Contemporary Engineering Sciences, Vol. 2, 2009, no. 9, 433 - 440

Elastic-Plastic Transition of Transversely Isotropic Thin Rotating Disc

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Abstract

Elastic-plastic stresses have been derived by using Seth's transition theory. Results obtained have been discussed numerically and depicted graphically. Thin rotating disc made of transversely isotropic material yields at a higher angular speed as compared to disc made of isotropic material. Rotating disc made of isotropic material required high percentage increase in angular speed to become fully plastic from its initial yielding as compared to disc made of transversely isotropic material. Rotating disc made of transversely isotropic material is on the safer side of design as compared to rotating disc made of isotropic material.

Mathematics Subject Classification: 74C20, 74C99

Keywords: Elastic-plastic, Transition, Transversely Isotropic, Thin Rotating Disc.

1. Introduction

Disc plays an important role in machine design. Stress analysis of rotating discs has an important role in engineering design. Rotating discs are the most critical part of rotors, turbines, flywheel etc. The problem of thin rotating flat discs made of isotropic material has been studied extensively [1-3]. Chakrabarty [1] and Heyman [2] solved the problem for the plastic state by utilizing the solution in the elastic state and consider the plastic range with the help of Tresca's yield condition. Further, to obtain the elastic-plastic stresses, these authors matched the elastic and plastic stresses at the same radius r = c of the disc. Perfectly elasticity and ideal plasticity are two extreme properties of the material and the use of ad-hoc rule like yield condition amounts to divide the two extreme properties by a sharp line, which is not physically possible. The transition theory does not require these assumptions and thus solves a more general problem from which cases pertaining to above assumption can be worked out. The transition theory utilizes the concept of generalized principal strain measure and asymptotic solution at critical points of the differential equation defining the deformed field and has been successfully applied to a large number of problems [4-9]. The generalized principal strain measure [4] is defined as

$$e_{ii} = \int_0^{e_{ii}^A} [1-2e_{ii}^A]^{\frac{n}{2}-1} de_{ij}^A = \frac{1}{n}[1-(1-2e_{ii}^A)^{\frac{n}{2}}], \qquad (i, j = 1,2,3) \tag{1.1}$$

where n is the measure and e_{ij}^A is the principal Almansi finite strain components.

In this paper an attempt has been made to study the behavior of transversely isotropic thin rotating disc using transition theory.

2. Governing Equations

We consider a thin disc of constant density made of transversely isotropic material with internal and external radii 'a' and 'b' respectively. The disc is rotating with angular velocity 'ω' about an axis perpendicular to its plane and passing through the centre of the disc. The disc is thin and is effectively in a state of plane stress. The displacement components in cylindrical co-ordinates are given [4-9] by,

$$u = r(1-\beta), \; v = 0, \; w = dz \tag{2.1}$$

where β is a function of $r = \sqrt{x^2 + y^2}$ only and d is a constant.

The finite components of strain [4-9] are given as,

$$\begin{aligned} e_{rr}^A &= (1/2)[1-(r\beta' + \beta)^2], \\ e_{\theta\theta}^A &= (1/2)[1-\beta^2], \\ e_{zz}^A &= (1/2)[1-(1-d)^2], \\ e_{r\theta}^A &= e_{\theta z}^A = e_{zr}^A = 0 \end{aligned} \tag{2.2}$$

where $\beta' = d\beta / dr$

Substituting equation (2.2) in equation (1.1), we get the generalized components of strain as

$$\begin{aligned} e_{rr} &= (1/n)[1-(r\beta' + \beta)^n], \\ e_{\theta\theta} &= (1/n)[1-\beta^n], \\ e_{zz} &= (1/n)[1-(1-d)^n], \\ e_{r\theta} &= e_{\theta z} = e_{zr} = 0 \end{aligned} \tag{2.3}$$

The stress-strain relations for transversely isotropic material are

$$\begin{aligned} T_{rr} &= C_{11}e_{rr} + (C_{11} - 2C_{66})e_{\theta\theta} + C_{13}e_{zz}, \\ T_{\theta\theta} &= (C_{11} - 2C_{66})e_{rr} + C_{11}e_{\theta\theta} + C_{13}e_{zz} \\ T_{zz} &= C_{13}e_{rr} + C_{13}e_{\theta\theta} + C_{33}e_{zz} = 0 \end{aligned}$$

$$e_{zz} = -(C_{13}/C_{33})e_{rr} - (C_{13}/C_{33})e_{\theta\theta} \quad ; \quad T_{zr} = T_{\theta z} = T_{r\theta} = 0 \qquad (2.4)$$

where C_{ij}'s are material constants.

