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Free Vibrations of Lightly Stretched and Lightly Coupled Elastically Connected Euler-Bernoulli Beams

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Abstract

The general theory for systems of stretched elastically connected Euler-Bernoulli beams is modified to develop approximate solutions for the natural frequencies and mode shapes several special cases. The beams are lightly stretched when the normal stress due to the axial load is much smaller than the normal stress due to bending. An asymptotic expansion in terms of the ratio of these stresses leads to a set of equations corresponding to elastically coupled, but unstretched Euler-Bernoulli beams for the lowest-order approximation. These solutions are organized as sets of n intramodal frequencies and mode shapes. A solvability condition is used to determine first-order corrections. A free end is an exceptional case.

The beams are lightly coupled when the elastic layers have little effect on the non-dimensional natural frequencies. An asymptotic expansion leads to an uncoupled set of equations at the lowest order. First-order corrections are obtained from application of a solvability condition. If any of the beams are identical the lowest-order solution is degenerate and a special form of the solvability condition is used to determine first-order corrections.

Keywords: Vibrations, elastically connected beams, solvability condition, degenerate conditions

1 Introduction

Free vibrations of elastically connected beams have been proposed as a model for analyzing the vibrations of multi walled carbon nanotubes where the connection is through the van der Waals forces [5],[7],[8]. The tensile strength of carbon nanotubes has been shown to increase when the tubes are stretched [3],[4].

As noted in [7] numerical difficulties occur in obtaining results for the natural frequencies for a single axially loaded beam. A Rayleigh-Ritz method for the vibrations using the mode shapes of unstretched beams to analyze the natural frequencies and mode shapes of a set of stretched beams is derived [2]. The purpose of this paper is to derive an approximate solution for the cases when the beams are lightly stretched or lightly coupled. The former can be applied to carbon nanotubes while the later applies to high frequency vibrations of nanotubes.

2 Problem formulation

The non-dimensional equations governing the motion of n elastically connected stretched beams on a Winkler foundation is presented in (1). A normal-mode solution is assumed resulting in a set of coupled ordinary differential equations which form an eigenvalue and eigenvector problem to solve for the natural frequencies and mode shapes. The resulting equations are summarized by

$$\mu_1 \frac{d^4 w_1}{dx^4} - \varepsilon \frac{d^2 w_1}{dx^2} + \lambda_0 w_1 + \lambda_1 (w_1 - w_2) - \beta_1 \omega^2 w_1 = 0, \qquad (1a)$$

$$\mu_j \frac{d^4 w_j}{dx^4} - \varepsilon \frac{d^2 w_j}{dx^2} + \lambda_{j-1} (w_j - w_{j-1}) + \lambda_j (w_j - w_{j+1}) - \beta_j \omega^2 w_j = 0, \qquad (1b)$$

for $j = 2,3,\ldots,n-1$ and

$$\mu_n \frac{d^4 w_n}{dx^4} - \varepsilon \frac{d^2 w_n}{dx^2} + \lambda_{n-1} (w_n - w_{n-1}) + \lambda_n w_n \beta_n \omega^2 w_n = 0. \qquad (1c).$$

The non-dimensional parameter ε is the ratio of the normal stress due to the axial load to the normal stress in the first beam due to bending. Equations (1) are summarized in a matrix form by

$$\mathbf{K} + \mathbf{K_c u} - \omega^2 \mathbf{Mu} = \mathbf{0}, \qquad (2)$$

where $\mathbf{u}$ is a vector whose components are the spatial mode shapes, $\mathbf{K}$ is the structural stiffness operator matrix which can be written as

$$\mathbf{K} = \mathbf{K_b} + \mathbf{K_a}, \qquad (3)$$

in which $\mathbf{K_b}$ is a diagonal operator matrix representing the bending stiffness with

$$(k_b)_{i,i} = \mu_i \frac{\partial^4}{\partial x^4}, \qquad (4)$$

$\mathbf{K_a}$ is a diagonal operator matrix representing the axial stiffness with

$$(K_a)_{i,i} = -\varepsilon \frac{\partial^2}{\partial x^2}, \qquad (5)$$

$\mathbf{K_c}$ is a tri-diagonal matrix of coupling stiffnesses due to the elastic layers with

$$(k_c)_{i,i-1} = -\lambda_{i-1} \quad i = 2,3,\ldots,n,$$
$$(k_c)_{i,i} = \lambda_{i-1} + \lambda_i \quad i = 1,2,\ldots,n, \qquad (6)$$
$$(k_c)_{i,i+1} = -\lambda_i \quad i = 1,2,\ldots n-1,$$

and $\mathbf{M}$ is a diagonal mass matrix with

$$m_{i,i} = \beta_i\,. \qquad (7)$$

The natural frequencies are the square roots of the eigenvalues of the operator $\mathbf{M}^{-1}(\mathbf{K_a} + \mathbf{K_b} + \mathbf{K_c})$ and the mode shapes are their corresponding eigenvectors.

While an exact solution can be obtained for Eq. (2) very large exponential terms lead to difficulty in solving transcendental equations to determine the natural frequencies and then to find their corresponding mode shapes. If all beams are identical the natural frequencies and mode shapes can be obtained by solving a matrix eigenvalue problem. A Rayleigh-Ritz solution is developed in Part 1 to avoid the numerical difficulties for the general case. Application of the Rayleigh-Ritz method to an illustrative example shows that the natural frequencies are approximately linear with ε, especially for small ε.

Two special cases in which asymptotic expansions lead to natural frequency and mode shape approximations are presented his paper. If ε is small the beams are lightly stretched. If the coupling does not significantly change the natural frequencies the beams are lightly coupled. In both cases an asymptotic expansion in terms of a small parameter leads to the application of a solvability condition [2] to determine approximations for the natural frequencies. The corresponding mode shape vectors are then obtained using the expansion theorem.

