where $\overrightarrow{A}$, the potential vector, is solution to a boundary partial differential equation in the considered domain.

2 Modeling

The model of electromagnetism is well known in the literature. But in order to facilitate the justification of work in other sections we present the Maxwell equations.
Let us begin with the acknowledgement that the expression of the Lorentz force

$$\vec{f}(P) = q\{\overrightarrow{E}(P) + \vec{v} \wedge \overrightarrow{B}(P)\} \tag{2}$$

not only gives meaning to the fields solutions of Maxwell equations when applied to point charges , but yields new predictions. The velocity appearing in the expression of Lorentz force is the velocity of the charge.
$\overrightarrow{B}$ defines an axis of rotation, $f(P)$: Strength exercised on the particle P, $\overrightarrow{E}(P)$: electric field (volts per meter), $\overrightarrow{B}(P)$ magnetic induction (areal density of magnetic flux q is related to P : electric charge of the particle.
In the vacuum in the sense we Condensed Matter

1. Electric field E_0 creates a point M by a stationary charge q, located at P.

$$\overrightarrow{E}_0(M) = \frac{q}{4\pi\varepsilon_0}\frac{\overrightarrow{PM}}{|PM|^3} \tag{3}$$

   ε_0 : dielectric constant of vacuum; $\varepsilon_0 = \frac{10^{-9}}{36\pi}$ Farad/metter

2. Electric field $E_{\vec{v}}$ creates at M by a charge q located at P and with a velocity v straight $\vec{v} = v\vec{k}$

$$E_{\vec{v}}(M) = (1 - \frac{v^2}{c^2})^{1/2}(E_0.\vec{k})\vec{k} + (1 - \frac{v^2}{c^2})^{-1/2}(E_0 - (E_0.\vec{k})\vec{k}) \tag{4}$$

   c speed of light in vacuum $\sim 3.10^8 m/s$

3. Magnetic induction $B_{\vec{v}}$ located at M by a charge q located at P and with a velocity v straight $\vec{v} = v\vec{k}$

$$\overrightarrow{B}(M) = \frac{\vec{v}(P) \wedge \overrightarrow{E}_{\vec{v}}(M)}{c^2} \tag{5}$$

4. Electric field $\overrightarrow{E}$ created at M by a volume charge density$\rho(x)$

$$\overrightarrow{E}(\underline{x}) = \frac{1}{4\pi\varepsilon_0}\int_\Omega \frac{\rho(\xi)(\xi_i - x_i)\vec{k}_i}{|\xi - x|^3}d\xi \tag{6}$$

All that precedes is valid provided reflect all charges (the charges of polarization and those of magnetization)

1. Approximation of $\overrightarrow{E}_{\bar{v}}$ and $\overrightarrow{B}_{\bar{v}}$:
   In a conductor, $\vec{v} \sim cm/s \Longrightarrow \frac{v^2}{c^2}$ is important in front of 1 then $\overrightarrow{E}_{\vec{v}} \neq \overrightarrow{E}_0$

$$\overrightarrow{B}_{\vec{v}}(M) = \frac{\mu_0}{4\pi}q\frac{\vec{v}(P)\wedge\overrightarrow{PM}}{|PM|^3} \tag{7}$$

$$\mu_0 = \frac{1}{\varepsilon_0 c^2} \tag{8}$$

   μ_0 : Magnetic permeability of the space $\sim 4\pi 10^{-7}H/m$, H: Henry

2. Strength of current I in a conductor
   The material contains positive and negative ions. The strength of current in a driver is the quantity of loads(responsibilities) which crosses the straight(right) section of a driver by unit of time. The intensity is expressed in ampere.

   Remark 1 *In electromagnetism the canonical greatness is L : length, T : time, M : the mass I : the intensity.*

3. Current Density

$$J = \frac{I}{S} \tag{9}$$

   The current vector is going to take into account the direction of the movement of loads.

$$\vec{J}(\underline{x}) \;=\; \rho(\underline{x})\vec{v} \tag{10}$$

   The variation of intensity is

$$dI = \vec{J}.\vec{n}dS = J.d\vec{S} \tag{11}$$

2.1 General laws of the electromagnetism

The law of preservation of the electricity is given by the relation between $\vec{J}$ and ρ In the Maxwell equations the unknowns are $\overrightarrow{E}$ and $\overrightarrow{B}$
The law of conservation of the electricity is given by the following relation

$$Q = \int_{\mathcal{D}} \rho dV, \quad \phi = \int_{\partial\mathcal{D}} \vec{J}.\vec{n} dS \tag{12}$$

where $\mathcal{D}$ is an arbitrary domain in $\mathbb{R}^3$, ρ the volume density of charges.
All charges through $\partial\mathcal{D}$ is exactly offset by the change in the total load.

$$\frac{d}{dt} Q + \phi = 0$$

$$\frac{d}{dt} \int_{\mathcal{D}} \rho(x,t) dv + \int_{\partial\mathcal{D}} \vec{J}(x,t).\vec{n} dS = 0 \;\forall\; \mathcal{D} \subset \mathbb{R}^3$$

$$\frac{\partial\rho}{\partial t} + div\vec{J} = 0 \tag{13}$$

There may be a source term, but it is not natural.