Using equation (2.3) in equation (2.4) we have

$$T_{rr} = (A/n)\left[2 - \beta^n\{1 + (1+P)^n\}\right] - (2C_{66}/n)(1-\beta^n)$$
$$T_{\theta\theta} = (A/n)\left[2 - \beta^n\{1 + (1+P)^n\}\right] - (2C_{66}/n)(1-\beta^n(1+P)^n) \qquad (2.5)$$
$$T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0$$

where $A = (C_{11} - (C_{13}^2/C_{33}))$.

The equations of equilibrium are all satisfied except

$$\frac{d}{dr}(rT_{rr}) - T_{\theta\theta} + \rho\omega^2 r^2 = 0 \qquad (2.6)$$

where ρ is the density of the material.

Using equation (2.5) in equation (2.6) we get a non-linear differential equation in β as,

$$\beta^{n+1}P(1+P)^{n-1} = \left[\frac{\rho\omega^2 r^2}{A} + \beta^n\left\{\frac{2C_{66}}{nA}(1 + nP - (1+P)^n) - P(1 + (1+P)^n)\right\}\right]\frac{d\beta}{dP} \qquad (2.7)$$

where $r\beta' = \beta P$.

The transitional points of β in equation (2.7) are $P = -1$ *and* $P = \pm\infty$.

The boundary conditions are

$$T_{rr} = 0 \quad \text{at} \quad r = a \ \text{ and } r = b. \qquad (2.8)$$

3. Solution of Problem

It has been shown [4-7] that the asymptotic solution through the principal stress leads from elastic to plastic state at the transition point $P \to \pm\infty$.

We define the transition function 'R' as,

$$R = T_{\theta\theta} = (A/n)\left[2 - \beta^n\{1 + (1+P)^n\}\right] - (2C_{66}/n)\left[1 - \beta^n(1+P)^n\right] \qquad (3.1)$$

Taking the logarithmic differentiation of equation (3.1) w.r.t. 'r', we get

$$\frac{d}{dr}(\log R) = \frac{-A}{Rr}\left[\beta^n P\{1 + (1+P)^n\} + \beta^{n+1}P(1+P)^{n-1}\frac{dP}{d\beta}\right] +$$
$$\frac{2C_{66}}{Rr}\left[\beta^n P(1+P)^n + \beta^{n+1}P(1+P)^{n-1}\frac{dP}{d\beta}\right] \qquad (3.2)$$

Substituting the value of $dP/d\beta$ from equation (2.7) in equation (3.2) and taking asymptotic value as $P \to \pm\infty$, we get after integration,

$$R = A_1 r^{-C_2} \qquad (3.3)$$

where A_1 is constant of integration and $C_2 = 2C_{66}/A$.

Using equation (3.3) in equation (3.1), we have

$$T_{\theta\theta} = A_1 r^{-C_2} \qquad (3.4)$$

Substituting equation (3.4) in equation (2.6), we get after integration

$$T_{rr} = \frac{A_2}{r} + A_1 \frac{r^{-C_2}}{-C_2+1} - \rho\omega^2 \frac{r^2}{3} \tag{3.5}$$

where A_2 is constant of integration.

Using boundary condition (2.8) in equation (3.5), we get

$$A_1 = \frac{\rho\omega^2}{3}\frac{(a^3-b^3)(-C_2+1)}{(a^{-C_2+1}-b^{-C_2+1})}; \qquad A_2 = \frac{\rho\omega^2}{3}\frac{(b^3a^{-C_2+1}-a^3b^{-C_2+1})}{\left(a^{-C_2+1}-b^{-C_2+1}\right)} \tag{3.6}$$