3 Asymptotic analysis

The natural frequencies of a single axially loaded beam are determined by solving a transcendental equation involving the parameters

$$\tau_{1,2} = \frac{1}{\sqrt{2}}\left[\sqrt{\varepsilon^2 + 4\delta^2} \pm \varepsilon\right]^{\frac{1}{2}} \qquad (8)$$

Binomial expansions of the left hand side of Eq. (8) assuming $\left|\frac{\varepsilon}{2\delta}\right| << 1$ leads to

$$\tau_{1,2} = \sqrt{\delta}\left[1 \pm \frac{\varepsilon}{4\delta} + \frac{\varepsilon^2}{32\delta^2}\right] + O\left(\varepsilon^3\right) \quad (9)$$

If $\varepsilon = 0$ the natural frequencies are those of an unstretched beam which are tabulated [1]. The expansions obtained in Eq. (9) suggest that an asymptotic expansion in terms of the parameter ε should lead to accurate results when $\left|\frac{\varepsilon}{4\delta}\right| << 1$. The lowest natural frequency of an unstretched beam can be used as a benchmark for allowable values of ε in an asymptotic expansion for the mode shapes and natural frequency.

Now consider the effect of coupling. The analysis of a set of identical elastically connected and axially loaded beams in [1] leads to an equation for their natural frequencies as

$$\omega_{k,j} = \sqrt{\delta_k^2 + \nu_j} \quad (10)$$

where $\delta_k, k = 1,2,...$ are the natural frequencies of a single axially loaded beam and ν_j, $j = 1,2,...,n$ are the eigenvalues of the coupling matrix. Defining $\kappa = \max_{j=1,...,n} \nu_j$ and $\sigma_j = \frac{\nu_j}{\kappa}$, Eq. (10) is rewritten as

$$\omega_{k,j} = \delta_k \sqrt{1 + \frac{\kappa}{\delta_k^2}\sigma_j} \quad (11)$$

The magnitude of the effect of the coupling on a set of natural frequencies for a set of identical beams is determined by the magnitude of $\frac{\kappa}{\delta_k^2}$. Thus a specific mode is said to be lightly coupled if $\frac{\kappa}{\delta_k^2} << 1$. In this case define $\eta_k = \frac{\kappa}{\delta_k^2}$ and $\lambda_i = \sigma_i \eta_k \delta_k^2$. The set of n coupled differential equations can be summarized as

$$\mathbf{Ku} + \eta_k \delta_k^2 \hat{\mathbf{K}}_\mathbf{c}\mathbf{u} - \omega^2 \mathbf{Mu} = \mathbf{0} \quad (12)$$

where $\mathbf{K}$ is a nxn diagonal operator matrix, $\hat{\mathbf{K}}_\mathbf{c}$ is a scaled coupling matrix and $\mathbf{M}$ is a diagonal mass matrix.

4. Lightly stretched beams

A. Asymptotic expansions

An asymptotic expansion for a mode shape for the ith beam when the beams are lightly stretched is

$$u_i(x) = u_{i,0}(x) + \varepsilon u_{i,1}(x) + \ldots \quad (13)$$

Correspondingly the natural frequency is expanded as

$$\omega = \omega_0 + \varepsilon\omega_1 + \ldots \quad (14)$$

Substitution of Eqs. (13) and (14) into Eq. (2), collecting coefficients of like powers of ε and taking the limit as powers of ε approach zero leads to the following sets of hierarchal equations:

O(1):

$$\mathbf{K_b u}_0 + \mathbf{K_c u}_0 - \omega^2 \mathbf{M u}_0 = \mathbf{0}, \quad (15)$$

and

O(ε): $$\mathbf{K_b u}_1 + \mathbf{K_c u}_1 - \omega^2 \mathbf{M u}_1 = \mathbf{u}_0'' + 2\omega_0\omega_1 \mathbf{M u_0}, \quad (16)$$

where $\mathbf{u_0} = \left[u_{1,0}(x) \quad u_{2,0}(x) \quad \ldots \quad u_{n,0}(x)^T\right]$ and $\mathbf{u}_1 = \left[u_{1,1}(x) \quad u_{2,1}(x) \quad \ldots \quad u_{n,1}(x)\right]^T$ and a prime represents differentiation of every component of the vector with respect to x.

The O(1) equations are those governing the free vibrations of a set of elastically coupled Euler-Bernoulli beams without stretching. They are of the case in [7] where the individual operators are proportional, $\mathbf{L_i} = \mu_i \mathbf{L_1}$. The nondimensional eigenvalue problem for the unstretched Euler-Bernoulli beam equation is of the form

$$\phi^{iv} - \sigma^2\phi = 0, \quad (17)$$

and is subject to appropriate boundary conditions. There are an infinite, but countable number of natural frequencies, σ_k $k = 1,2,\ldots$. Each natural frequency has a corresponding mode shape, $\phi_k(x)$. The mode shapes satisfy the orthogonality condition

$$\int_0^1 \phi_i(x)\phi_j(x)dx = 0, \quad \mathrm{i} \neq \mathrm{j}, \quad \mathrm{i,j} = 1,2,... \quad (18)$$

and are normalized by requiring .

$$\int_0^1 \beta_k \phi_k^2(x)dx = 1, \quad \mathrm{k} = 1,2,... \quad (19)$$

For each k there is a set of n intramodal frequencies, $\omega_{k,j}$ $j = 1,2,\ldots,n$ and corresponding mode shapes of the form $\mathbf{u}_{\mathbf{o}_{\mathbf{k},j}} = \mathbf{a_{k,j}}\phi_k(x)$ where $\mathbf{a_{k,j}}$ is a nx1 vector of constants. These intramodal natural frequencies and mode shapes are obtained by solving the matrix eigenvalue problem

$$\mathbf{Ka} = \omega^2 \mathbf{Ma}, \tag{20}$$

where **K** is a tri-diagonal matrix whose elements are

$$k_{i,i} = \mu_i \sigma^2 + \lambda_{i-1} + \lambda_i \quad \mathrm{i = 1,2,...,n} \tag{21a}$$

$$k_{i,j-1} = -\lambda_{i-1} \quad \mathrm{i = 2,3,...,n} \tag{21b}$$

$$k_{i,j+1} = -\lambda_i \quad \mathrm{i = 1,2,...,n-1} \tag{21c}$$

The intramodal mode shapes satisfy orthogonality conditions

$$(\mathbf{a}_{\mathbf{k},\mathbf{j}})^T \mathbf{Ma}_{\mathbf{k},\ell} = 0 \quad j \neq \ell \tag{22a}$$

and

$$(\mathbf{a}_{\mathbf{k},\mathbf{j}})^T \mathbf{Ka}_{\mathbf{k},\ell} = 0 \quad j \neq \ell \tag{22b}$$