2.2 Non stationary Maxwell's Equations

The unknowns are $\overrightarrow{E}$ the electric field $\overrightarrow{B}$ the magnetic field and $\vec{J}$ the electric current density with J_{total} being the total current including the displacement current. The generalized Ampere theorem is given by

$$\frac{\partial\rho}{\partial t} + div\vec{J} = 0 \tag{14}$$

We find $\vec{\alpha}$ such that

$$\overrightarrow{rot}\,\overrightarrow{B} = \mu_0(\vec{J} + \vec{\alpha}) \tag{15}$$

The complete set of Maxwell's equations are

$$\left\{ \begin{array}{rcl} div\,\overrightarrow{B} & = & 0 \\ div\,\overrightarrow{E} & = & \frac{\rho_{tot}}{\epsilon_0} \\ \overrightarrow{rot}\,\overrightarrow{B} & = & \mu_0(\vec{J} + \epsilon_0 \frac{\partial \overrightarrow{E}}{\partial t}) \\ \overrightarrow{rot}\,\overrightarrow{E} & = & -\frac{\partial \overrightarrow{B}}{\partial t} \end{array} \right. \tag{16}$$

According to Maxwell's equations, we have $\overrightarrow{\alpha} = \epsilon_0 \frac{\partial \overrightarrow{E}}{\partial t}$. By Helmholtz's theorem $\overrightarrow{B}$ can be written in terms of vector field $\overrightarrow{A}$, called the magnetic potential

$$\overrightarrow{B} = \overrightarrow{rot}\,\overrightarrow{A}. \tag{17}$$

Plugging this relation into Faraday's law, we get

$$\overrightarrow{rot}(\overrightarrow{E} + \frac{\partial \overrightarrow{A}}{\partial t}) = 0. \tag{18}$$

By Helmhotz's theorem the quantity in parenthesis can be written in terms of scalar function V called the electric potential

$$\overrightarrow{E} + \frac{\partial \overrightarrow{A}}{\partial t} = -\overrightarrow{grad}\ V, \text{ that is } \overrightarrow{E} = -\overrightarrow{grad}\ V - \frac{\partial \overrightarrow{A}}{\partial t} \tag{19}$$

We also have

$$\overrightarrow{rot}(\overrightarrow{B}) = \overrightarrow{rot}\ \overrightarrow{rot}(\vec{A}) = \mu_0(J + \varepsilon_0 \frac{\partial \overrightarrow{E}}{\partial t}) = \overrightarrow{grad}\ div\vec{A} - \Delta\vec{A} = \nabla(\nabla \cdot \overrightarrow{A}) - \nabla^2 \overrightarrow{A}. \tag{20}$$

The equation (20) implies

$$\nabla(\nabla \cdot \overrightarrow{A}) - \nabla^2 \overrightarrow{A} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial \nabla V}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \overrightarrow{A}}{\partial t^2}. \tag{21}$$

The equality (21) implies

$$\mu_0 \varepsilon_0 \frac{\partial^2 \overrightarrow{A}}{\partial t^2} - \nabla^2 \overrightarrow{A} = \mu_0 J - \nabla(\nabla \cdot \overrightarrow{A} + \mu_0 \varepsilon_0 \frac{\partial \overrightarrow{V}}{\partial t}). \tag{22}$$

A more sensible choice is the so called Lorentz gauge

$$div \overrightarrow{A} = \nabla \cdot \overrightarrow{A} = -\varepsilon_0 \mu_0 \frac{\partial \overrightarrow{V}}{\partial t}. \tag{23}$$

If we adopt the Lorentz gauge the last term on the right hand side of (22) becomes zeros. Substituting the Lorentz gauge condition into the expression

$$-\nabla^2 V - \frac{\partial}{\partial t} div \overrightarrow{A} - \frac{\rho}{\varepsilon_0} \tag{24}$$

we obtain

$$\varepsilon_0 \mu_0 \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \frac{\rho}{\varepsilon_0}. \tag{25}$$

Thus we find that Maxwell's equations reduce to the following

$$\varepsilon_0 \mu_0 \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \frac{\rho}{\varepsilon_0} \varepsilon_0 \mu_0 \tfrac{\partial^2 A}{\partial t^2} - \nabla^2 A = \mu_0 J. \tag{26}$$

The Maxwell's equations can be expressed by

$$V = \left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta\right] V = \frac{\rho_{tot}}{\varepsilon_0} \overrightarrow{A} = \left[\tfrac{1}{c^2} \tfrac{\partial^2}{\partial t^2} - \Delta\right] \overrightarrow{A} = \mu_0 \vec{J}_{tot} \tag{27}$$

where $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ is the vaccum velocity of the light and is the d'Alembertian operator.

2.2.1 General laws and special laws of behavior

1. **Electric polarization** The electric polarization or polarization density or simply polarization is the vector field that expressed the density of permanent or induced electric dipole moment per unit volume.
   The behavior of electric fields (E and D), magnetic field B, charge densityρ and current density J by Maxwell's equations.
   The polarization density P defines the electric displacement field D

$$D \;=\; \varepsilon_0 E + P \tag{28}$$

   where ε_0 is the electric permeability.
   Electric polarization corresponds to rearrangement of the bounds electrons in the material, which creates on additional charge, known as the bound charge density ρ_b : $\rho_b = -\nabla \cdot \overrightarrow{P} = -div\overrightarrow{P}$ so that the total charge density that enters Maxwell's equations is given by

$$\rho_{tot} = \rho_{ext} + \rho_b \tag{29}$$

   where ρ_{ext} is the free charge density describing charges brought from outside. We obtain the continuity equation. That is the charge being conserved, the net flow of chosen volume must equal to the net charge in charge held inside the volume

$$\int_S J.dA - \frac{d}{dt}\int_V \rho_{tot} dV - \int_V \frac{\rho_{tot}}{\partial t} dV \tag{30}$$

   where ρ_{tot} is the charge density per unit volume and dA is surface element of the surface S enclosing the volume V. From the divergence theorem, we have

$$\int_S JdS = \int_V divJdV \tag{31}$$

   Hence

$$\int_V divJdV = -\int_V \frac{\partial \rho_{tot}}{\partial t} dV \tag{32}$$

   This relation is valid for any volume, we obtain the continuity's equation

$$div\vec{J} = -\frac{\partial \rho_{tot}}{\partial t} = -\frac{\partial \rho_{ext}}{\partial t} - \frac{\partial \rho_b}{\partial t} = -(\tfrac{\partial \rho_{ext}}{\partial t} - div\tfrac{\partial P}{\partial t}). \tag{33}$$

   By including

$$\overrightarrow{rot}\,\overrightarrow{B} = \mu_0[\vec{J} + \varepsilon_0 \frac{\partial \overrightarrow{E}}{\partial t}]$$

we have

$$div\vec{J} = div(\frac{\overrightarrow{rot}\,\overrightarrow{B}}{\mu_0} - \varepsilon_0\frac{\partial\overrightarrow{E}}{\partial t}) = -\frac{\partial\rho_{ext}}{\partial t} + div\frac{\partial\vec{P}}{\partial t} \tag{34}$$