Substituting the values of A_1 and A_2 in equations (3.4) and (3.5), we get

$$T_{rr} = \frac{\rho\omega^2/3}{a^{-C_2+1}-b^{-C_2+1}}\left[\frac{(b^3a^{-C_2+1}-a^3b^{-C_2+1})}{r} + (a^3-b^3)r^{-C_2} - r^2(a^{-C_2+1}-b^{-C_2+1})\right]$$

$$T_{\theta\theta} = \frac{\rho\omega^2}{3}\frac{(a^3-b^3)(-C_2+1)}{a^{-C_2+1}-b^{-C_2+1}} r^{-C_2} \tag{3.7}$$

It is found from equation (3.7) that $T_{\theta\theta}$ is maximum at the internal surface (r = a), therefore yielding of the disc will take place at the internal surface, we have

$$\left|T_{\theta\theta}\right|_{r=a} = \left|\frac{\rho\omega^2}{3}\frac{(a^3-b^3)(-C_2+1)}{\left(a^{-C_2+1}-b^{-C_2+1}\right)} a^{-C_2}\right| \equiv Y \quad (say) \tag{3.8}$$

The angular speed required for initial yielding is given by

$$\Omega_i^2 = \frac{\rho\omega_i^2 b^2}{Y} = \frac{3(R_0^{-C_2+1}-1)}{(R_0^3-1)(-C_2+1)} R_0^{C_2} \tag{3.9}$$

And transitional stresses (3.7) become

$$\sigma_r = \frac{T_{rr}}{Y} = \frac{\Omega_i^2}{3(R_0^{-C_2+1}-1)}\left[R_0^3R^{-1}(R_0^{-C_2-2}-1) + (R_0^3-1)R^{-C_2} - R^2(R_0^{-C_2+1}-1)\right]$$

$$\sigma_\theta = \frac{T_{\theta\theta}}{Y} = \frac{\Omega_i^2}{3}\frac{(R_0^3-1)(-C_2+1)}{(R_0^{-C_2+1}-1)} R^{-C_2} \tag{3.10}$$

where $R_0 = (a/b)$ *and* $R = (r/b)$.

If the speed of rotation is further increased, the yielding in the disc will spread gradually; the disc is rendered more and more plastic and finally become fully plastic at some value Ω_f^2 of the angular speed. For fully plastic state i.e. $C_2 \to 0$, equation (3.7) become

$$\left|T_{\theta\theta}\right|_{r=b} = \left|\frac{\rho\omega_f^2}{3}\frac{(a^3-b^3)}{(a-b)}\right| \equiv Y^*(say) \tag{3.11}$$

and the angular velocity Ω_f^2 required for fully plastic is given by

$$\Omega_f^2 = \frac{\rho\omega_f^2 b^2}{Y^*} = \frac{3(R_0-1)}{(R_0^3-1)} \tag{3.12}$$

and stresses (3.7) for fully plastic state become,

$$\sigma_r = \frac{T_{rr}}{Y^*} \equiv \frac{\Omega_f^2}{3}\left[(R_0^2 + R_0 + 1) - \frac{R_0(R_0+1)}{R} - R^2\right]$$
$$\sigma_\theta = \frac{T_{\theta\theta}}{Y^*} \equiv \frac{\Omega_f^2}{3}(R_0^2 + R_0 + 1) \tag{3.13}$$

4. Isotropic Material

Elastic-plastic transitional stresses (3.10) for isotropic material become,

$$\sigma_r = \frac{\Omega_i^2}{3R}\left[\frac{(1-R_0^3)(R^{C_1} - R_0^{C_1})}{(1-R_0^{C_1})} - (R^3 - R_0^3)\right]$$
$$\sigma_\theta = \frac{\Omega_i^2 C_1}{3}\left[\frac{(1-R_0^3)}{(1-R_0^{C_1})}\right]R^{-\frac{1}{2-C}} \tag{4.1}$$

where $C_1 = (1-C)/(2-C)$.

and angular speed required for initial yielding is given by,

$$\Omega_i^2 = \frac{3}{C_1}\left[\frac{R_0^{C_1} - 1}{R_0^3 - 1}\right]R_0^{\frac{1}{2-C}} \tag{4.2}$$

For fully plastic state $(C \to 0)$ equations (4.1) become,

$$\sigma_r = \frac{\Omega_f^2 R^{-1}}{3}\left[\frac{(1-R_0^3)}{(1-R_0^{1/2})}\left\{(R^{1/2} - R_0^{1/2}) - (R^3 - R_0^3)\right\}\right]$$
$$\sigma_\theta = \frac{\Omega_f^2 R^{-1/2}}{6}\left[\frac{1-R_0^3}{1-R_0^{1/2}}\right] \tag{4.3}$$

and angular speed required for fully plastic state is given by,

$$\Omega_f^2 = 6\left[\frac{1-R_0^{1/2}}{1-R_0^3}\right] \tag{4.4}$$

Equation (4.3) and equation (4.4) are same as obtained by Gupta and Shukla [6].

