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Limiting Properties of the Doubling Algorithm for Solving the Discrete Time Riccati Equation

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Abstract

Doubling algorithm provides the steady state solution of the discrete time Riccati equation. In this paper it is shown that Doubling algorithm also provides the steady state solution of a related dual Riccati equation. This result is established for the Riccati equation emanating from the Kalman filter or the Lainiotis filter equations. This result is also confirmed for zero or non-zero initial conditions.

Keywords: Kalman filter, Lainiotis filter, Riccati equation, Doubling algorithm

1 Introduction

The discrete time Riccati equation arises in linear estimation, namely in the implementation of the discrete time Kalman filter [3] and the discrete time Lainiotis filter [5]. It is associated with time invariant systems described by the following state space equations for $k \geq 0$:

$$x(k+1) = Fx(k) + w(k) \tag{1}$$

$$z(k) = Hx(k) + v(k) \tag{2}$$

where $x(k)$ is the $nx1$ dimensional state vector at time k, $z(k)$ is the $mx1$ dimensional measurement vector, F is the system transition matrix, H is the output matrix, $\{w(k)\}$ and $\{v(k)\}$ are independent Gaussian zero-mean white and uncorrelated random processes, $Q(k)$ and $R(k)$ are the plant and measurement noise covariance matrices respectively, and $x(0)$ is a Gaussian random process with mean x_0 and covariance P_0.

The discrete time Kalman filter [3] and Lainiotis filter [5] are well known algorithms that solve the filtering problem: to calculate the estimate $x(L/L)$ and the estimation error covariance $P(L/L)$ at time L. Kalman filter computes the prediction $x(L+1/L)$ and the prediction error covariance $P(L+1/L)$ as well.

For time invariant systems, it is well known [1] that if the signal process model is asymptotically stable (i.e. all eigenvalues of F lie inside the unit circle), then there exist a steady state value P_e of the estimation error covariance matrix and a steady state value P_p of the prediction error covariance matrix.

Doubling algorithm is the most famous recursive algorithm for solving the discrete time Riccati equation. Doubling algorithm is associated with the discrete time Riccati equation in linear systems problems and provides its steady state solution through the limit of the prediction error covariance matrix or the estimation error covariance matrix.

In this paper the limits of all the parameters involved in the Doubling algorithm are determined. It is shown that the Doubling Algorithm also provides the steady state solution of a related dual Riccati equation. This result is established established for the Riccati equation emanating from the Kalman filter or the Lainiotis filter equations for zero or non-zero initial conditions. This result is confirmed through simulation results.

2 Riccati equations

The steady state prediction error covariance can be calculated by solving the discrete time Riccati equation emanating from Kalman filter (**REKF**):

$$P(k+1/k) = Q + F[I + P(k/k-1)H^T R^{-1} H]^{-1} P(k/k-1) F^T \quad (3)$$

with initial condition $P(0/-1) = P_0$.

In the zero initial condition case $P(0/-1) = 0$, REKF can be written as:

$$P(k+1/k) = Q + F[P^{-1}(k/k-1) + H^T R^{-1} H]^{-1} F^T \quad (4)$$

with initial condition $P(1/0) = Q$.

The steady state estimation error covariance matrix can be calculated by solving the discrete time Riccati equation emanating from Lainiotis filter (**RELF**):

$$P(k+1/k+1) = P_n + F_n[I + P(k/k)O_n]^{-1} P(k/k) F_n^T \quad (5)$$

with initial condition $P(0/0) = [I + P_0 H^T R^{-1} H]^{-1} P_0$, where

$A = [HQH^T + R]^{-1}$, $K_n = QH^T A$ and

$P_n = Q - K_n HQ$, $F_n = F - K_n HF$, $O_n = F^T H^T AHF$.

In the zero initial condition case $P(0/0) = 0$, RELF can be written as:

$$P(k+1/k+1) = P_n + F_n[P^{-1}(k/k) + O_n]^{-1} F_n^T \quad (6)$$

with initial condition $P(1/1) = P_n$.

