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DEA and Multi-Objective Shortest Path Problems

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Abstract

In this paper we are concerned with a particular multi-criteria optimal path problem, known as the multi-objective shortest path problem. The purpose of this paper is to study a data envelopment analysis (DEA) model that allows us to determine the set of all non-dominated paths on a network. We prove that the non-dominated path in multi-objective shortest path problem is equivalent to corresponding strongly efficient unit in its DEA model without output with actually observed units.

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1 Introduction

The optimal solution of the classic shortest path problem with a Min-Sum objective function is a single shortest route in a directed graph. This problem may be solved by a label-correcting algorithm [14] or for non-negative costs solved by a label-setting algorithm such as Dijkstra's algorithm [6]. We can also consider several other objective functions such as Max-Sum, Max-Production or Max-Min function (See Ahuja et al. [15]). However, considering one objective function such as minimizing the sum of the arcs' costs of the path may not be sufficient to describe real world problems. Suppose that each of the arcs of the graph has a number of both kinds of criteria, costs and benefits. In general, no single route

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will have simultaneously the least costs and the most benefits. The solution to an instance of this multi-criteria problem will be a set of Pareto optimal routes. The multi-objective shortest path problem (MSPP) was proposed first by Vincke in1974 [12]. When all the criterions lead only to Min-Sum objective functions, under assumptions concerning the cost values, the set of non-dominated paths can be computed by a multiple objective extension of the Dijkstra's algorithm, i.e., Martins' label-setting algorithm [4]. For the same problem, Martins and Santos [5] and Guerriero and Musmanno [7], also give a label-correcting algorithm. Several other objective functions such as bottleneck objective functions are discussed in [11, 19].

Data envelopment analysis (DEA) is a mathematical programming-based technique widely used to measure relative efficiencies of decision making units (DMUs) such as evaluating the efficiencies of paths on a road network. Unlike parametric models, DEA does not require the analyst to pre-specify the functional form of the efficient frontier (production function). Instead of a pre-specified function form, the frontier is convex and based on the construction of a piece-wise linear combination of the efficient units. That "efficiency" in DEA and "convex efficiency" in MCDM are equivalent is not a new fact. A number of papers [2, 16, and 17] have discussed the analysis of the links between DEA and MCDM.

This paper deals with the use of a technique based on an adaptation of DEA model of Caporaletti et al. [10] to distinguish between efficient and not efficient DMUs. Although Cardillo et al. [3] and Jahanshahloo et al. [8] have presented techniques to evaluate and identify the efficient paths on a road network; they do not take into account the number of paths in a multi-criteria optimal path problem, specifically in multi-objective shortest path problem. We prove that the efficient solutions in DEA model without output with actually observed units are equivalent to the non-dominated paths in MSPP.

2 Background

In this section, we introduce the notation and definitions needed to define the shortest path problem with multiple objectives. A network $\mathcal{N}$ is defined as a directed and connected graph $G=(X,U)$, where $X=\{1,2,\cdots,t\}$ is vertex set with *t* vertices (none of which is isolated) and $U=\{e_1,e_2,\cdots,e_u\}$ is arc set with *u* vertices. Each arc e_l *for all* $l=1,\cdots,u$, is denoted by an ordered pair (v,w), where $v,w\in U$ such that $v\neq w$. It is supposed that there is only one directed arc (v,w) from *v* to *w*. In this network, we specify a source vertex *s* and a terminal vertex *t*, and *m* is the number of criteria with the corresponding *m* dimensional function vector cost assigned to each arc,

$$\begin{aligned} c:\ & U\to \mathbf{R}^m \\ & (v,w)\mapsto c(v,w)=\mathbf{c}_{v,w}=(c^1_{v,w},\cdots,c^m_{v,w}) \end{aligned} \quad (1)$$

A path p is a sequence of vertices and arcs from s to t. We assume that the vertices appeared on the path are all distinct. A cycle C is a path with non-repeated vertices except the source and terminal vertices that are coincident. Given the source vertex s and a vertex v in X, the set of all paths from s to v is denoted by $P^{s,v}$. We denote the set of all paths from s to t by P, instead of $P^{s,t}$, in order to simplify the notation.