Let us introduce $\vec{J}_{ext}$ defined by

$$div\vec{J}_{ext} = -\frac{\partial\rho_{ext}}{\partial t} \tag{35}$$

conduction current density.

$$div[\frac{\overrightarrow{rot}\,\overrightarrow{B}}{\mu_0} - \varepsilon_0\frac{\partial\overrightarrow{E}}{\partial t} - \vec{J}_{ext} - \frac{\partial\vec{P}}{\partial t}] = 0 \tag{36}$$

There exists $\overrightarrow{M}$ such that

$$\frac{\overrightarrow{rot}\,\overrightarrow{B}}{\mu_0} - \varepsilon_0\frac{\partial\overrightarrow{E}}{\partial t} - \vec{J}_{ext} - \frac{\partial\vec{P}}{\partial t} = \overrightarrow{rot}\vec{M} \tag{37}$$

Let

$$J_p = \frac{\partial\vec{P}}{\partial t} \tag{38}$$

This is the current density due to variation of system.

$$\vec{J}_a = \vec{J}_{ext} + \vec{J}_P \tag{39}$$

2. **Linear laws:**
   $D_i = \varepsilon_{ij}E_j \quad \varepsilon_{ij} = \varepsilon_0\delta_{ij} + P_{ij}$
   $B_i = \mu_{ij}H_j \quad \mu_{ij} = \mu_0\delta_{ij} + M_{ij}$
   $J_i = \sigma_{ij}E_j$ σ_{ij} is the conduction tensor.

3. **Isotropic law** $P_{ij} = \pi\delta_{ij}$
   $M_{ij} = \mu\delta_{ij}$
   $\sigma_{ij} = \sigma\delta_{ij}$

3 Topological optimization problem

The goal of the topological optimization problem is to find an optimal design with an a priori poor information on the optimal shape of the structure. The shape optimization problem consists in minimizing a functional $j(\Omega) = J(\Omega; u_\Omega)$ where the function u_Ω is defined, for example, on an open and bounded subset of $\mathbb{R}^N$. For $\epsilon > 0$; let $\Omega_\epsilon = \Omega \backslash \bar{\omega}_\epsilon$ be the set obtained by removing a small part, where Ω and ω are two regular subset containing the origin. Then, using general adjoint method, an asymptotic expansion of the function will be obtained in the following form

$$j(u_{\Omega_\epsilon}) - j(\Omega) = f(\epsilon)g(x_0) + o(f(\epsilon), \ lim_{\epsilon \to 0} f(\epsilon). = 0 \tag{40}$$

The topological sensitivity $g(x_0)$ provides information when the creating a small hole located at x_0 : Hence, the function g will be used as descent direction in the optimization process.

The model of the electromagnetism gives

$$\Delta \overrightarrow{A} - \frac{1}{c^2} \frac{\partial^2 \overrightarrow{A}}{\partial t^2} = -\mu_0 J_{tot} \tag{41}$$

where

$$\overrightarrow{A} = (A_1, A_2, A_3) \tag{42}$$

is the potential vector. Let us take for $j = 1, 2, 3$

$$A_j = \Phi_j(x) e^{i w_j t} \tag{43}$$

then we obtain the following equation

$$\Delta \Phi_j + w_j^2 \Phi_j = \frac{\mu_0}{J_{tot}}, \ j = 1, 2, 3. \tag{44}$$

In this equation we will add a Dirichlet boundary condition $\Phi_j = g_i$ on $\partial\Omega$ or a Neumann condition $\frac{\partial \Phi_j}{\partial n} = 0$ sur $\partial\Omega$ Then we obtain the following equation

$$\begin{cases} \Delta \Phi + w^2 \Phi = \frac{\mu_0}{Re(e^{iwt})} J_{tot} \text{ in } \Omega \\ \Phi = g \text{ or } \frac{\partial \Phi}{\partial n} = g \text{ on } \partial\Omega \end{cases} \tag{45}$$

where $g \in (H^{-\frac{1}{2}})^3$ and $\Delta \Phi = (\Delta \Phi_1, \Delta \Phi_2, \Delta \Phi_3)$ is the vectorial Laplacian and $w^2 = (w_1^2, w_2^2, w_3^2)$

3.1 Position of the problem

Consider a regular and bounded domain $\Omega \subset \mathbb{R}^N (N = 2$ or $3)$ for example of class $\mathcal{C}^2$, Φ is solution to (45). Let $\omega \subset \Omega$ a bounded open set of $\mathbb{R}^N$ containing the origin and $x_0 \in \Omega$. For all $\epsilon > 0$ let us defined $\omega_\epsilon = x_0 + \epsilon\omega$ and $\Omega_\epsilon = \Omega \backslash \bar{\omega}_\epsilon$. Let Φ_ϵ be the solution in Ω_ϵ of:

$$\begin{cases} \Delta\Phi_\epsilon + w^2\Phi_\epsilon = \frac{\mu_0}{Re(e^{iwt})} J_{tot} \text{ in } \Omega \\ \frac{\partial\Phi_\epsilon}{\partial n} = 0 \text{ on } \partial\Omega \\ \Phi_\epsilon = 0 \text{ or } \frac{\partial\Phi_\epsilon}{\partial n} = 0 \text{ on } \partial\omega_\epsilon \end{cases} \tag{46}$$

Consider the coast functional defined by

$$J(\Phi_\epsilon) = \int_{\Omega_\epsilon} |\overrightarrow{rot}(\Phi_\epsilon) - B_d|^2 \tag{47}$$

where B_d, a reference magnetic field, can be expressed in the form $B_d = \overrightarrow{rot} A_d$ with A_d a reference potential vector.
Our aim is to evaluate the difference $J(\Phi_\epsilon) - J(\Phi_0)$ if ϵ tends to zero by using the generalized adjoint method.
Multiplying (46) by a test function v and integrating we have

$$\int_{\Omega_\epsilon} \nabla\Phi_\epsilon : \nabla v - w^2\Phi_\epsilon v + \int_{\partial\Omega} \frac{\partial\Phi_\epsilon}{\partial n}.v + \int_{\partial\omega_\epsilon} \frac{\partial\Phi_\epsilon}{\partial n}.v = -\int_{\omega_\epsilon} \frac{\mu_0}{\cos wt} J_{tot} v$$