Note that if $P(0/-1) = 0$ then $P(0/0) = 0$.

The steady state prediction and estimation error covariance matrices are connected through the following relationships:

$$P_e = [I + P_p H^T R^{-1} H]^{-1} P_p \quad (7)$$

$$P_p = Q + F P_e F^T \quad (8)$$

3 Doubling algorithms

The discrete time Riccati equation has attracted enormous attention. In view of the importance of the Riccati equation, there exists considerable literature both on its algebraic solutions as well as on its recursive solutions concerning per step or doubling algorithms [1], [2], [4], [6]. Doubling algorithm is the most famous recursive algorithm for solving the discrete time Riccati equation, due to the fact that it is numerically robust and converges fast to the steady state solution [1].
Doubling algorithm associated with the Riccati equation emanating from the Kalman filter (**DAREKF**) has as follows:

$$\begin{aligned}
a_{KF}(k+1) &= a_{KF}(k)\left[I + b_{KF}(k)c_{KF}(k)\right]^{-1} a_{KF}(k) \\
b_{KF}(k+1) &= b_{KF}(k) + a_{KF}(k)\left[I + b_{KF}(k)c_{KF}(k)\right]^{-1} b_{KF}(k)a_{KF}^{T}(k) \\
c_{KF}(k+1) &= c_{KF}(k) + a_{KF}^{T}(k)c_{KF}(k)\left[I + b_{KF}(k)c_{KF}(k)\right]^{-1} a_{KF}(k) \\
P_{KF}(k+1) &= c_{KF}(k) + a_{KF}^{T}(k)P_{KF}(k)\left[I + b_{KF}(k)P_{KF}(k)\right]^{-1} a_{KF}(k)
\end{aligned} \tag{9}$$

with initial conditions

$$\begin{aligned}
a_{KF}(1) &= F^{T} \\
b_{KF}(1) &= H^{T}R^{-1}H \\
c_{KF}(1) &= Q \\
P_{KF}(1) &= P(1/0) = Q + F[I + P_0 H^{T}R^{-1}H]^{-1}P_0 F^{T}
\end{aligned} \tag{10}$$

and limiting solution

$$\lim P_{KF}(k) = P_p \tag{11}$$

In the zero initial condition case we have:

$$c_{KF}(k) = P_{KF}(k) \tag{12}$$

Doubling algorithm associated with the Riccati equation emanating from the Lainiotis filter (**DARELF**) has as follows:

$$\begin{aligned}
a_{LF}(k+1) &= a_{LF}(k)\left[I + b_{LF}(k)c_{LF}(k)\right]^{-1} a_{LF}(k) \\
b_{LF}(k+1) &= b_{LF}(k) + a_{LF}(k)\left[I + b_{LF}(k)c_{LF}(k)\right]^{-1} b_{LF}(k)a_{LF}^{T}(k) \\
c_{LF}(k+1) &= c_{LF}(k) + a_{LF}^{T}(k)c_{LF}(k)\left[I + b_{LF}(k)c_{LF}(k)\right]^{-1} a_{LF}(k) \\
P_{LF}(k+1) &= c_{LF}(k) + a_{LF}^{T}(k)P_{LF}(k)\left[I + b_{LF}(k)P_{LF}(k)\right]^{-1} a_{LF}(k)
\end{aligned} \tag{13}$$

with initial conditions

$$\begin{aligned}
a_{LF}(1) &= F_n^{T} \\
b_{LF}(1) &= O_n \\
c_{LF}(1) &= P_n \\
P_{LF}(1) &= P(1/1) = P_n + F_n[I + P_0 O_n]^{-1}P_0 F_n^{T}
\end{aligned} \tag{14}$$

and limiting solution

$$\lim P_{LF}(k) = P_e \tag{15}$$

In the zero initial condition case we have:

$$c_{LF}(k)=P_{LF}(k) \tag{16}$$

4 Dual Riccati equations

Consider the following dual Riccati equation for REKF (**dual REKF**):

$$\Pi(k+1/k)=H^T R^{-1} H + F^T [I+\Pi(k/k-1)Q]^{-1}\Pi(k/k-1)F \tag{17}$$

with initial condition $\Pi(0/-1)=\Pi_0$.