Let $c^i(p)$ denotes the value of i-th criterion of a path p, for each $i=1,\cdots,m$. The cost vector $C(p)$ of path p is the sum of the cost vectors of its arcs, i.e.

$$C(p)=(c^1(p),c^2(p),\cdots,c^m(p))=\sum_{(v,w)\in p} c(v,w), \tag{2}$$

while on the contrary, the cost vector for bottleneck functions is,

$$C(p)=(c^1(p),c^2(p),\cdots,c^m(p))=\min_{(v,w)\in p} c(v,w). \tag{3}$$

Definition 2.1 If *the objective vector is to be minimized, a path* $p\in P^{v,w}$ *from vertex v to vertex w dominates another path* $q\in P^{v,w}$ *if and only if the objective vector* $C(p)$ *dominates* $C(q)$ *in the sense that* $C_k(p)\le C_k(q), k=1,\cdots,m$, *with strict inequality holding for at least one k.*

Definition 2.2 A *path* $p\in P^{v,w}$ *from vertex v to vertex w is non-dominated path in* $P^{v,w}$ *if and only if there exists no other path* $q\in P^{v,w}$ *which dominates p. The set of all non-dominated paths from v to w is denoted by* $ND^{v,w}$ *and ND will be used for* $ND^{s,t}$.

Let $G=(X,U)$ denote a directed graph consisting of a finite set $X=\{1,2,\cdots,t\}$ of vertices and a finite set $U=\{e_1,e_2,\cdots,e_u\}$ of u arcs. There is a cost vector $\mathbf{c}_{v,w}\in\mathbf{R}^m$ associated with each arc, where m represents the number of criteria. Let $C(p)$ be the cost vector of a path p in G. The multi-objective shortest path problem is to find the set of all Pareto optimal paths from the source vertex s to all other vertices in G, with respect to the cost vector $C(p)$, so the version considered in this paper is as follows:

$$\mathop{Min}_{p\in P^{s,t}} \sum_{(v,w)\in p} c(v,w) \qquad \forall t\in X\setminus\{s\} \tag{4}$$

in which, $c(v,w)=(c^1_{v,w},\cdots,c^m_{v,w})=\mathbf{c}_{v,w}$ is the cost vector of $(v,w)\in p$.

The original DEA model, introduced by Charnes et al. [1] in 1978 and referred to as the CCR model, optimizes the fractional output per input of each DMU. We suppose that there exist n DMUs, indexed by $k=1,2,\cdots,n$; y_{rk} is the quantity of output $r, r=1,2,\cdots,s$, produced by the decision making unit p, x_{ik} the quantity of input i, $i=1,2,\cdots,m$, used by DMU_k, u_r the weight associated with output r, v_i the weight associated with input *i*. The optimization program to assess the efficiency of DMU_p, $p \in \{1, 2, \cdots, n\}$, is defined as follows:

$$\begin{aligned} z^p &= Max \ \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip}} \\ s.t \ & \frac{\sum_{r=1}^{s} u_r y_{rk}}{\sum_{i=1}^{m} v_i x_{ik}} \le 1, \quad k = 1, 2, \cdots, n. \\ & u_i, v_r \ge 0, \qquad for\ all\ \ i, r \end{aligned} \tag{5}$$

If $z^{p^*} = 1$, then DMU_p is efficient.

Now, we present a definition of efficiency proposed by Caporaletti et al. [10].

Definition 2.3 *The DUM_k is an efficient unit, if and only if there exists no convex combination of the other DMUs such that all criteria of the convex combination is less than or equal to the criteria of DMU_k and at least one criterion is not equal.*

3 DEA and Multi-Objective Shortest Path Problems

In this section, we show that a non-dominated path in multi-objective shortest path problem is equivalent to efficient DMUs according to definition 2.3 in DEA model without output with actually observed units. Using DEA terminology, we observe that in the well-known multi-objective shortest path problem, the paths in the network $\mathcal{N}$ can be considered as the decision making units (DMUs) without outputs, by replacing its cost vectors to be minimized with inputs. Let the DMU_j to be evaluated on any trial be designated as DMU_p where *p* ranges over $1, 2, \cdots, |P^{s,v}|$ in which *v* is a vertex in $X \setminus \{s\}$. Since we consider DMUs as network's paths and actually we have to choose only one real route, the convex

linear combination of paths is not permissible. This intuition is the best lead to deal with free disposal hull (FDH) models.

FDH is a special case of DEA that employs a smaller set of units when defining the efficiency frontier. Instead of DEA's piecewise linear frontier, FDH employs a stepwise (or staircase) frontier that ensures that efficiency evaluations are effected by only actually observed performances [18]. Because the FDH frontier is either identical to or interior to the DEA frontier, FDH will typically give larger estimates of average efficiency than DEA. Unlike DEA, FDH is not restricted to convex technologies.