Let

$$\mathcal{V}_\epsilon = \{v \in H^1(\Omega_\epsilon),\ v = 0 \text{ on } \partial\omega_\epsilon\},$$

$$a_\epsilon(\Phi_\epsilon, v) = \int_{\Omega_\epsilon} \nabla\Phi_\epsilon : \nabla v - w^2\Phi_\epsilon v$$

and

$$L_\epsilon(v) = -\int_{\Omega_\epsilon} \frac{\mu_0}{\cos wt} J_{tot} v$$

Following the same idea, in Ω, let

$$\mathcal{V} = \{v \in H^1(\Omega),\ v = 0\ \partial\Omega\},$$

$$a_0(\Phi, v) = \int_\Omega \nabla\Phi : \nabla v - w^2\Phi v,$$

$$a_\epsilon(\Phi_\epsilon, v) - a_0(\Phi, v) = \int_{\Omega_\epsilon} \nabla(\Phi_\epsilon - \Phi)\nabla V - w^2(\Phi_\epsilon - \Phi)v + \int_{\bar{\omega}_\epsilon} \nabla\Phi\nabla v - w^2 \int_{\bar{\omega}_\epsilon} \Phi v$$

3.2 A generalized adjoint method

The mathematical framework for domain parametrization introduced by the Murat-Simon work [13] cannot be used here. Alternatively, it is possible however to invoke the adjoint method, as described in [21], in application to topological optimization. A basic feature of the adjoint method is yield of an asymptotic expansion of a functional $J(\Phi_\epsilon)$ which depends of a parameter Φ, using a adjoint state V which does not depend on the parameter. This implies the need to solve a certain system of equations in order to obtain an approximation of the topological gradient $g(x)$; accordingly, let $\mathcal{V}$ be a fixed Hilbert space and $\mathcal{L}(\mathcal{V})$ $(resp\mathcal{L}_2(\mathcal{V}))$ denotes the spaces of linear (resp bilinear) forms on $\mathcal{V}$. We are able then to state the following hypothesis:

1. $\mathbf{H-1}$ There exists a real function f defined in $\mathbb{R}^+$, a bilinear and continuous form a_0 defined in $\mathcal{L}_2(\mathcal{V})$ and a linear form δ_a such that:

$$\lim_{\epsilon \to 0} f(\epsilon) = 0 \tag{48}$$

$$\|a_\epsilon - a_0 - f(\epsilon)\delta_a\|_{\mathcal{L}_2(\mathcal{V})} \; = \; 0, \tag{49}$$

$$\|l_\epsilon - l_0 - f(\epsilon)\delta_J\|_{\mathcal{L}(\mathcal{V})} \; = \; 0, \tag{50}$$

2. $\mathbf{H-2}$ The bilinear form a_0 is coercive: There exists a constant $\alpha > 0$ such that

$$a_0(u,u) \geq \alpha \|u\|^2, \qquad \forall u \in \mathcal{V} \tag{51}$$

According to (51), the bilinear form a_ϵ depends continuously on ϵ; hence there exists ϵ_0 and $\beta > 0$ such that $\forall\ \epsilon \in [0;\epsilon_0]$ the following uniform coercivity condition holds.

$$a_\epsilon(u,u) \geq \alpha \|u\|^2, \qquad \forall u \in \mathcal{V} \tag{52}$$

Using inequality (52) for $\Phi_1 = \Phi_\epsilon - \Phi_0$, for all $\epsilon \geq 0$, the function Φ_ϵ is solution to (46), the inequality (48) and (49) and the continuity of δ_a, we obtain the following lemma

Lemma 1 *We have*

$$\|\Phi_\epsilon - \Phi\| = O(f(\epsilon)) \tag{53}$$

3. $\mathbf{H-3}$ Consider a cost function $j(\epsilon) = J(\Phi_\epsilon)$; where the functional J is differentiable. For $u \in \mathcal{V}$ there exists a linear and continuous form $DJ(u) \subset L(\mathcal{V})$ and δ_J such that:

$$J_\epsilon(v) - J_0(u) = DJ_0(u)(v-u) + f(\epsilon)\delta_J + o(\|v-u\| + f(\epsilon) \tag{54}$$

The Lagrangian is defined by

$$\mathcal{L}_\epsilon(u,v) = J_\epsilon(u) - a_\epsilon(u,v) - l_\epsilon(v)\ \forall u,\ v\ \in \mathcal{V} \tag{55}$$

Theorem 3.1 *If hypothesis* $\mathbf{H-1}$, $\mathbf{H-2}$ *and* $\mathbf{H-3}$ *are satisfied and let* Φ_ϵ *the solution to (46). Then the functionals* J_ϵ *admits the following asymptotic expansion*

$$J_\epsilon(\Phi_\epsilon) - J_0(\Phi) \;=\; f(\epsilon)\delta_{\mathcal{L}}(\Phi, U) + o(f(\epsilon))$$

where $\delta_{\mathcal{L}}(u,v) = \delta_J(u) + \delta_a(u,v) - \delta_l$ *and* U *is the solution of the adjoint problem: to look for* $U \in \mathcal{V}$ *such that*

$$a_0(W,U) = -DJ_0(\Phi)W, \;\forall W \in \mathcal{V}$$

In order to get the asymptotic expansion of the cost functional, we will use the fact that variation of the Lagrangian is equal to the one of the cost functional. Then

$$J(\Phi_\epsilon) - J(\Phi) = \mathcal{L}_\epsilon(\Phi_\epsilon, v) - \mathcal{L}_0(\Phi, v), \;\forall\, v \;\in\; \mathcal{V} \tag{56}$$