In the zero initial condition case $\Pi(0/-1)=0$, dual REKF can be written as:

$$\Pi(k+1/k)=H^T R^{-1} H + F^T [\Pi^{-1}(k/k-1)+Q]^{-1} F \tag{18}$$

with initial condition $\Pi(1/0)=H^T R^{-1} H$.

Due to the fact that the matrices F and F^T have the same eigenvalues, all eigenvalues of F^T lie inside the unit circle; thus there exists a unique steady state or limiting solution Π_p of the dual REKF.

Consider the following dual Riccati equation for RELF (**dual RELF**):

$$\Pi(k+1/k+1)=O_n + F_n^T [I+\Pi(k/k)P_n]^{-1}\Pi(k/k)F_n \tag{19}$$

with initial condition $\Pi(0/0)=[I+\Pi_0 O_n]^{-1}\Pi_0$.

In the zero initial condition case $\Pi(0/0)=0$, RELF can be written as:

$$\Pi(k+1/k+1)=O_n + F_n^T [\Pi^{-1}(k/k)+P_n]^{-1} F_n \tag{20}$$

with initial condition $\Pi(1/1)=O_n$.

There exists a unique steady state or limiting solution Π_e of the dual RELF.

5 Dual Doubling algorithms

Dual Doubling algorithm associated with the dual Riccati equation emanating from the Kalman filter (**dual DAREKF**) has as follows:

$$\begin{aligned}
\alpha_{KF}(k+1) &= \alpha_{KF}(k)\left[I+\beta_{KF}(k)\gamma_{KF}(k)\right]^{-1}\alpha_{KF}(k)\\
\beta_{KF}(k+1) &= \beta_{KF}(k)+\alpha_{KF}(k)\left[I+\beta_{KF}(k)\gamma_{KF}(k)\right]^{-1}\beta_{KF}(k)\alpha_{KF}^T(k)\\
\gamma_{KF}(k+1) &= \gamma_{KF}(k)+\alpha_{KF}^T(k)\gamma_{KF}(k)\left[I+\beta_{KF}(k)\gamma_{KF}(k)\right]^{-1}\alpha_{KF}(k)\\
\Pi_{KF}(k+1) &= \gamma_{KF}(k)+\alpha_{KF}^T(k)\gamma_{KF}(k)\left[I+\beta_{KF}(k)\Pi_{KF}(k)\right]^{-1}\alpha_{KF}(k)
\end{aligned} \tag{21}$$

with initial conditions

$$\begin{aligned}
\alpha_{KF}(1) &= F\\
\beta_{KF}(1) &= Q\\
\gamma_{KF}(1) &= H^T R^{-1} H\\
\Pi_{KF}(1) &= \Pi(1/0)=H^T R^{-1} H + F^T[I+\Pi_0 Q]^{-1}\Pi_0 F
\end{aligned} \tag{22}$$

and limiting solution

$$\lim \Pi_{KF}(k) = \Pi_p \tag{23}$$

In the zero initial condition case we have:

$$\gamma_{KF}(k) = \Pi_{KF}(k) \tag{24}$$

Dual Doubling algorithm associated with the dual Riccati equation emanating from the Lainiotis filter (**dual DARELF**) has as follows:

$$\begin{aligned}
\alpha_{LF}(k+1) &= \alpha_{LF}(k)\left[I+\beta_{LF}(k)\gamma_{LF}(k)\right]^{-1}\alpha_{LF}(k)\\
\beta_{LF}(k+1) &= \beta_{LF}(k)+\alpha_{LF}(k)\left[I+\beta_{LF}(k)\gamma_{LF}(k)\right]^{-1}\beta L_{LF}(k)\alpha_{LF}^{T}(k)\\
\gamma_{LF}(k+1) &= \gamma_{LF}(k)+\alpha_{LF}^{T}(k)\gamma_{LF}(k)\left[I+\beta_{LF}(k)\gamma_{LF}(k)\right]^{-1}\alpha_{LF}(k)\\
\Pi_{LF}(k+1) &= \gamma_{LF}(k)+\alpha_{LF}^{T}(k)\gamma_{LF}(k)\left[I+\beta_{LF}(k)\Pi_{LF}(k)\right]^{-1}\alpha_{LF}(k)
\end{aligned} \tag{25}$$

