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On the Construction of Nearly Optimal Golomb Rulers by Unwrapping Costas Arrays

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Abstract

We show that stacking the columns of a Costas array one below the other yields a Golomb ruler, provided several blank rows have been appended at the bottom of the array first, and we prove rigorously an upper bound for the necessary number of rows. We then provide a method to determine the numbers of blank rows appended for which the construction succeeds, and we also determine by simulation the smallest such number over all Costas arrays of a given order. We argue that these Golomb rulers, though suboptimal, have applications in channels affected by multi-path interference and periodic bursts of noise, thanks to their special structure. We finally study briefly alternative unwrapping strategies.

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Keywords: Costas arrays, Golomb rulers, Sidon sets, unwrapping

1 Introduction

Costas arrays, namely square arrangements of dots and blanks such that there lies exactly one dot per row and column, and such that no four dots form a parallelogram and no three dots lying on a straight line are equidistant, appeared for the first time in 1965 in the context of SONAR detection [8, 9],

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when J.P. Costas, seeking to improve the performance of SONAR systems, used them to describe a novel frequency hopping pattern for SONARs with optimal auto-correlation properties. About two decades later, Prof. S. Golomb published two generation techniques [12, 17, 18] for Costas arrays, both based on the theory of finite fields, known as the Welch and the Golomb method, respectively. These are still the only general construction methods for Costas permutations available today.

The analog of a Costas array in one dimension is a Golomb ruler, namely a linear arrangement of dots and blanks such that no distance between two dots is repeated twice. It is actually an older concept than Costas arrays themselves: in the equivalent form of Sidon sets, i.e. integer sets of distinct pairwise sums, Golomb rulers appeared for the first time in the literature of harmonic analysis in 1932 [30]. They were rediscovered, in an engineering context, in 1953, when they were used for radio-frequency allocation avoiding third-order interference [1], and also in 1976, in the context of graph theory [7]. Other important applications of Golomb rulers in engineering include generating convolutional self-orthogonal codes [28] and the formation of optimal linear telescope arrays in radio-astronomy [6].

Any diagonal of a Costas array defines a Golomb ruler; this is a direct consequence of the definitions. But can the entire Costas array be transformed into a Golomb ruler? Perhaps the simplest transformation one can think of to this effect is to stack the columns of the array one below the other ("unwrap the array vertically"). In this paper we show that this direct approach actually succeeds for small orders only. There is a possible generalization, however, which we prove to be always successful, namely to append $m-n$ blank rows at the bottom of the array, for a suitably chosen m, and then unwrap. We further determine an upper bound to the value of m needed and argue that the Golomb rulers so constructed have useful properties for use in channels affected by multi-path interference and periodic burst noise.

The paper is organized as follows: Section 2 reviews the basic definitions for Costas arrays and Golomb rulers; Section 3 presents vertical (column-wise) unwrapping of Costas arrays and proves the method successfully produces Golomb rulers provided the array is padded with sufficiently many blank rows at the bottom, and investigates further exactly how many blank rows are necessary; Section 4 describes a potential application of rulers produced by the vertical unwrapping of Costas arrays in communications; and Appendix A presents alternative unwrapping strategies.

2 Basics

In this section we collect a set of properties and results used throughout the paper. For $n \in \mathbb{N}$, we denote the set of integers $\{1, \ldots, n\}$, which will be appearing quite often, by $[n]$.

2.1 Costas permutations

Let us begin with the definition of a Costas function/permutation [8, 9, 12]:

Definition 1. Consider a bijection $f : [n] \to [n]$; f is a Costas permutation iff:

$$\forall i, j, k \text{ such that } 1 \leq i, j, i+k, j+k \leq n : \\ f(i+k) - f(i) = f(j+k) - f(j) \Rightarrow i = j \text{ or } k = 0.$$

A permutation f corresponds to a permutation array $A_f = [a^f_{i,j}]$ by setting the elements of the permutation to denote the positions of the (unique) 1 in the corresponding column of the array, counting from top to bottom: $a^f_{f(i),i} = 1$. It is customary to represent the 1s of a permutation array as "dots" and the 0s as "blanks". From now on the terms "array" and "permutation" will be used interchangeably, in view of this correspondence.

Figure 1 shows an example of a Costas array of order 27. Incidentally, all Costas arrays of order 27 have been discovered through exhaustive search, and this turned out to be the only one that was not already known [13].