The first-order equations corresponding to the zeroth-order natural frequency $\omega_{0_{k,j}}$ are

$$\mathbf{K_b u}_{1_{k,j}} + \mathbf{K_c u_1} - \omega_{o_{k,j}}^2 \mathbf{Mu}_{1_{k,j}} = \mathbf{u}_0'' + 2\omega_0 \omega_1 \mathbf{Mu}_{\mathbf{0}_{k,j}} \tag{23}$$

Equation (23) represents a system of non-homogeneous equations for which a non-trivial solution of the corresponding homogeneous system exists. Thus a solution to Eq. (23) exists if and only if a solvability condition is satisfied. The operator $\mathbf{K_b} + \mathbf{K_c} - \omega_{o_{k,j}}^2 \mathbf{M}$ is self-adjoint with respect to the standard inner product on W, defined by

$$(\mathbf{f}, \mathbf{g})_W = \int_0^1 \mathbf{g}^{\mathbf{T}} \mathbf{f} dx, \tag{24}$$

where **f** and **g** are arbitrary nx1 vectors of functions all of which satisfy all of the boundary conditions.

Taking the inner product of both sides of Eq. (23) with $\mathbf{u}_{\mathbf{0}_{\mathbf{k},\mathbf{j}}}$ leads to

$$\left(\mathbf{K_b u}_{1_{k,j}} + \mathbf{K_c u_1} - \omega_{o_{k,j}}^2 \mathbf{Mu}_{1_{k,j}}, \mathbf{u}_{\mathbf{0}_{\mathbf{k},\mathbf{j}}}\right)_W = \left(\mathbf{u}_{\mathbf{0}k,i}'', \mathbf{u}_{\mathbf{o}_{\mathbf{k},\mathbf{j}}}\right)_W + 2\omega_{0_{k,j}} \omega_{1_{k,j}} \left(\mathbf{Mu}_{\mathbf{0}_{\mathbf{k},\mathbf{j}}}, \mathbf{u}_{0_{\mathbf{k},\mathbf{j}}}\right)_W. \tag{25}$$

Since the operator is self adjoint, Eq. (25) becomes

$$\left(\mathbf{u}_{\mathbf{1}_{k,j}}, \mathbf{K_b u}_{0_{k,j}} + \mathbf{K_c u_1} - \omega_{o_{k,j}}^2 \mathbf{Mu}_{0_{k,j}}\right)_W = \left(\mathbf{u}_{\mathbf{0}k,i}'', \mathbf{u}_{\mathbf{o}_{\mathbf{k},\mathbf{j}}}\right)_W + 2\omega_{0_{k,j}} \omega_{1_{k,j}} \left(\mathbf{Mu}_{\mathbf{0}_{\mathbf{k},\mathbf{j}}}, \mathbf{u}_{0_{\mathbf{k},\mathbf{j}}}\right)_W, \tag{26}$$

which in view of Eq. (15) leads to the solvability condition

$$\omega_{1_{k,\ell}} = -\frac{\left(\mathbf{u}_{\mathbf{0}k,i}'', \mathbf{u}_{0_{\mathbf{k},\mathbf{j}}}\right)_W}{2\omega_{0_{k,j}} \left(\mathbf{Mu}_{\mathbf{0}_{\mathbf{k},\mathbf{j}}}, \mathbf{u}_{\mathbf{o}_{\mathbf{k},\mathbf{j}}}\right)_W}. \tag{27}$$

Solving for $\omega_{1_{k,j}}$, assuming the mode shapes are normalized according to Eq. (19), gives

$$\omega_{1_{k,j}} = -\frac{1}{2\omega_{0_{k,j}}}\int_0^1 \phi_k''(x)\phi_k(x)dx. \qquad (28)$$

When $\omega_{1_{k,j}}$ is chosen according to Eq. (28), Eq. (16) has a non-unique solution as any constant multiple of $\mathbf{u}_{\mathbf{0}_{\mathbf{k},\mathbf{j}}}$ can be added to the solution. The perturbation in the mode shape vector is obtained using the expansion theorem

$$\mathbf{u}_{\mathbf{1}_{\mathbf{k},\mathbf{j}}} = \sum_{\ell=1}^{\infty}\sum_{m=1}^{n}\alpha_{\ell,m}\mathbf{a}_{\ell,m}\phi_\ell(x). \qquad (29)$$

The summations in Eq. (29) are carried out for all ℓ and m except for the case when $\ell = k$ and $m = j$ as this term corresponds to the homogeneous solution.