According to the FDH model by Tulkens [9] the input-oriented BCC model without output is as follows,

$$
\begin{array}{lll}
Min & \theta^p & \\
s.t & \sum_{j=1}^{|P^{s,v}|} \lambda_j x_{ij} + s_i = \theta^p x_{ip} \ , & i=1,2,\cdots,m \\
 & \sum_{j=1}^{|P^{s,v}|} \lambda_j = 1 & \\
 & \lambda_j \in \{0,1\} \ , & j=1,2,\cdots,|P^{s,v}| \\
 & s_i \geq 0 \ , & i=1,2,\cdots,m
\end{array} . \tag{6}
$$

Theorem 3.1 *The VRS FDH model without output is feasible and bounded.*

Proof: The VRS FDH model has a feasible solution $\lambda_p = 1$ and $\lambda_j = 0 \ (j \neq p)$. Note that the model (6) is a mixed integer programming but we can consider the following equivalent input-oriented model by Agrell et al. [13],

$$
\begin{array}{lll}
Min & \sum_{j=1}^{|P^{s,v}|} \theta_j^p & \\
s.t & \sum_{j=1}^{|P^{s,v}|} \lambda_j x_{ij} \leq \theta_j^p x_{ip}, & i=1,2,\cdots,m \\
 & \sum_{j=1}^{|P^{s,v}|} \lambda_j = 1 & \\
 & \lambda_j \geq 0 & , j=1,2,\cdots,|P^{s,v}|
\end{array} . \tag{7}
$$

Consider the dual form of (7), we have:

$$
\begin{array}{lll}
Max & z^p & \\
s.t & \sum_{i=1}^{m} x_{ip} v_{ij} = 1 \ , & j=1,2,\cdots,|P^{s,v}| \\
 & \sum_{i=1}^{m} x_{ij} v_{ij} \geq z^p \ , & j=1,2,\cdots,|P^{s,v}| \\
 & v_{ij} \geq 0, \quad i=1,2,\cdots,m \ , & j=1,2,\cdots,|P^{s,v}|
\end{array} . \tag{8}
$$

It is obvious that problem (8) is feasible and $z^* \leq 1$, so the model (6) is bounded.□

When in model (6), $\theta^{p^*}=1$ and if $s_i=0$ for all $i=1,2,\cdots,m$ then DMU_p is called *strongly efficient*, else if there exists $s_i>0$, $i\in\{1,2,\cdots,m\}$ then DMU_p is called *weakly efficient*.

Theorem 3.2 *The non-dominated paths in the multi-objective shortest path problem are equivalent to the strongly efficient decision making units in VRS FDH model without output according to definition 2.3.*

Proof: ($\Rightarrow$) Let there exists a strongly efficient DMU_p of model (6) such that it is not a non-dominated path. Therefore, there exists $x_k=(x_{1k},x_{2k},\cdots,x_{mk})^T\in S$ such that $x_k\le x_p$ with at least one strict inequality. By setting $\lambda_k=1, \lambda_j=0\ (j\ne k)$, we have:

$$\sum_j \lambda_j x_j = x_k \qquad s.t \qquad \sum_j \lambda_j = 1 \tag{9}$$

that is a contradiction.

($\Leftarrow$) On the contrary, we prove that if DMU_p is not strongly efficient in VRS FDH model without output, then, it would not be non-dominated path. Assume that DMU_p is either inefficient or weakly efficient unit. If DMU_p is an inefficient unit, so $\theta_p^*<1$, and there exists a linear combination such as

$$x=\sum_j \lambda_j x_j \qquad s.t \qquad \sum_j \lambda_j = 1 \quad \& \quad \lambda_j\in\{0,1\} \tag{10}$$

such that $x\le x_p \quad \& \quad x\ne x_p$. Hence, there exists $k\in\{1,2,\cdots,|P^{s,v}|\}$ such that

$$\lambda_k=1 \quad \& \quad \frac{x_{ik}}{x_{ip}}\le\theta_p^*<1 \quad \textit{for all} \quad i=1,2,\cdots,m\,. \tag{11}$$

Consequently, we have $x_{ik}<x_{ip}$ for all $i=1,2,\cdots,m$.