3.3 Variation of the cost functional

$$J(\overrightarrow{A}) = \int_\Omega \|\overrightarrow{rot}(\overrightarrow{A}) - B_d\|^2 = \int_\Omega \sum_{i=1}^{3}((\overrightarrow{rot}\,\overrightarrow{A})_i - B_{d_i})^2$$

with

$$\overrightarrow{rot}(\overrightarrow{A}) = \begin{pmatrix} \partial_2 A_3 - \partial_3 A_2 \\ \partial_3 A_1 - \partial_1 A_3 \\ \partial_1 A_2 - \partial_2 A_1 \end{pmatrix} \tag{57}$$

Let $\overrightarrow{h} = (h_1, h_2, h_3)$ in $H_0^1(\Omega, \mathbb{R}^3)$ then we have

$$J(\overrightarrow{A}+\overrightarrow{h}) = \int_\Omega \|\overrightarrow{rot}(\overrightarrow{A})+\overrightarrow{rot}(\overrightarrow{h})-B_b\|^2 dx = \int_\Omega \sum_{i=1}^{3}((\overrightarrow{rot}\,\overrightarrow{A})_i - B_{d_i} + (\overrightarrow{rot}(\overrightarrow{h}))_i)^2$$

$$= \int_\Omega \sum_{i=1}^{3}\{(\overrightarrow{rot}\,\overrightarrow{A})_i - B_{d_i})^2 + (\overrightarrow{rot}(\overrightarrow{h}))_i)^2 + 2(\overrightarrow{rot}\,\overrightarrow{A})_i - B_{d_i})(rot(\overrightarrow{h}))_i)$$

$$J(\overrightarrow{A}+\overrightarrow{h}) - J(\overrightarrow{A}) = 2\int_\Omega \sum_{i=1}^{3}(\overrightarrow{rot}\,\overrightarrow{A})_i - B_{d_i})(\overrightarrow{rot}(h))_i) + \int_\Omega \sum_{i=1}^{3}(\overrightarrow{rot}(\overrightarrow{h}))_i)^2 dx$$

$$= 2\int_\Omega \overrightarrow{rot}(\overrightarrow{h})(\overrightarrow{rot}(\overrightarrow{A}) - B_d)dx + \int_\Omega \|\overrightarrow{rot}(\overrightarrow{h})\|^2 dx$$

But we get

$$\int_\Omega \overrightarrow{rot}(\overrightarrow{h})(\overrightarrow{rot}(\overrightarrow{A}) - B_d)dx = \int_\Omega (\partial_2 h_3 - \partial_3 h_2)K_1 + (\partial_3 h_1 - \partial_1 h_3)K_3 + (\partial_1 h_2 - \partial_2 h_1)K_3$$

where $K_i = \overrightarrow{rot}(\overrightarrow{A}))_i - B_{d_i}$ This gives

$$\int_\Omega \overrightarrow{rot}(h)(\overrightarrow{rot}(\overrightarrow{A}) - B_d)dx = \int_\Omega -h_3\partial_2 K_1 + h_2\partial_3 K_1 - h_1\partial_3 K_2 + h_3\partial_1 K_2 - h_2\partial_1 K_3 + h_1\partial_2 K_3$$

$$= \int_\Omega h_1(\partial_2 K_3 - \partial_3 K_2) + h_2(\partial_3 K_1 - \partial_1 K_3) + h_3(\partial_1 K_2 - \partial_2 K_1)$$

then

$$J(\overrightarrow{A} + \overrightarrow{h}) - J(\overrightarrow{A}) = 2\int_\Omega \overrightarrow{rot}(\overrightarrow{rot}(\overrightarrow{A}) - B_d)\,\overrightarrow{h}$$

$$+ \underbrace{\int_\Omega \Big[(\partial_2 h_3 - \partial_3 h_2)^2 + (\partial_3 h_1 - \partial_1 h_3)^2 + (\partial_1 h_2 - \partial_2 h_1)^2}_{o(\|h\|^2_W)}$$

where $W = H^1_0(\Omega, \mathbb{R}^3)$ is a functional space; then $DJ(\overrightarrow{A}) = 2\overrightarrow{rot}(\overrightarrow{rot}(\overrightarrow{A}) - B_d$ and the adjoint problem takes the form

$$\begin{cases} -\Delta U = 2(\overrightarrow{rot}\ \overrightarrow{rot}(\overrightarrow{A}) - B_d) & \text{in } \Omega \\ U = 0 \quad \text{or } \frac{\partial U}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{58}$$

3.4 Variation of the bilinear form

Let $w_\epsilon = \Phi_\epsilon - \Phi$, then w_ϵ is solution to

$$\begin{cases} \Delta w_\epsilon + w^2 w_\epsilon = 0 & \text{in } \Omega_\epsilon \\ \frac{\partial w_\epsilon}{\partial n} = 0 & \text{on } \partial\Omega \\ \frac{\partial w_\epsilon}{\partial n} = -\frac{\partial \Phi}{\partial n} & \text{on } \partial\omega_\epsilon \end{cases} \tag{59}$$

Let us approximate the solution w_ϵ by h_ϵ solution to

$$\begin{cases} \Delta h_\epsilon + w^2 h_\epsilon = 0 & \text{in } \mathbb{R}^2 \backslash \omega_\epsilon \\ \frac{\partial h_\epsilon}{\partial n} = -\frac{\partial \Phi}{\partial n} & \text{in } \partial\omega_\epsilon \\ h_\epsilon = 0 \text{ at } & \infty \end{cases} \tag{60}$$

Let $h_\epsilon = \epsilon H_\epsilon(\frac{x}{\epsilon})$ then we get

$$\begin{cases} \Delta H = 0 & \text{in } \mathbb{R}^2 \backslash \bar{\omega} \\ \frac{\partial H}{\partial n} = -\frac{\partial \Phi}{\partial n} & \text{on } \partial\omega \\ H = 0 & \text{at } \infty \end{cases} \tag{61}$$