Substituting Eq. (29) into Eq.(16) leads to

$$(\mathbf{K_b}+\mathbf{K_c})\left(\sum_{\ell=1}^{\infty}\sum_{m=1}^{n}\alpha_{\ell,m}\mathbf{a}_{\ell,m}\phi_\ell(x)\right) - \omega_{0_{k,j}}^2\mathbf{M}\left(\sum_{\ell=1}^{\infty}\sum_{m=1}^{n}\alpha_{\ell,m}\mathbf{a}_{\ell,m}\phi_\ell(x)\right) = \mathbf{u}''_{\mathbf{o}_{\mathbf{k},\mathbf{j}}} + 2\omega_{0_{k,j}}\omega_{1_{k,j}}\mathbf{Mu_0} \;. \qquad (30)$$

It can be shown that the set of mode shape vectors is complete in V. Thus the summations converge and the order of summation and operation can be interchanged. Performing this step and taking the standard inner product of both sides of Eq.(30) with $\mathbf{w}_{\mathbf{o}_{\mathbf{p},\mathbf{q}}} = \mathbf{a}_{\mathbf{p},\mathbf{q}}\phi_p(x)$ for arbitrary p and q, except for the choice of $p = k$ and $q = j$, leads to

$$\sum_{\ell=1}^{\infty}\sum_{m=1}^{n}\alpha_{\ell.m}\left[\left(\omega_{0_{\ell,m}}^2 - \omega_{0_{k,jq}}^2\right)\left(\mathbf{a}_{\ell,\mathbf{m}}\phi_\ell, \mathbf{a}_{\mathbf{p},\mathbf{q}}\phi_p\right)_W\right] =$$
$$\left(\mathbf{a}_{\mathbf{k},\mathbf{j}}\phi_k'', \mathbf{a}_{\mathbf{p},\mathbf{q}}\phi_p\right)_W + 2\omega_{0_{k,j}}\omega_{1_{k,j}}\left(\mathbf{Ma}_{\mathbf{k},\mathbf{j}}\phi_k, \mathbf{a}_{\mathbf{p},\mathbf{q}}\phi_p\right)_W. \qquad (31)$$

Due to mode shape orthogonality the only non-zero term on the left-hand side of Eq. (31) corresponds to $\ell = p$ and $m = q$ and the second-term on the right hand side is identically zero. Thus

$$\alpha_{p,q} = \frac{\left(\mathbf{a}_{\mathbf{k},\mathbf{j}}\phi_k'', \mathbf{a}_{\mathbf{p},\mathbf{q}}\phi_q\right)_W}{\omega_{0_{p,q}}^2 - \omega_{0_{k,j}}^2}. \qquad (32)$$

Note that $\alpha_{p,q} = 0$ when $k = p$. This implies that $\mathbf{u}_{\mathbf{1}_{\mathbf{k},\mathbf{i}}}$ is orthogonal to all zeroth-order modes within its own set of intramodal vectors.

B. Free ends

An exceptional case occurs when one end of each of the beams is free. If the ends at x=1 are free, the boundary conditions at these ends are

$$\frac{d^2u_i}{dx^2}(1) = 0, \qquad (33a)$$

and

$$\mu_i \frac{d^3 u_i}{dx^3}(1) - \varepsilon \frac{du_i}{dx}(1) = 0, \qquad (33b)$$

for $i = 1,2,...,n.$ However the boundary conditions at x=1 for the unstretched Euler-Bernoulli beams are $\frac{d^2 u_i}{dx^2}(1) = 0$ and $\frac{d^3 u_i}{dx^3}(1) = 0$. The zeroth-order problem is unchanged and if the beam is fixed at x=0, then $\omega_{o_{k,j}}$ and $\phi_{k,j}$ will be the natural frequencies and mode shapes for the fixed-free Euler-Bernoulli beam. However the boundary condition satisfied by $u_{i,j}$ is $\mu_i \frac{d^3 u_{1,i}}{dx^3}(1) = \frac{du_{0,1}}{dx}(1)$.

The analysis to determine $\omega_{1_{k,j}}$ for the case of the fixed-free beam begins with Eq. (16), but since $\mathbf{u_1}$ does not satisfy the same boundary conditions as $\mathbf{u_o}$, Eq. (15). Noting that

$$\left(\mathbf{K_b w}_{1_{k,j}} + \mathbf{K_c w}_{1_{k,j}} - \omega^2_{o_{k,j}} \mathbf{M w}_{1_{k,j}}, \mathbf{w_{0_{k,j}}}\right)_W = \int_0^1 \left(\mathbf{w_{0_{k,j}}}\right)^T \left[\left(\mathbf{K_b} + \mathbf{K_c}\right)\mathbf{w}_{1_{k,j}} - \omega^2_{o_{k,j}} \mathbf{M w}_{1_{k,j}}\right] dx$$

$$= \sum_{s=1}^{n} \int_0^1 \left(\mathbf{w_{0_{k,j}}}\right)_s \left[\mu_s \frac{d^4 \left(\mathbf{w}_{1_{k,j}}\right)_s}{dx^4} - \omega^2_{o_{k,j}} \beta_s \left(\mathbf{w}_{1_{k,j}}\right)_s \right] dx, \qquad (34)$$

application of integration by parts four times to the integral of Eq. (34) leads to

$$\left(\mathbf{K_b w}_{1_{k,j}} + \mathbf{K_c w}_{1_{k,j}} - \omega^2_{o_{k,j}} \mathbf{M w}_{1_{k,j}}, \mathbf{w_{0_{k,j}}}\right)_W =$$

$$\sum_{s=1}^{n} \left\{ \int_0^1 \left(\mathbf{w_{1_{k,j}}}\right)_s \left[\mu_s \frac{d^4 \left(\mathbf{w}_{0_{k,j}}\right)_s}{dx^4} - \omega^2_{o_{k,j}} \beta_s \left(\mathbf{w}_{0_{k,j}}\right)_s - \lambda_{s-1} \left(\mathbf{w}_{0_{k,j}}\right)_{s-1} + \left(\lambda_{s-1} + \lambda_s\right)\left(\mathbf{w}_{0_{k,j}}\right)_s - \lambda_s \left(\mathbf{w}_{0_{k,j}}\right)_{s+1} \right] dx \right.$$

$$\left. - \mu_s \frac{d\left(\mathbf{w}_{0_{k,j}}\right)_s}{dx}(1) \left(\mathbf{w}_{0_{k,j}}\right)_s (1) \right\}. \qquad (35)$$