The Costas property is invariant under horizontal and vertical flips, as well as transposition (and therefore also under rotations of the array by multiples of 90^o, which can be expressed as combinations of the previous two operations), hence a Costas array gives birth to an equivalence class that contains either eight Costas arrays, or four if the array happens to be symmetric.

As we have mentioned several equivalent versions of the definition of the Costas permutation, it is perhaps worth discussing the equivalence in some more detail. Starting with the array A_f, observe that its dots lie at positions $(f(i), i)$, $i \in [n]$, hence the vectors between them are $(f(i) - f(j), i - j)$, $i > j$, $i, j \in [n]$. If two such vectors are equal, then either they share a common endpoint, in which case they lie on the straight line and they are equidistant, or else the four distinct endpoints from a parallelogram. Hence, Definition 1 is equivalent to the definition mentioned in the Introduction.

2.2 Golomb rulers

Geometrically, the Golomb ruler can be interpreted as a sequence of dots/1s (called the *markings* of the ruler) and blanks/0s, so that all distances between

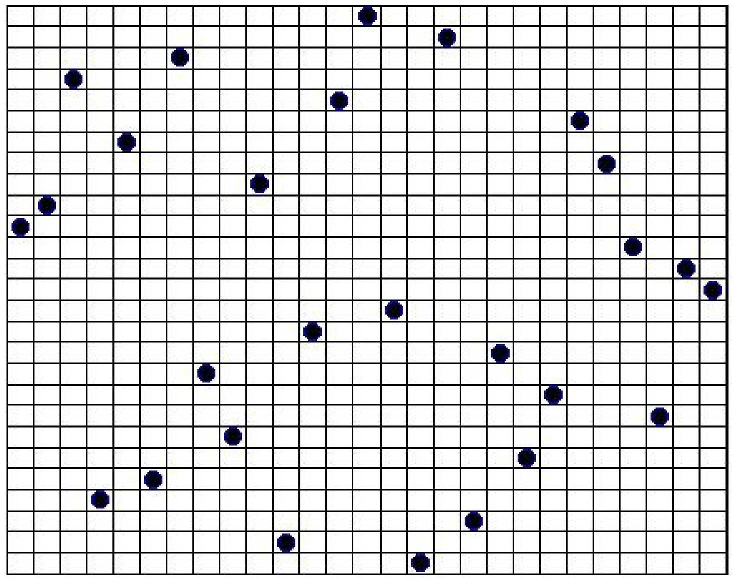

Figure 1: An example of a Costas array of order 27

pairs of dots are distinct.

Definition 2. Let $m, n \in \mathbb{N}$, and let $f : [m] \to [n+1]$ be monotonic and injective with $f(1) = 1$, $f(m) = n + 1$ (whence $m \leq n + 1$); f is a Golomb ruler of length n with m markings (also designated as being of order m) iff

$$\forall i, j, k, l \in [m],\ f(i) - f(j) = f(k) - f(l) \Leftrightarrow i = k \wedge j = l.$$

The positions of the dots in this binary sequence g_f correspond to the range of f.

For example, Figure 2 shows B_f, where $f([8]) = \{1, 2, 5, 10, 16, 23, 33, 35\}$ is a Golomb ruler.

Remark 1. Unfortunately, the established range conventions for Costas arrays and Golomb rulers in the literature are incompatible, as a Golomb ruler of length n is usually taken to begin at 0 and end at n ($f(1) = 0$ and $f(m) = n$). Our choice to start Golomb rulers at 1 agrees with the convention used for Costas arrays.

Two important questions arise:

1. What is the maximal m possible for given n?

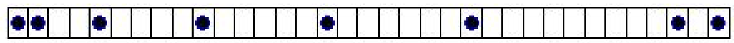

Figure 2: An example of an optimal Golomb ruler with 8 markings (at 1, 2, 5, 10, 16, 23, 33, and 35) of length 34

2. What is the minimal n possible for given m?

A Golomb ruler is *optimal* iff it satisfies either of the two conditions above (though some authors restrict the term to the latter condition only). For neither case, however, do we have closed form answers, although several estimates exist: the simplest one is that a Golomb ruler of length n defines at most n possible distances, and, in order to have m points, the $\binom{m}{2}$ distances they define will need to be unique. It follows that

$$\binom{m}{2} = \frac{m(m-1)}{2} \leq n \Rightarrow m \leq \sqrt{2n} \text{ asymptotically.} \tag{1}$$