Using potential theory the solution H of problem (61) is obtained by

$$H(x) = \int_{\partial\omega} \lambda(y) E(x - y) ds(y)\ \forall x \in \mathbb{R}^2 \backslash \bar{\omega} \tag{62}$$

where $\lambda \in H_0^{-1/2}(\partial\omega)$ is the unique solution of the integral equation

$$\frac{\lambda(x)}{2} + \sum \lambda(y)\partial_{n_x} E(x-y)ds(y) = -\nabla\Phi_0.n \tag{63}$$

If $\omega = B(0,1)$ one proves that $\delta_a = -2\pi\nabla\Phi(0)\nabla V(0) + k^2 mes(\omega)\Phi(0)V(0)$ where V is solution to adjoint problem (66). For the functional we have

$$J_\epsilon(\Phi_\epsilon) - J_0(\Phi) = \int_{\Omega_\epsilon} |\Phi_\epsilon - \Phi_d|^2 - \int_\Omega |\Phi - \Phi_d|^2$$

$$= \int_{\omega_\epsilon} |\Phi - \Phi_d|^2$$

Using Taylor expansion of Φ and a change of variables we obtain

$$\int_{\omega_\epsilon} |\Phi - \Phi_d|^2 = -\pi\epsilon^n |\Phi(0) - \Phi_d(0)|^2 + \epsilon^n$$

Thus we have

$$J_\epsilon(\Phi_\epsilon) - J_0(\Phi) = \epsilon^n(-\pi|\Phi(0) - \Phi_d(0)|^2) + o(\epsilon^n) \tag{64}$$

Then $\delta_J = -\pi|\Phi(0) - \Phi_d(0)|^2$. We can proof easily that $\delta_l = 0$

Theorem 3.2 *Let J_ϵ the functional defined by (47) where Φ_ϵ is solution to (46) with $\frac{\partial\Phi_\epsilon}{\partial n} = 0$ on $\partial\omega_\epsilon$. The following asymptotic expansion holds*

$$J_\epsilon(\Phi_\epsilon) - J(\Phi) = \epsilon^3(-2\pi\nabla\Phi(0)\nabla U(0) + k^2 mes(\omega)\Phi(0)U(0) - \pi|\Phi(0) - \Phi_d(0)|^2) + o(\epsilon^n) \tag{65}$$

where Φ is solution to (45) and U is solution to the adjoint state

$$\begin{cases} -\Delta U = 2(\overrightarrow{rot}\ \overrightarrow{rot}(\overrightarrow{\Phi}) - B_d) & in\ \Omega \\ \frac{\partial U}{\partial n} = 0 & on\ \partial\Omega. \end{cases} \tag{66}$$

Remark 2 *In the case for Diriclet condition on $\partial\omega_\epsilon$ and $\omega = B(0,1)$, the topological derivative reads $\mathcal{G}(x_0) = -2\pi\Phi(x_0)U(x_0)$ where Φ is solution to (45) and V is solution to the adjoint problem*

$$\begin{cases} -\Delta U = 2(\overrightarrow{rot}\ \overrightarrow{rot}(\overrightarrow{A}) - B_d) & in\ \Omega \\ U = 0 & on\ \partial\Omega. \end{cases} \tag{67}$$

3.5 Permanent Case

Let us consider the initial problem

$$\Delta\overrightarrow{A} - \frac{1}{c^2}\frac{\partial^2\overrightarrow{A}}{\partial t^2} = -\mu_0 J_{tot} \tag{68}$$

in which we have a Dirichlet or Neumann condition B^Ω in Ω, where

$$\overrightarrow{A} = (A_1, A_2, A_3) \tag{69}$$

design the potential vector. In the permanent case $\frac{\partial}{\partial t} = 0$, we obtain

$$\begin{cases} -\Delta\overrightarrow{A} = -\mu_0 J_{tot} \quad \text{in } \Omega \\ B^\Omega\overrightarrow{A} = g \quad \text{on } \partial\Omega \end{cases} \tag{70}$$

In Ω_ϵ

$$\begin{cases} -\Delta A^\epsilon = -\mu_0 J_{tot} \quad \text{in } \Omega_\epsilon \\ B^\Omega A^\epsilon = g \quad \text{on } \partial\Omega \\ A^\epsilon = 0 \quad \text{on } \partial\omega_\epsilon \end{cases} \tag{71}$$

Multiplying (71) by a test function v and integrating we get

$$\int_{\Omega_\epsilon} \nabla A^\epsilon : \nabla v dx - \int_{\partial\Omega} \frac{\partial A^\epsilon}{\partial n}.v - \int_{\partial\omega_\epsilon} \frac{\partial A^\epsilon}{\partial n} v = \int_{\Omega_\epsilon} \mu_0 J_{tot} v dx \tag{72}$$

Consider the functional space V_p defined by

$$V_p = \{v \in H^1(\Omega_\epsilon);\ v = 0 \text{ on } \partial\Omega,\ v = 0 \text{ on } \omega_\epsilon\},$$

then the bilinear form associated to the operator is the following

$$a_\epsilon(A^\epsilon, v) = \int_{\Omega_\epsilon} \nabla A^\epsilon : \nabla v dx$$

and the linear form is given by

$$l_\epsilon = \int_{\Omega_\epsilon} \mu_0 J_{tot} v dx$$

In the same way multiplying (70) by a test function $v \in (H_0^1(\Omega))^3$ and integrating in Ω we have

$$a_0(A, v) = l_0(v) \quad \forall v \in (H_0^1(\Omega))^3 \tag{73}$$

where

$$a_0(A, v) = \int_\Omega \nabla A : \nabla v, \quad l_0(v) = \int_\Omega \mu_0 J_{tot} v dx \tag{74}$$

Consider the exterior problem defined in $\mathbb{R}^n \backslash \omega$ by

$$\begin{cases} -\Delta\mathcal{A} = \mu_0 J_{tot} \text{ in } \mathbb{R}^n \backslash \omega \\ \mathcal{A} = 0 \text{ on } \partial\omega \end{cases} \tag{75}$$

Using the potential theory the problems (70),(71) and (75) admit solutions see [7]