The integrand on the right-hand side of Eq. (35) is identically zero leading to

$$\left(\mathbf{K_b w}_{1_{k,j}} + \mathbf{K_c w}_{1_{k,j}} - \omega^2_{o_{k,j}} \mathbf{M w}_{1_{k,j}}, \mathbf{w_{0_{k,j}}}\right)_W = -\left(\mathbf{a_{k,j}}\right)^T \Delta \mathbf{a_{k,j}} \phi_k'(1) \phi_k(1), \qquad (36)$$

where Δ is a diagonal matrix with $\Delta_{i,i} = \mu_i$. Substituting Eq. (36) into Eq. (35) and rearranging leads to

$$\omega_{1_{k,j}} = -\frac{1}{2\omega_{0_{k,j}}} \int_0^1 \phi_k''(x)\phi_k(x)dx - \frac{\left(\mathbf{a_{k,j}}\right)^T \Delta \mathbf{a_{k,j}}}{2\omega_{0_{k,j}} \left(\mathbf{a_{k,j}}\right)^T \mathbf{a_{k,j}}} \phi_k'(1)\phi_k(1). \qquad (37)$$

Since $\mathbf{w}_{\mathbf{1}_{k,j}}$ does not satisfy the same boundary conditions as $\mathbf{w}_{0_{k,j}}$ the expansion theorem cannot be directly used to determine the mode shape perturbations. However defining

$$\mathbf{v}_{k,j} = \mathbf{w}_{\mathbf{1_{k,j}}} + \frac{1}{12}\mathbf{w}'_{\mathbf{0_{k,j}}}(1)x^2(x-1)^3, \qquad (38)$$

leads to $\mathbf{v}_{k,j}(0)=0$ $\mathbf{v}'_{k,j}(0)=0$ $\mathbf{v}''_{k,j}(1)=0$ $\mathbf{v}'''_{k,j}(1)=0$. Thus, since $\mathbf{v}_{k,j}$ satisfies the same boundary conditions as $\mathbf{w}_{0_{k,j}}$, it can be expanded using the zeroth-order mode shapes

$$\mathbf{v}_{\mathbf{1_{k,j}}} = \sum_{\ell=1}^{\infty}\sum_{m=1}^{n} \mu_{\ell,m}\mathbf{a}_{\ell,m}\phi_\ell(x), \qquad (39)$$

where the forms of $\mu_{\ell,m}$ are determined by rewriting Eq.(32) using $\mathbf{v}_{\mathbf{1_{k,j}}}$ as dependent variables and using the procedure as before.

5 Lightly coupled beams

A General case

The set of identical beams considered in Section 3 is, of course, a special case of the more general problem. As evidenced by Eq. (8) the coupling stiffens the system and leads to increases in natural frequencies. However, if all beams are not identical, the uncoupled natural frequencies for the individual beams are different. Let ϕ_k be the smallest natural frequency of all of the beams for the kth mode. For this general case define $\eta = \frac{\kappa}{\phi_k^2}$. Asymptotic expansions for the general case are

$$\mathbf{u}(x) = \mathbf{u}_0(x) + \eta\mathbf{u}_1(x) + \cdots \qquad (40)$$

$$\omega = \omega_0 + \eta\omega_1 \ldots \qquad (41)$$

The resulting hierarchal equations are

O(1)

$$\mathbf{Ku_0} - \omega_0^2\mathbf{Mu_0} = \mathbf{0} \qquad (42)$$

and

O(η)

$$\mathbf{Ku}_1 - \omega_0^2\mathbf{Mu}_1 = -\hat{\mathbf{K}}_\mathbf{c}\mathbf{u_0} + 2\omega_1\omega_0\mathbf{Mu_0} \qquad (43)$$

The differential equations represented by Eq. (42) are uncoupled and of the form

$$\mu_i u_i^{iv} - \varepsilon u_i'' - \omega^2\beta_i u_i = 0 \qquad (44)$$

Equation (44) can be rewritten as

$$u_i^{iv} - \hat{\varepsilon} u_i'' - \hat{\omega}_i^2 u_i = 0 \qquad (45)$$

where

$$\hat{\varepsilon} = \frac{\varepsilon}{\mu_i} \qquad (46)$$

and

$$\hat{\omega}_i^2 = \frac{\beta_i}{\mu_i}\omega^2 \qquad (47)$$

Thus, for given end conditions the natural frequencies and mode shapes for each beam are obtained from the natural frequencies and mode shapes of a single axially loaded beam. Each beam has an infinite number of natural frequencies and corresponding mode shapes. In contrast with the notation of Section 3, the natural frequencies are written as $\omega_{o_{i,j}}$ $\mathrm{i} = 1,2,\ldots,\mathrm{n}$ $\ j = 1,2,\ldots$. Each natural frequency has a corresponding mode shape of

$$\mathbf{u}_{\mathbf{0}_{\mathbf{i,j}}} = \mathbf{a}_{\mathbf{i}}\Phi_{i,j}(x) \qquad (48)$$

where $\mathbf{a}_{\mathbf{i}}$ is a nx1 vector with $(\mathbf{a}_{\mathbf{i}})_k = \delta_{i,k}$, the Kronecker delta, and $\Phi_{i,j}(x)$ is the jth stretched beam mode shape corresponding to parameters used for the ith beam.

The first-order equation corresponding to a specific natural frequency becomes

$$\mathbf{K}\mathbf{u}_{1_{i,j}} - \omega_{0_{i,j}}^2 \mathbf{M}\mathbf{u}_{1_{i,j}} = -\hat{\mathbf{K}}_{\mathbf{c}}\mathbf{a}_{\mathbf{i}}\Phi_{i,j} + 2\omega_{0_{i,j}}\omega_{1_{i,j}}\mathbf{M}\mathbf{a}_{\mathbf{i}}\Phi_{i,j}(x) \qquad (49)$$

While the first-order equations are not coupled, the first-order mode shape for the frequency $\omega_{0_{i,j}}$ involves beams $i-1, i$ and $i+1$. Equation (49) represents a non-homogeneous system of equations for which a non-trivial solution of the corresponding homogeneous system exists. Thus a solution of Eq. (49) exists if and only if a solvability condition is satisfied. Two cases are considered: the case where no two beams are identical and the case where two or more beams are identical. In the former case all natural frequencies are distinct. For the latter case, if beams indexed as p and q are identical, when two beams are identical they have the same set of natural frequencies, but their mode shapes are different because $\mathbf{a}_{\mathbf{p}} \neq \mathbf{a}_{\mathbf{q}}$. This is a degenerate case which is treated separately.