This turns out to be a very generous upper bound: improved arguments show that $m < \sqrt{n} + O(n^{1/4})$ [14] and even better that $m < \sqrt{n} + \sqrt[4]{n} + 1$ [19]. Furthermore, the maximal m for a given n satisfies asymptotically $m > \sqrt{n} - O(n^{5/16})$ [14], but it is conjectured to satisfy $m > \sqrt{n}$ [10]. Not surprisingly, the main emphasis of Golomb ruler research has been on the discovery of optimal Golomb rulers (see e.g. [12, 26, 27, 29] and many more), which parallels the effort for the complete enumeration of Costas arrays of a certain order (see e.g. [2, 13, 22]). For example, the Golomb ruler shown in Figure 2 is an optimal Golomb ruler with 8 markings, whose length is 34: no shorter Golomb ruler with 8 markings exists.

In this work, we will call a family of Golomb rulers *nearly (asymptotically) optimal* iff $\lim_n \frac{\sqrt{n}}{m(n)} \to c > 1$.

Many of the results above were actually formulated in the context of Sidon sets [14], which historically precede Golomb rulers. At some point the two research communities realized that both terms describe the very same object, but this observation was not made immediately, and, in the meantime, different bodies of literature developed, including duplicate results [10].

A very comprehensive source of information about Golomb rulers and Sidon sets is A. Dimitromanolakis' diploma thesis [10]. There are various construction methods available for Golomb rulers [4, 5, 23, 24, 31], but we won't have the occasion to use them in this work. It was recently shown [21] that various problems related to the construction of Golomb rulers are NP-complete.

3 Results

We show that, if enough blank rows are appended at the bottom of a Costas array, its vertical unwrapping results in a Golomb ruler, and we further discuss this result in a series of remarks.

Theorem 1. Let $f : [n] \to [n]$, $n \in \mathbb{N}$, be a Costas permutation, and consider the function $g : [n] \to [nm]$, where $m \in \mathbb{N}$ as well, so that

$$g(i) = (i-1)m + f(i),\ i \in [n]. \tag{2}$$

If $m \geq 2n-2$, g is a Golomb ruler.

Proof. The values of g are clearly distinct. To verify the defining property of the Golomb ruler, we consider the equation:

$$g(i) - g(j) = g(u) - g(v) \Leftrightarrow f(i) - f(j) = f(u) - f(v) + m(u - v + j - i). \tag{3}$$

Given that $f(i) - f(j) \in \{-(n-1), \ldots, -1, 1, \ldots, n-1\}$, it follows that $f(i) - f(j) - (f(u) - f(v)) \in \{-2n+3, \ldots, -1, 1, \ldots, 2n-3\}$ (note that the values $\pm(2n-2)$ are not attainable since f is a permutation). If then $m \geq 2n-2$, (3) can only have a solution if $u - v + j - i = 0 \Leftrightarrow i - j = u - v$; but, in this case, (3) becomes

$$f(i) - f(j) = f(u) - f(v),\ i - j = u - v, \tag{4}$$

which, since f has the Costas property, affords $i = j$, $u = v$ or $i = u$, $j = v$ as its only solutions. This completes the proof. □

Remark 2. The value $m = 2n-2$ leads to nearly optimal Golomb rulers: the ruler's length is at most $(n-1)m + n = 2(n-1)^2 + n$ and at least $(n-2)m + 2 = 2(n-1)(n-2) + 2$, and therefore asymptotically we expect about $\sqrt{2n^2} = \sqrt{2}n$ points in optimal rulers of this length. These rulers have n points instead, which is within a constant multiplicative factor $\sqrt{2}$ of the optimal value. For general m, the length of these rulers is asymptotically nm, and optimal rulers for this length are expected to have $\sqrt{mn}$ points.

Remark 3. When $n \leq m < 2n-2$, (3) can have a solution iff $u - v + j - i = 0, \pm 1$: g will fail to be a Golomb ruler iff (3) has a solution for $u - v + j - i = \pm 1$. When considering the entire equivalence class of a Costas permutation, we may restrict our attention to $+1$.