with initial conditions

$$\begin{aligned}
\alpha_{LF}(1) &= F_n\\
\beta_{LF}(1) &= P_n\\
\gamma_{LF}(1) &= O_n\\
\Pi_{LF}(1) &= \Pi(1/1) = O_n + F_n^T[I+\Pi_0 P_n]^{-1}\Pi_0 F_n
\end{aligned} \tag{26}$$

and limiting solution

$$\lim \Pi_{LF}(k) = \Pi_e \tag{27}$$

In the zero initial condition case we have:

$$\gamma_{LF}(k) = \Pi_{LF}(k) \tag{28}$$

6 Limits of Doubling algorithms parameters

The Doubling algorithm and the dual Doubling algorithm associated with the Riccati equation emanating from the Kalman filter are connected with the following interesting relations:

$$\begin{aligned}
\alpha_{KF}(k) &= a_{KF}^{T}(k)\\
\beta_{KF}(k) &= c_{KF}(k)\\
\gamma_{KF}(k) &= b_{KF}(k)
\end{aligned} \tag{29}$$

Proof. (by induction)

From (10) and (22) it is obvious that equations (29) hold for $k=1$.

Let that equations (29) hold for $k=k$.

Then, form (9) and (21) for $k=k+1$ we have:

$$\begin{aligned}
\alpha_{KF}(k+1) &= \alpha_{KF}(k)\left[I+\beta_{KF}(k)\gamma_{KF}(k)\right]^{-1}\alpha_{KF}(k)\\
&= a_{KF}^{T}(k)\left[I+c_{KF}(k)b_{KF}(k)\right]^{-1}a_{KF}^{T}(k)\\
&= a_{KF}^{T}(k)\left[I+b_{KF}^{T}(k)c_{KF}^{T}(k)\right]^{-T}a_{KF}^{T}(k)\\
&= [a_{KF}(k)\left[I+b_{KF}(k)c_{KF}(k)\right]^{-1}a_{KF}(k)]^{T} = a_{KF}^{T}(k+1)
\end{aligned}$$

$$\beta_{KF}(k+1)=\beta_{KF}(k)+\alpha_{KF}(k)\left[I+\beta_{KF}(k)\gamma_{KF}(k)\right]^{-1}\beta_{KF}(k)\alpha_{KF}^{T}(k)=$$
$$=\beta_{KF}(k)+\alpha_{KF}(k)\left[\beta_{KF}(k)^{-1}+\gamma_{KF}(k)\right]^{-1}\alpha_{KF}^{T}(k)=$$
$$=c_{KF}(k)+a_{KF}^{T}(k)\left[c_{KF}(k)^{-1}+b_{KF}(k)\right]^{-1}a_{KF}(k)=$$
$$=c_{KF}(k)+a_{KF}^{T}(k)c_{KF}(k)\left[I+b_{KF}(k)c_{KF}(k)\right]^{-1}a_{KF}(k)=c_{KF}(k+1)$$

and

$$\gamma_{KF}(k+1)=\gamma_{KF}(k)+\alpha_{KF}^{T}(k)\gamma_{KF}(k)\left[I+\beta_{KF}(k)\gamma_{KF}(k)\right]^{-1}\alpha_{KF}(k)=$$
$$=\gamma_{KF}(k)+\alpha_{KF}^{T}(k)\left[\gamma_{KF}(k)^{-1}+\beta_{KF}(k)\right]^{-1}\alpha_{KF}(k)=$$
$$=b_{KF}(k)+a_{KF}(k)\left[b_{KF}(k)^{-1}+c_{KF}(k)\right]^{-1}a_{KF}^{T}(k)=$$
$$=b_{KF}(k)+a_{KF}(k)\left[I+b_{KF}(k)c_{KF}(k)\right]^{-1}b_{KF}(k)a_{KF}^{T}(k)=b_{KF}(k+1)$$