For the case where no two beams are identical, taking the n-dimensional inner product with $\mathbf{u}_{\mathbf{0}_{\mathbf{i,j}}}$ and following the steps shown in Section 3 leads to the solvability condition

$$\omega_{1_{i,j}} = \frac{\kappa_{i-1} + \kappa_i}{2\omega_{0_{i,j}}} \qquad (50)$$

The mode shape perturbation is assumed to be of the form

$$\mathbf{u}_{\mathbf{1}i,\mathbf{j}} = \sum_{p=1}^{\infty}\sum_{q=1}^{n} \alpha_{p,q} \mathbf{a}_{\mathbf{q}} \Phi_{q,p}(x) \qquad (51)$$

Substitution of Eq. (51) into Eq. (43) leads to

$$\alpha_{p,q} = \frac{-\left(\mathbf{K}_{\mathbf{c}}\mathbf{a}_i \Phi_{i,j}(x), \mathbf{a}_{\mathbf{q}}\Phi_{q,p}(x)\right)}{\omega_{0,p,q}^2 - \omega_{0,i,j}^2} \qquad (52)$$

B. Degenerate case

For the degenerate case assume beams p and q are identical. The most general mode shape for the frequency $\omega_{o_{p,j}} = \omega_{o_{q,j}}$ is a linear combination of the linearly independent mode shapes

$$\mathbf{u}_0 = C_1 \mathbf{a}_{\mathbf{p}} \Phi_{p,j} + C_2 \mathbf{a}_{\mathbf{q}} \Phi_{q,j} \qquad (53)$$

Taking the n-dimensional inner product of Eq. (43) using i=p with the mode shape of Eq. (53) leads to

$$\left(\mathbf{K}\mathbf{u}_{1_{p,j}} - \omega_{0_{pi,j}}^2 \mathbf{M}\mathbf{u}_{1_{pi,j}}, C_1\mathbf{a}_{\mathbf{p}}\Phi_{p,j} + C_2\mathbf{a}_{\mathbf{q}}\Phi_{q,j}\right) =$$
$$-\left(\hat{\mathbf{K}}_{\mathbf{c}}\mathbf{a}_p\Phi_{p,j}, C_1\mathbf{a}_{\mathbf{p}}\Phi_{p,j} + C_2\mathbf{a}_{\mathbf{q}}\Phi_{q,j}\right) + 2\omega_{0_{p,j}}\omega_1\left(\mathbf{M}\mathbf{a}_p\Phi_{p,j}(x), C_1\mathbf{a}_{\mathbf{p}}\Phi_{p,j} + C_2\mathbf{a}_{\mathbf{q}}\Phi_{q,j}\right) \qquad (54)$$

Due to self-adjointness of the operator, the left-hand side of Eq. (54) is zero. Noting that $\mathbf{a}_q^{\mathbf{T}}\mathbf{a}_{\mathbf{p}} = 0$ and $\mathbf{a}_p^{\mathbf{T}}\mathbf{a}_{\mathbf{p}} = 1$, Eq. (54) leads to

$$\left(-\mathbf{a}_{\mathbf{P}}^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_{\mathbf{p}} + 2\omega_{0_{p,j}}\omega_1\right)C_1 + \left(-\mathbf{a}_{\mathbf{q}}^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_{\mathbf{p}}\right)C_2 = 0 \qquad (55)$$

Taking the n-dimensional inner product of Eq. (55) using i=q with the mode shape of Eq. (53) leads to

$$\left(-\mathbf{a}_{\mathbf{P}}^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_q\right)C_1 + \left(-\mathbf{a}_{\mathbf{q}}^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_q + 2\omega_{0_{p,j}}\omega_1\right)C_2 = 0 \qquad (56)$$

Equations (55) and (56) have a non-trivial solution only if

$$\begin{vmatrix} -\mathbf{a}_{\mathbf{P}}^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_{\mathbf{p}} + 2\omega_{0_{p,j}}\omega_1 & -\mathbf{a}_{\mathbf{q}}^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_{\mathbf{p}} \\ -\mathbf{a}_{\mathbf{P}}^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_q & -\mathbf{a}_{\mathbf{q}}^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_q + 2\omega_{0_{p,j}}\omega_1 \end{vmatrix} = 0 \qquad (57)$$

Evaluation of the determinant leads to

$$\omega_{1_{p,j}} = \frac{1}{4\omega_{0_{p,j}}}\left\{\mathbf{a}_{\mathbf{P}}^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_{\mathbf{p}} + \mathbf{a}_q^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_q \pm \sqrt{\left(\mathbf{a}_{\mathbf{P}}^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_{\mathbf{p}} - \mathbf{a}_q^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_q\right)^2 + 4\left(\mathbf{a}_{\mathbf{P}}^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_q\right)^2}\right\} \qquad (58)$$

If the beams are not adjacent then $\mathbf{a}_{\mathbf{P}}^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_q = 0$ and the natural frequency perturbations are calculated as in the non-degenerate case. Otherwise, assuming q=p+1, Eq. (58) becomes

$$\omega_{1_{p,j}} = \frac{1}{4\omega_{0_{p,j}}}\left\{\kappa_{p-1} + 2\kappa_p + \kappa_{p+1} \pm \sqrt{\left(\kappa_{p-1} - \kappa_{p+1}\right)^2 + 4\kappa_p^{\,2}}\right\} \qquad (59)$$