Remark 4. This result can be extended to a more general class of arrays which satisfy the Costas property without necessarily representing permutations (namely multiple dots per row and column are now allowed, as well as non-square arrays). The proof above is still valid in this context with only minor notational changes. J.P. Robinson in 1997 actually studied empirically this transformation over such arrays (termed "Golomb rectangles") [25], which he viewed as "folded" Golomb rulers. He also considered higher-dimensional (3D, 4D,...) folds. Furthermore, a similar result in the context of PPM sequences had appeared even earlier in 1987 [15]. Finally, truncated Costas arrays extended with blank columns have been used in the construction of Optical Orthogonal Codes (OOC) [20].

The theorem demonstrates in particular that, for any Costas array f, the set of values of $m \geq n$ for which the construction yields a Golomb ruler is nonempty. What is the smallest possible value of m? This value is of interest because it leads to the densest possible constructions. In order to find it, we note that, according to Remark 3, $f(i) - f(j) - (f(u) - f(v)) = m$, where $u - v + j - i = 1$ and at most two among i, j, u, v are equal, provide all values of m for which the Golomb ruler property fails. It suffices then to pick the values of m in the range $\{n, n+1, \dots, 2n-2\}$ that do not appear in this list, and then pick the smallest one. For an entire family of Costas arrays, it is of interest to determine the minimum of this minimal value over all Costas arrays of the family.

This double minimum, evaluated for the family of all Costas arrays of a fixed order n, where $3 \leq n \leq 27$, is shown in Table 1, which was obtained by exhaustive computer search. It is clear from the table that the minimal value of m allowed is systematically below $2n - 2$ within the given range of n. For $n = 1, 2$ unwrapping obviously works without any blank rows, as the unwrapped array contains at most one distance, hence these values are not shown. The table also proves $m = n$ to be possible also when $n = 3, 4, 6$.

Table 2 shows the full histogram of Costas arrays of order n over the allowed values of m: more specifically, the entry corresponding to n and $m - n$ shows the number of (equivalence classes of) Costas arrays of order n for which appending $m - n$ blank rows at the bottom an unwrapping leads to a Golomb ruler. In accordance with Theorem 1, the maximal nonzero entry of the row corresponding to order n lies at $m - n = n - 2$ and equals the total number of Costas arrays of order n: larger m are not shown, as the entry values remain constant from then on. The minimal value of m for which the entry is nonzero, on the other hand, has been recorded in Table 1. To prevent possible misconceptions, let us state explicitly that the fact that $m = M$ leads to a Golomb ruler does *not* necessarily imply that $m = l(n), l(n) + 1, \dots, M - 1$ also do ($l(n)$ denoting the smallest allowed m for n): the allowed values of m for a given array do *not* have to be consecutive.

Note that unwrapping two Costas arrays that are horizontal and/or vertical flips of each other leads to the same set of distance vectors between the dots

n	m	n	m	n	m	n	m	n	m
3	3	8	9	13	16	18	23	23	35
4	4	9	10	14	19	19	27	24	38
5	6	10	12	15	20	20	29	25	38
6	6	11	14	16	21	21	31	26	42
7	8	12	14	17	24	22	30	27	42

Table 1: The minimal allowed value for m in Theorem 1 as a function of the order n; it is smaller than $2n-2$ in all cases.

in the ruler: in other words, unwrapping leads to a Golomb ruler for the one iff it does for the other. It follows that the number of Costas arrays for which a specific m is allowed includes "half" equivalence classes (unwrapping is not compatible with transposition, so the other half of the equivalence class may behave differently) and is thus divisible by 4, as can be verified in Table 2.

Do Costas arrays exist for which the smallest allowed value of m is $m = 2n-2$? We will call these Costas arrays *incompressible*, in the sense that the white space needed between columns when unwrapping cannot be further compressed with respect to the theoretical bound proved in Theorem 1. Exhaustive search showed that incompressible Costas arrays exist indeed, and yielded the results contained in Table 3, along with some interesting further observations (which can be extrapolated as conjectures to all $n > 27$ as well):

1. Costas arrays with a corner dot are "mostly" not incompressible: only 12 exceptions were found, 4 of order 6 and 8 of order 14, their three unique representatives being shown in Table 4.

2. Costas arrays generated by either the Golomb or the Welch method are never incompressible.

3. Symmetric Costas arrays are "mostly" not incompressible: only two exceptions were found, both of order 13, shown in Table 5.