Then, the limits of Doubling algorithms parameters are determined:

$$\lim a_{KF}(k)=0,\ \ \lim b_{KF}(k)=\Pi_p,\ \ \lim c_{KF}(k)=P_p,\ \ \lim P_{KF}(k)=P_p \tag{30}$$

and

$$\lim \alpha_{KF}(k)=0,\ \ \lim \beta_{KF}(k)=P_p,\ \ \lim \gamma_{KF}(k)=\Pi_p,\ \ \lim \Pi_{KF}(k)=\Pi_p \tag{31}$$

Proof.

Due to the fact that all eigenvalues of F lie inside the unit circle, we have: $\lim a_{KF}(k)=0$ and $\lim \alpha_{KF}(k)=0$.

Then, from the Doubling algorithm equations and the dual Doubling algorithm equations, we have: $\lim c_{KF}(k)=\lim P_{KF}(k)=P_p$ and $\lim \gamma_{KF}(k)=\lim \Pi_{KF}(k)=\Pi_p$.

Then from (29) we derive $\lim b_{KF}(k)=\Pi_p$ and $\lim \beta_{KF}(k)=P_p$.

It becomes obvious that the Doubling algorithm and the dual Doubling algorithm associated with the Riccati equation emanating from the Lainiotis filter are connected with the following interesting relations:

$$\alpha_{LF}(k)=a_{LF}^{T}(k),\ \ \beta_{LF}(k)=c_{LF}(k),\ \ \gamma_{LF}(k)=b_{LF}(k) \tag{32}$$

Then, the limits of Doubling algorithms parameters are determined:

$$\lim a_{LF}(k)=0,\ \ \lim b_{LF}(k)=\Pi_e,\ \ \lim c_{LF}(k)=P_e,\ \ \lim P_{LF}(k)=P_e \tag{33}$$

and

$$\lim \alpha_{LF}(k)=0,\ \ \lim \beta_{LF}(k)=P_e,\ \ \lim \gamma_{LF}(k)=\Pi_e,\ \ \lim \Pi_{LF}(k)=\Pi_e \tag{34}$$

In the zero initial conditions case, the same relations hold.

7 Simulation results

The limits of Doubling algorithms are verified through the following simulation example: a simple scalar time invariant model is assumed with $n=1$ and $m=1$, where $F=0.8$, $H=4$, $Q=2$ and $R=10$.

Using DAREKF and dual DAREKF we compute the limits of Doubling algorithms parameters:

$$\lim a_{KF}(k) = \lim \alpha_{KF}(k) = 0$$

$$\lim b_{KF}(k) = \lim \gamma_{KF}(k) = \lim \Pi_{KF}(k) = \Pi_p = 1.8250$$

$$\lim c_{KF}(k) = \lim P_{KF}(k) = \lim \beta_{KF}(k) = P_p = 2.3150$$

Using DARELF and dual DAREKF we compute the limits of Doubling algorithms parameters:

$$\lim a_{LF}(k) = \lim \alpha_{LF}(k) = 0$$

$$\lim b_{LF}(k) = \lim \gamma_{LF}(k) = \lim \Pi_{LF}(k) = \Pi_e = 0.2520$$

$$\lim c_{LF}(k) = \lim P_{LF}(k) = \lim \beta_{LF}(k) = P_e = 0.4921$$

Observe that the steady state prediction and estimation error covariance matrices satisfy relations (7) and (8).

8 Conclusions

Doubling algorithm provides the steady state solution of the discrete time Riccati equation as well as the steady state solution of a related dual Riccati equation. This result is established for the Riccati equation emanating from the Kalman filter or the Lainiotis filter equations and is confirmed for zero as well as for non-zero initial conditions.

The relations of the parameters of the Doubling algorithm and the dual Doubling algorithm are derived. The limits of Doubling algorithms parameters are determined.

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