If DMU_p is weakly efficient then $\theta_p^*=1$, and

$$\exists k\ (\ \exists l\ \ (\ k\in\{1,2,\cdots,|P^{s,v}|\}\wedge k\ne p\ \wedge\ l\in\{1,2,\cdots,m\})\ s.t\ \ s_l>0)\,. \tag{12}$$

Since $\theta_p^*=1$, we have:

$$\sum_{j=1}^{|P^{s,v}|}\lambda_j x_{ij}+s_i=x_{ip} \quad \textit{for all} \quad i=1,2,\cdots,m\,. \tag{13}$$

Moreover, we have $\lambda_k=1$ and $s_i\ge 0$, $i=1,2,\cdots,m$ in which $s_l>0$. Hence, there exists $x_k\in S$ such that $x_k\le x_p$ with at least one strict inequality, therefore, DMU_p is dominated by DMU_k. Consequently, it was proved that DMU_p is not a non-dominated unit and equivalently it is not a non-dominated solution to the MSPP. So eventually, the above theorem proves the assumption. □

4 Illustrative example

As an example to illustrate that strongly efficient paths in VRS FDH without output model and non-dominated paths in MSPP with Min-Sum objective function are equivalent. We consider the network $\mathscr{N}$ with five vertices and eight arcs with two dimensional cost vector, shown in Fig.1.

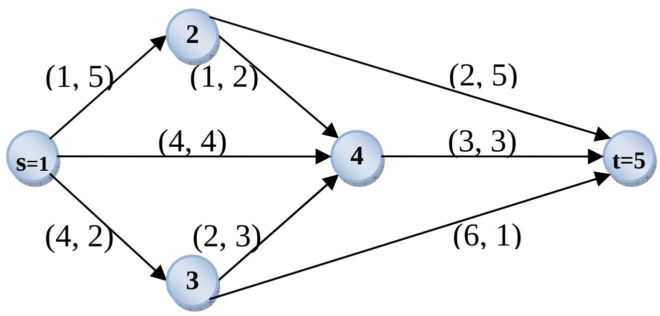

Figure 1 Example of MSPP with Min-Sum objective function

There exist five paths from vertex s to vertex t which three of them, obtained by Martins' label-setting algorithm (presented to solve the MPSS), are in the set $ND^{s,t}$ as shown in Table 1.

Table 1 Paths from s to t on the network $\mathscr{N}$

Path (p)	**Input (Cost) vector, $C(p)$**	**Martins' Algorithm**	**Efficiency (θ)**	**Slack Variables (s)**	**Status**
p_1**:1 →2→5**	$C(p_1)=(3, 10)$	$p_1 \in ND^{s,t}$	$\theta^*=1$	$\forall j \ \forall i \ \ s_i=0$	Strongly Efficient
p_2**:1→2→4→5**	$C(p_2)=(5, 10)$	Dominated by p_1	$\theta^*=1$	$j=1, i=1 \ s_i=2$	Weakly Efficient
p_3**:1→4→5**	$C(p_3)=(7, 7)$	$p_3 \in ND^{s,t}$	$\theta^*=1$	$\forall j \ \forall i \ \ s_i=0$	Strongly Efficient
p_4**:1→3→4→5**	$C(p_4)=(9, 8)$	Dominated by p_3	$\theta^*=0.875$	$j=3 \begin{cases} i=1 & s_i=2 \\ i=2 & s_i=1 \end{cases}$	Inefficient
p_5**:1→3→5**	$C(p_5)=(10, 3)$	$p_5 \in ND^{s,t}$	$\theta^*=1$	$\forall j \ \forall i \ \ s_i=0$	Strongly Efficient

According to the DEA model proposed in this paper, we obtain four efficient paths with $\theta_p^* = 1$. In spite of the fact that the efficiency of path p_4 equals one, its cost vector is dominated by the cost vector of path p_3. On investigating the path p_4 carefully, we find that, for i=1 and j=3, the corresponding slack variable of the constraint in model (6) is not equal to zero. Hence, p_4 is weakly efficient path.

5 Conclusions

In this paper we discover which DEA model's solutions correspond to the Pareto optimal solution of a MSPP. Also, we have explained by The VRS FDH model without output that the non-dominated paths obtained by label-setting Martins' algorithm are equivalent to the efficient DMUs in DEA environment. The DEA technique provided powerful vision to peruse other multi-criteria optimal path problems, which have not been solved by an efficient labeling algorithm, in which both kinds of criteria, costs and benefits exist on each arc. Under concepts presented here, by considering costs as inputs and benefits as outputs, we might be able to solve multi-criteria optimal path problems. This study will be done in a future paper soon under way.

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