3.5.1 Polarization matrix and topological derivative

Consider the basis of homogeneous polynomials in $\mathbb{R}^3$ of degrees inferior or equal to 1 satisfying $\Delta P = 0$. Let $U^1, \dots, U^N$ the basis of such polynomials. The polynomials satisfy

$$\left\{ \begin{array}{c} U^i(zx) = z^{s_i} U^i(x) \text{ with } s_i \text{ the degree of } U^i \\ \sum_{i=}^{N} U_j^k U_j^h = \delta_{hk} \end{array} \right. \tag{76}$$

As we have a Dirichlet condition on $\partial\omega,\ N = 3$ then we can take $U^i(x) = e^j$ where e^j is the vectors of the canonical basis of $\mathbb{R}^3$. Thus $s^i = 0,\ i = 1, 2, 3$. Let Φ^i the fundamental matrix associated to the vectorial Laplacian ie satisfying

$$\Delta \Phi^j = e^j \delta_x,\ j = 1, 2, 3 \tag{77}$$

where δ_x is the dirac mass concentrated at x, then $\Phi^j(x) = \frac{1}{K_3|x|} e^j,\ j = 1, 2, 3$. We define the vectors $U^{-i},\ i = 1, 2, 3$ by the relation

$$U^{-i}(x) = \sum_{j=1}^{3} U_j^i(x) \Phi^j = \Phi^i \tag{78}$$

Using the theory developed in [22, 23] and considering the following problem

$$\left\{ \begin{array}{c} -\Delta \zeta^j = 0 \text{ in } \mathbb{R}^3 \backslash \omega \\ \zeta^j = 0 \text{ on } \partial\omega \end{array} \right. \tag{79}$$

the solution ζ^j of (79) can be written as follows

$$\zeta^j = U^j + z^j = U^j + \sum_{j=1}^{3} m_{jk}^{\omega} U^{-j} + \tilde{z^j}$$

where z^j is solution to problem

$$\left\{ \begin{array}{c} -\Delta z^j = 0 \text{ in } \mathbb{R}^3 \backslash \omega \\ z^j = -U^j \text{ on } \partial\omega \end{array} \right. \tag{80}$$

where $\tilde{z^j}$ is the remainder of the Taylor development of z^j at the origin. The terms m_{jk}^{ω} are the coefficients of the polarization matrix.
In the following we give a theorem which give the topological derivative.

Theorem 3.3 *Let A^ϵ be the solution to (71) and J_ϵ the functional defined by (47). Then J_ϵ admits the following asymptotic expansion*

$$J_\epsilon(A_\epsilon) - J_0(A) = -\epsilon A \mathcal{A}^{\omega} U + o(f(\epsilon)) \tag{81}$$

where A is solution to problem (70) and U the solution to the adjoint problem defined by

$$\begin{cases} -\Delta U = 2\overrightarrow{rot}\overrightarrow{A}(\overrightarrow{rot}\overrightarrow{A} - \overrightarrow{B}_d) \ in \ \Omega \\ U = 0 \ on \ \partial\Omega \end{cases} \tag{82}$$

$\mathcal{A}^\omega$ denotes the polarization matrix.

Proof 1 *The proof of theorem 3.3 is standard in the literature see [23, 9, 13]*

4 Numerical simulations

In this section we consider a reference magnetic field $\overrightarrow{B}_d$ in a square $\Omega = [-1,1]\times[-1,1]$. Our aim is to compare the magnetic field $\overrightarrow{B} = \overrightarrow{rot}(\overrightarrow{A})$ where $\overrightarrow{A}$ is solution to (45) with the reference magnetic field $\overrightarrow{B}_d$ in different cases. Let ν be the frequency the pulsation is given by $2\pi\nu$. In the case of atomic physics the frequency ν is between $32768Hz$ and $9,192631770GHz$. we take a particular value $\nu = 40000Hz$. In the of geophysic the frequency ν is between $0,01$ and $10Hz$ and we take a particular value $\nu = 1Hz$.
The reference magnetic field is given by $B_d = (2x, 2x+y, 2x+2)$

4.1 Neumann condition

We simulate the topological derivative given in theorem 3.2 at a point x_0 by

$$-2\pi(\nabla\Phi(0)\nabla U(x_0) + k^2\Phi(0)U(x_0) - |\Phi(x_0) - \Phi_d(x_0)|^2)$$

where Φ is solution to (45) and U the adjoint state solution to (66). The topological derivative in both case is given by figure 1.

4.2 Dirichlet condition on the bounds of ω_ϵ

The topological derivative is given by in Remark 3.3 by $\mathcal{G}(x_{x_0}) = -2\pi\Phi.U$ where Φ is solution to (45) and U the adjoint state given by (67). The topological derivative is given in both cases by figure 2.

4.3 Permanent Case

We simulate the topological derivative $\mathcal{G}(x_0)$ given by theorem 3.3 where A is solution to (70) and U solution to (82). The topological gradient is given by figure 3.

Remark 3 *Note that in all figures where the topological derivative approaches zero, the magnetic field $\overrightarrow{B}$ is close to the reference magnetic field B_d. When they are very far away it means that the two fields do not approach.*