6 Examples

A. Lightly stretched pinned-pinned beams

Consider a set of n lightly stretched pinned-pinned beams. Using the notation of Section 3 the natural frequencies and mode shapes for the corresponding unstretched beams are

$$\omega_{0k,j} = \sqrt{k^4\pi^4 + \nu_j} \quad (61)$$

$$\mathbf{u_{ok,j}} = \sqrt{2}\mathbf{a_j} \sin(k\pi x) \quad (62)$$

The first-order perturbations for the natural frequencies are calculated from Eq. (28) as

$$\omega_{1k,j} = -\frac{1}{2\omega_{ok,j}}\int_0^1 \left[-k^2\pi^2\sqrt{2}\sin(k\pi x)\right]\left[\sqrt{2}\sin(k\pi x)\right]dx$$
$$= \frac{k^2\pi^2}{2\omega_{0k,j}} \quad (63)$$

Thus the first-order natural frequency approximations obtained from Eq. (14) are

$$\omega_{k,j} = \sqrt{k^4\pi^4 + \nu_j} + \frac{\varepsilon k^2\pi^2}{2\sqrt{k^4\pi^4 + \nu_j}} \quad (64)$$

Equation (64) is identical to the approximation obtained using a binomial expansion on Eq. (62), keeping only through linear terms in ε.

Application of Eq. (32) to determine the components of the mode shape perturbations leads to $\alpha_{p,q} = 0$ for all p and q. This, of course, is consistent in that the mode shapes for a set of stretched pinned-pinned beams are the same as the mode shapes for the corresponding unstretched beams.

B. Lightly coupled pinned-pinned beams

Now assume pinned-pinned beams of Section 6.1 are lightly coupled. The O(1) natural frequencies and mode shapes are

$$\omega_{0k,j} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2} \quad k = 1,2,\ldots \quad j = 1,2,\ldots,n \quad (65)$$

and

$$\mathbf{u_{0k,j}} = \sqrt{2}\mathbf{b_j} \sin(k\pi x) \quad (66)$$

where $\left(b_j\right)_\ell = \delta_{k,\ell}$. Since all beams are identical all natural frequencies are degenerate. Indeed each has n linearly independent mode shapes. In this case the appropriate form of Equation (48) is $\mathbf{u_0} = \sum_{j=1}^{n} C_j \sqrt{2}\mathbf{b}_j \sin(k\pi x)$ which results in a solvability condition of the form

$$\begin{vmatrix} -\mathbf{a}_1^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_1 + 2\omega_{0_{p,j}}\omega_1 & -\mathbf{a}_2^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_1 & -\mathbf{a}_3^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_1 & \cdots & -\mathbf{a}_n^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_1 \\ -\mathbf{a}_1^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_2 & -\mathbf{a}_2^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_2 + 2\omega_{0_{p,j}}\omega_1 & -\mathbf{a}_3^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_2 & \cdots & -\mathbf{a}_n^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_2 \\ -\mathbf{a}_1^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_3 & -\mathbf{a}_2^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_3 & -\mathbf{a}_3^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_3 + 2\omega_{0_{p,j}}\omega_1 & \cdots & -\mathbf{a}_n^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mathbf{a}_1^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_n & -\mathbf{a}_2^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_n & -\mathbf{a}_3^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_n & \cdots & -\mathbf{a}_n^{\mathbf{T}}\hat{\mathbf{K}}_{\mathbf{C}}\mathbf{a}_n + 2\omega_{0_{p,j}}\omega_1 \end{vmatrix} = 0 \quad (67)$$

Evaluation of the inner products leads to

$$\begin{vmatrix} -\kappa_0 - \kappa_1 + 2\omega_0\omega_1 & -\kappa_1 & 0 & \cdots & 0 \\ -\kappa_1 & -\kappa_1 - \kappa_2 + 2\omega_0\omega_1 & -\kappa_2 & \cdots & 0 \\ 0 & -\kappa_2 & -\kappa_2 - \kappa_3 + 2\omega_0\omega_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\kappa_{n-1} - \kappa_n + 2\omega_0\omega_1 \end{vmatrix} = 0 \qquad (68)$$

Equation (68) shows that $2\omega_0\omega_1$ must equal the eigenvalues of $\hat{\mathbf{K}}_{\mathbf{c}}$, $\hat{\nu}_j = \dfrac{\nu_j}{\eta}$.

The first-order expansion for the natural frequencies becomes

$$\omega_{k,j} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2} + \frac{\eta\hat{\nu}_j}{2\sqrt{k^4\pi^4 + \varepsilon k^2\pi^2}} + O\left(\eta^2\right) \qquad (69)$$

Equation (69) is identical to the equation obtained by substituting $\hat{\nu}_j = \dfrac{\nu_j}{\eta}$ in Eq. (60) and using a binomial expansion for $\eta\hat{\nu}_j < k^4\pi^4 + \varepsilon k^2\pi^2$, keeping only through the linear terms.

Specifically, consider three identical elastically coupled pinned-pinned axially loaded beams with a coupling matrix of the form

$$\mathbf{K}_{\mathbf{c}} = \eta\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad (70)$$

The exact natural frequencies are

$$\omega_{k,1} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2}, \qquad (71a)$$

$$\omega_{k,2} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2 + \eta} \qquad (71b)$$

and

$$\omega_{k,3} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2 + 3\eta}\ . \qquad (71c)$$

The corresponding mode shapes are

$$\mathbf{u}_{\mathbf{k},\mathbf{1}} = \sqrt{2}\begin{bmatrix}0.577\\0.577\\0.577\end{bmatrix}\sin(k\pi x)\ \mathbf{u}_{\mathbf{k},2} = \sqrt{2}\begin{bmatrix}-0.707\\0\\0.707\end{bmatrix}\sin(k\pi x)\ \mathbf{u}_{\mathbf{k},3} = \sqrt{2}\begin{bmatrix}0.409\\-0.816\\0.409\end{bmatrix}\sin(k\pi x) \quad (72)$$