Note that, although Tables 2 and 3 are related, there is no direct link between them, as they display different information: the entries of Table 3 count the number of Costas arrays of order n that require the addition of at least $m-n = n-2$ rows in order to yield a Golomb ruler through unwrapping, whereas in Table 2 a Costas array contributes to multiple entries of row n, according to whether the addition of $m-n$ rows leads to a Golomb ruler through unwrapping. In other words, the entries of Table 3 count the number of Costas arrays of order n that contribute to the last entry $(m-n = n-2)$ in row n of Table 2 without contributing to any previous entry of that row, but this piece of information cannot be retrieved by studying Table 2.

$n \backslash m-n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
3	4	4																								
4	4	8	12																							
5		28	24	40																						
6	4	20	68	92	116																					
7		8	16	132	132	200																				
8		4	44	92	292	324	444																			
9		4	16	76	172	468	568	760																		
10			12	56	400	684	1488	1624	2160																	
11				48	140	820	1488	2876	3352	4368																
12			4	4	100	508	1572	2904	5512	6344	7852															
13				12	32	176	944	3168	5132	9120	10476	12828														
14						60	508	1532	4632	7240	12556	14204	17252													
15						12	84	764	1828	5936	8760	14636	16124	19612												
16						4	56	208	1164	2696	6756	9972	15836	17896	21104											
17								52	232	1092	2596	6104	8936	14160	15592	18276										
18						4	4	4	88	232	1076	2204	5592	7816	11584	12900	15096									
19									12	24	164	944	1776	4020	5160	7992	8816	10240								
20										20	60	172	612	1200	2484	3540	6160	5580	6464							
21											8	40	108	324	696	1416	1972	2840	3108	3536						
22									4	4	16	8	36	132	332	616	884	1312	1724	1864	2052					
23													8	12	24	108	240	448	516	692	792	872				
24															8	16	28	56	68	108	176	176	200			
25														4	8	12	16	20	36	40	60	84	68	88		
26																	4	16	12	32	32	48	48	52	56	
27																8	8	28	24	56	84	120	156	172	192	204

Table 2: The number of Costas arrays of order n for which appending $m-n$ blank rows at the bottom and unwrapping leads to a Golomb ruler; blank entries correspond to 0

n	#	n	#	n	#
5	4	11	108	17	104
6	4	12	84	18	84
7	4	13	160	19	68
8	12	14	192	20	32
9	20	15	136	21	4
10	40	16	104	22	4

Table 3: The number of incompressible Costas arrays of orders $5 \leq n \leq 22$; for all other orders $n \leq 27$ such arrays do not exist at all.

1	5	6	2	4	3								
1	9	11	4	14	10	13	2	6	3	8	7	5	12
1	11	8	13	14	5	4	2	9	3	12	7	10	6

Table 4: The three unique representatives under horizontal and/or vertical flips of incompressible Costas arrays with a corner dot, one of order 6 and two of order 14.

4 Applications

Research on Golomb rulers seems today to focus exclusively on optimal Golomb rulers: so much so, as a matter of fact, that the importance of the additional "degrees of freedom" offered by non-optimal Golomb rulers is not fully appreciated. Consider the common situation in communications where a message of n bits needs to be transmitted in a medium seriously affected by multi-path interference: positioning the transmissions as indicated by a Golomb ruler minimizes this sort of interference, as each position of the dominant signal will only be affected by at most one other copy of the signal; this is an immediate consequence of the definition of a Golomb ruler, and a well-known fact. Assume, without loss of generality, that each bit requires a unit of time (instant) to transmit.

Suppose though, in addition, that the channel is also affected by bursts of noise occurring almost periodically and within the last k instants of a periodic cycle lasting m instants, where m and n are comparable, so that we are forced to transmit bits in between the bursts (but obviously not during the bursts), while retaining the overall structure of a Golomb ruler throughout the transmission. The Golomb rulers constructed by vertically unwrapping Costas arrays, as described above, offer the ideal solution to this problem.

Channels with these two noise features (multi-path interference and periodic bursts of noise) actually become increasingly important in applications today, as indoor power-line channels fall in this category [3, 16].

8	11	3	4	13	9	7	1	6	12	2	10	5
9	4	12	2	8	13	7	5	1	10	11	3	6

Table 5: The two symmetric incompressible Costas arrays of order 13.