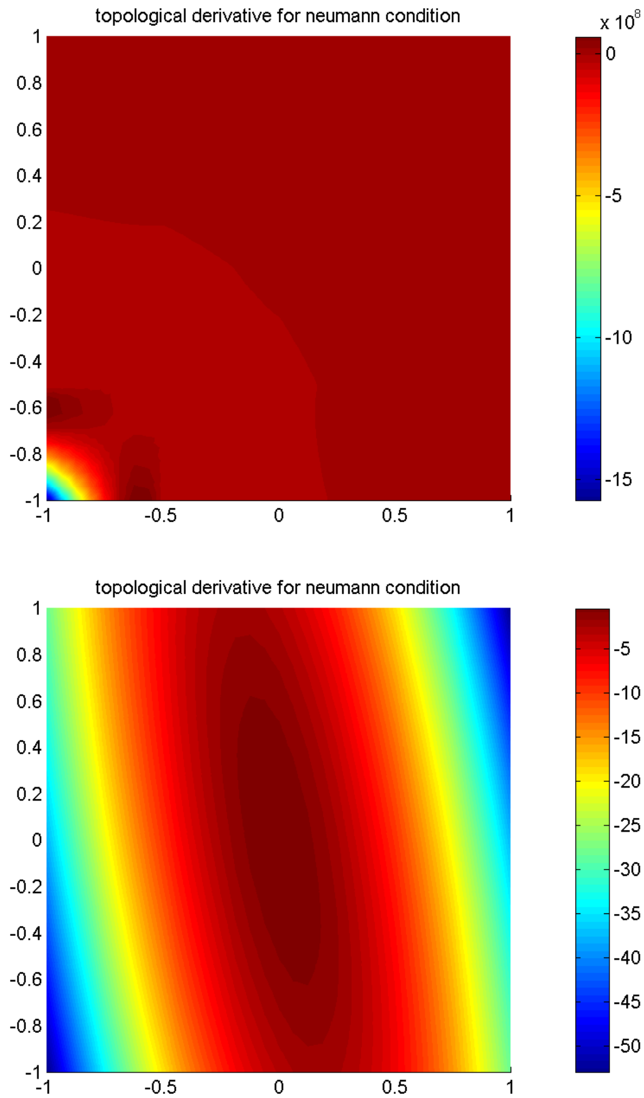

Figure 1: topological derivative in atomic physic at the top and in geophysic at the bottom

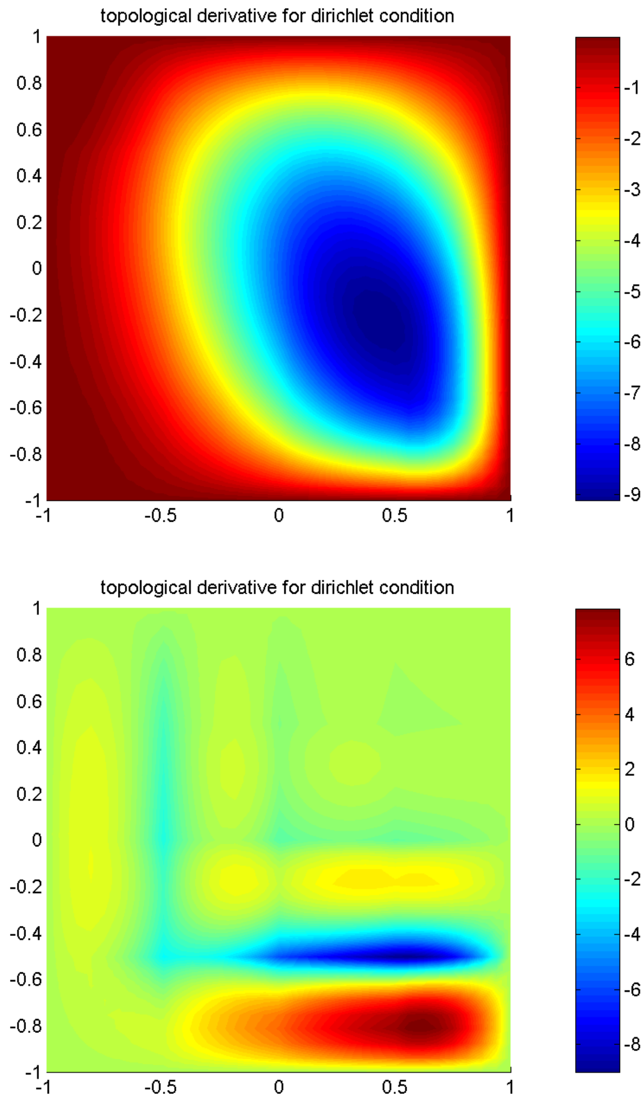

Figure 2: topological derivative in atomic physic at the top and in geophysic at the bottom

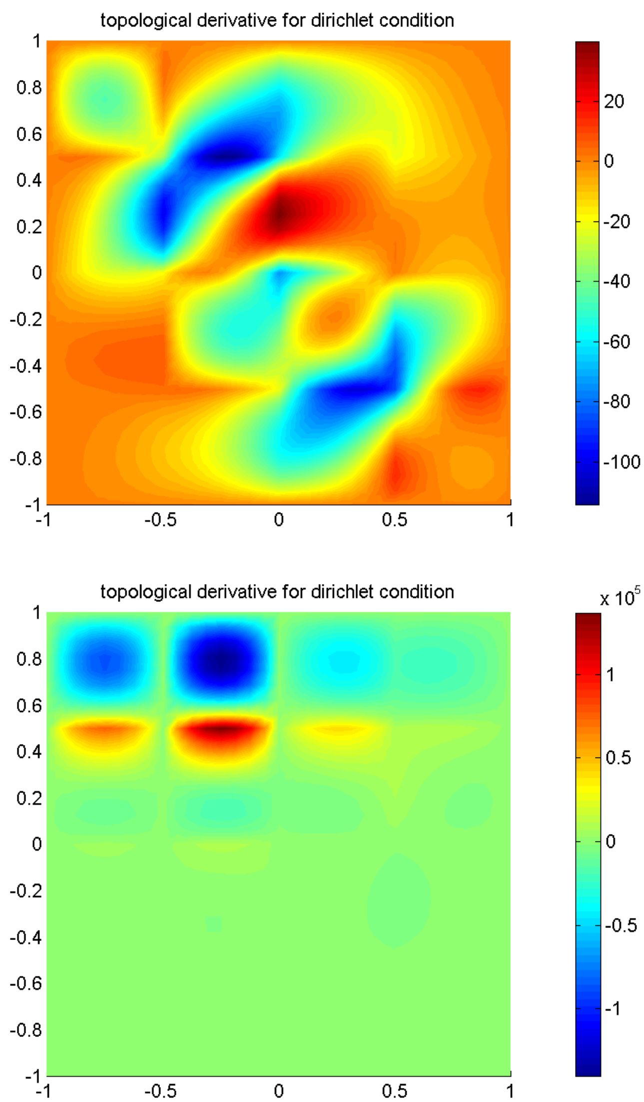

Figure 3: topological derivative in physic atomic at the top and in geophysic at the bottom

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