The first-order approximations for the natural frequencies for a lightly stretched beam are

$$\omega_{k,1} = k^4\pi^4 + \frac{\varepsilon}{2} + O\left(\varepsilon^2\right) \quad (73a)$$

$$\omega_{k,2} = \sqrt{k^4\pi^4 + \eta} + \frac{\varepsilon k^2\pi^2}{2\sqrt{k^4\pi^4 + \eta}} + O\left(\varepsilon^2\right) \quad (73b)$$

and

$$\omega_{k,3} = \sqrt{k^4\pi^4 + 3\eta} + \frac{\varepsilon k^2\pi^2}{2\sqrt{k^4\pi^4 + 3\eta}} + O\left(\varepsilon^2\right). \quad (73c)$$

The first-order approximations for the natural frequencies of a lightly coupled beam are

$$\omega_{k,1} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2} + O\left(\eta^2\right), \quad (74a)$$

$$\omega_{k,2} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2} + \frac{\eta}{2\sqrt{k^4\pi^4 + \varepsilon k^2\pi^2}} + O\left(\eta^2\right) \quad (74b)$$

and

$$\omega_{k,3} = \sqrt{k^4\pi^4 + \varepsilon k^2\pi^2} + \frac{3\eta}{2\sqrt{k^4\pi^4 + \varepsilon k^2\pi^2}} + O\left(\eta^2\right). \quad (74c)$$

Parameters may be selected such that intermodal degenerate cases occur. For example if $\eta = 5\pi^4 + \varepsilon\pi^2$ then $\omega_{1,3} = \omega_{2,1}$. The general mode shape is a linear combination of both modes

$$\mathbf{u} = C_1\sqrt{2}\begin{bmatrix}0.577\\0.577\\0.577\end{bmatrix}\sin(k\pi x) + \mathrm{C}_2\sqrt{2}\begin{bmatrix}0.409\\-0.816\\0.409\end{bmatrix}\sin(2\pi x) \quad (75)$$

C. Non-identical lightly stretched fixed-fixed beams

Consider three lightly stretched fixed-fixed beams with $\mu_1 = 1, \mu_2 = 2.5, \mu_3 = 4, \beta_1 = 1, \beta_2 = 2, \beta_3 = 3, \lambda_0 = 0, \lambda_1 = 500, \lambda_2 = 200, \lambda_3 = 0$. The mode shapes for fixed-fixed beams are

$$\phi_k(x) = \cosh\left(\sqrt{\delta_k}\,x\right) - \cos\left(\sqrt{\delta_k}\,x\right) - \frac{\cosh\left(\sqrt{\delta_k}\right) - \cos\left(\sqrt{\delta_k}\right)}{\sinh\left(\sqrt{\delta_k}\right) - \sin\left(\sqrt{\delta_k}\right)}\left[\sinh\left(\sqrt{\delta_k}\,x\right) - \sin\left(\sqrt{\delta_k}\,x\right)\right] \quad (76)$$

where δ_k is the kth solution of $\cos\left(\sqrt{\delta_k}\right)\cosh\left(\sqrt{\delta_k}\right) = 1$. The set of intramodal natural frequencies and mode shapes corresponding to $\delta_3 = 120.9$ are

$\omega_{o3,1} = 122.81,\ \omega_{03,2} = 136.64,\ \omega_{03,3} = 139.94$

$$\mathbf{w}_{\mathbf{o3},\mathbf{1}} = \begin{bmatrix} 0.9951 \\ 0.0698 \\ 0.0013 \end{bmatrix}\phi_3(x)\ \ \mathbf{w}_{\mathbf{o3},2} = \begin{bmatrix} 0.0984 \\ -0.6992 \\ -0.0645 \end{bmatrix}\phi_3(x)\ \ \mathbf{w}_{\mathbf{o3},3} = \begin{bmatrix} 0.0088 \\ -0.0788 \\ 0.5737 \end{bmatrix}\phi_3(x) \quad (77)$$

Application of Eq. (28) leads to the first-order expansions for the natural frequencies as

$$\omega_{3,1} = 122.8 + 0.3998\varepsilon + \ldots\ , \quad (78a)$$

$$\omega_{3,2} = 136.64 + 0.3593\varepsilon + \ldots \quad (78b)$$

and

$$\omega_{3,3} = 139.94 + 0.3509\varepsilon + \ldots\ . \quad (78c)$$

7 Conclusion

Asymptotic expansions are obtained for the natural frequencies and mode shapes for a set of elastically connected axially loaded beams when the beams are lightly stretched and when the beams are lightly coupled.

The natural frequencies and mode shapes for the lightly stretched beams are expanded in terms of the non-dimensional parameter ε which is the ratio of the normal stress due to the axial load to the normal stress due to bending. The zeroth-order problem is that of unstretched elastically connected Euler-Bernoulli beams. Analogous to the case of identical axially loaded beams the natural frequencies and mode shapes can be organized into intramodal sets. A solvability condition is used to determine the first-order correction. If an end of the beam is free the boundary conditions for the first-order problem differ from those defining the zeroth-order problem. A special form of the solvability condition is applied in this exceptional case.

A study of the natural frequencies and mode shapes for a set of identical axially loaded elastically connected beams suggests the development of a non-dimensional parameter as the ratio of the largest eigenvalue of the coupling matrix to the square of the smallest natural frequency. The beams are lightly coupled when this parameter is small in which case an asymptotic solution is employed to approximate the natural frequencies and mode shapes. The zeroth order problem is that of uncoupled axially-loaded Euler-Bernoulli beams. A solvability condition is used to determine the first-order correction for the natural frequencies. A special case in which the some zeroth-order natural frequencies have multiple mode shapes occurs when some of the beams are identical.

The results are applied to the case of identical pinned-pinned beams for which an exact solution is available. The asymptotic expansions for natural frequencies of both the lightly stretched case and the lightly coupled case are identical to the exact natural frequencies when expanded to first-order. The method is also applied to lightly coupled non-identical fixed-free beams.

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