5 Conclusion

We considered the construction of Golomb rulers through a transformation of Costas arrays, dubbed vertical unwrapping, whereby several blank rows are appended to the bottom of the array and then its columns are stacked one below the other. We proved this construction to be successful whenever at least $n-2$ extra blank rows are appended to a Costas array of order n, we determined through exhaustive search the minimal number of blank rows for which the construction is successful for a given order, and then provided a method to determine all allowed numbers of added blank rows for a given Costas array based on the entries of its difference triangle.

We noted that the resulting Golomb rulers, though suboptimal (but still nearly optimal), are naturally divided into n segments of $n+k$ positions each (assuming k blank rows were appended), so that there is exactly one dot per segment, which never lies within the final k positions. We finally argued that this property is desirable and useful for the reliable transmission of information over channels affected by multi-path interference and periodic bursts of noise: an increasingly important channel for everyday applications that fits this description is the indoor power-line channel.

Acknowledgements

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Method \ n	2	3	4	5	6	7	8	9
Diagonal	2	4	8	16	20	22	4	4
Spiral	2	3	4	9	11	4	1	0

Table 6: The number of Costas arrays, per order n and method, whose unwrapping results to a Golomb ruler

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A Alternative unwrapping strategies

In this appendix we present the results of two alternative unwrapping strategies, namely diagonal and spiral unwrapping, described below. The reason we chose not to include this material in the main body of the text is the lack of any theoretical justification why these methods should produce any Golomb rulers; we simply proceeded heuristically, encouraged by the fact that both techniques preserve a small part of the structure of the original Costas array, though perhaps less so than vertical unwrapping. We applied both methods on the database of all known Costas arrays of order 27 and below, testing the resulting rulers for the Golomb property: our conclusion was that both methods successfully produce Golomb rulers, but only for small orders (the results are shown in Table 6).

A.1 Diagonal unwrapping

Simply put, diagonal unwrapping stacks the anti-diagonals of an array one next to the other: if $A = [a_{ij}]$ is the array, diagonal unwrapping produces the sequence

$$a_{11}|a_{21}a_{12}|a_{31}a_{22}a_{13}|a_{41}\ldots \tag{5}$$

The vertical lines are added simply to enhance readability. If A is a Costas array, the result is a ruler, possibly a Golomb ruler, and the positions of the markings can be computed from the entries of the corresponding Costas permutation:

Algorithm 1. Let $f : [n] \to [n]$, $n \in \mathbb{N}$, be a Costas permutation. The ruler resulting from the diagonal unwrapping of f is given by the following:

$$D_f(i) = \begin{cases} \frac{(k-2)(k-1)}{2} + i, & \text{if } k := i + f(i) \le n \\ \frac{n(n+1)}{2} + \frac{(k-n-2)(3n+1-k)}{2} + n + 1 - f(i), & \text{if } k := i + f(i) > n \end{cases}$$

Clearly, working along the diagonals instead is completely equivalent.

A.2 Spiral unwrapping

Spiral unwrapping proceeds iteratively, by listing sequentially all boundary elements of a square array, starting at $(1, 1)$ and proceeding clock-wise, and then repeating for the interior of the array:

Algorithm 2. Let $A = [a_{ij}]$, $i, j \in [n]$, $n \in \mathbb{N}$ be a square array of order n. The ruler resulting from the spiral unwrapping of A is produced iteratively as follows:

$$S(A) = a_{1,1} \ldots a_{1,n-1} | a_{1,n} \ldots a_{n-1,n} | a_{n,n} \ldots a_{n,2} | a_{n,1} \ldots a_{2,1} | S([a_{ij}]_{i,j \in \{2,\ldots,n-1\}}).$$

The vertical lines group the array elements of the boundary into 4 sets of $n-1$ elements each and are added simply to enhance readability.

Example

Applying the algorithm when $n = 4$,

$$S(A) = a_{11}a_{12}a_{13}|a_{14}a_{24}a_{34}|a_{44}a_{43}a_{24}|a_{14}a_{13}a_{12}|a_{22}|a_{23}|a_{33}|a_{32}. \tag{6}$$

The Costas permutation 1243 is spirally unwrapped into the ruler

$$100|001|010|000|1|0|0|0 \text{ or } 1, 6, 8, 13,$$

if we represent the ruler by the positions of the nonzero elements (1s), as usual. This particular ruler fails to have the Golomb property, as $13-8 = 6-1 = 5$.

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