5. Numerical Illustration

As a numerical example, elastic constants C_{ij} have been given in table 1 for isotropic material (Brass $\sigma = 0.33$) and transversely isotropic materials (Mg and Beryl). In figure 1, curves have been drawn between angular speed Ω_i^2 required for initial yielding at the internal surface for various radii ratios. It has been observed that rotating disc made of transversely isotropic material require higher angular speed to yield as compared to disc made of isotropic material. Rotating disc made of transversely isotropic material (Mg) yields at higher angular speed $\left(\Omega_i^2 = 1.3969\right)$ for $R_0 = 0.35$ as compared to other values of radii ratios, while in case of isotropic material rotating disc yields at higher angular speed $\left(\Omega_i^2 = 1.3202\right)$ for radii ratio $R_0 = 0.4$ as compared to other values of radii ratios $\left(R_0 = a/b\right)$(see table 2). It means that rotating disc made of transversely isotropic material requires lesser radii ratio to yield at the internal surface as compared to rotating disc made of isotropic material. Rotating disc made of isotropic material require high percentage increase in angular speed to become fully plastic as compared to rotating disc made of transversely isotropic material (see table 3). In figure 2, curves have been drawn between stresses and radii ratio $\left(R = r/b\right)$. It is observed that circumferential stress for isotropic material is maximum at the internal surface as compared to transversely isotropic material. Therefore, rotating disc made of transversely isotropic material is on the safer side of the design as compared to rotating disc made of isotropic material.

Table 1: Elastic constants C_{ij} used (in units of 10^{10} N/m^2)

Materials	C_{44}	C_{11}	C_{12}	C_{13}	C_{33}
Brass (Isotropic Material)	0.999	3.0	1.0	1.0	3.0
Magnesium(Transversely Isotropic Material)	1.64	5.97	2.62	2.17	6.17
Beryl (Transversely Isotropic Material)	0.883	2.746	0.98	0.67	4.69

Table 2: Angular speed required for initial yielding for various radii ratios

	Brass	**Magnesium**	**Beryl**
R_0 =a / b	Ω_i^2	Ω_i^2	Ω_i^2
0.25	1.2624	1.3662	1.3553
0.3	1.2994	1.3909	1.3814
0.35	1.3170	1.3969	1.3886
0.4	1.3202	1.38934	1.3822
0.45	1.3127	1.3722	1.3660
0.5	1.2974	1.3480	1.3928

Table 3: Angular speed required for initial yielding and fully plastic state

		Ω_i^2	Ω_f^2	P %
$0.5 \le R \le 1$	Magnesium	1.3480	1.7143	12.77
	Beryl	1.3284	1.7143	13.59
	Brass	1.2983	2.0057	24.29

where P = $\left[\left(\sqrt{\Omega_f^2/\Omega_i^2}\right)-1\right] x\,100$ is the percentage increase in angular speed Ω_i^2 from initial yielding to fully plastic state Ω_f^2 .

Acknowledgment: The author wishes to acknowledge his sincere thanks to Prof. S.K. Gupta for his encouragement during the preparation of this paper.

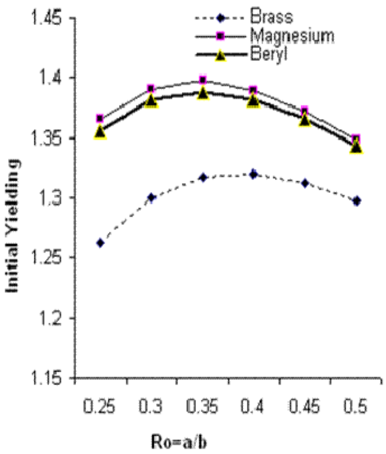

Fig. 1. Angular speed required for initial yielding for various radii ratios R_0 = a/b

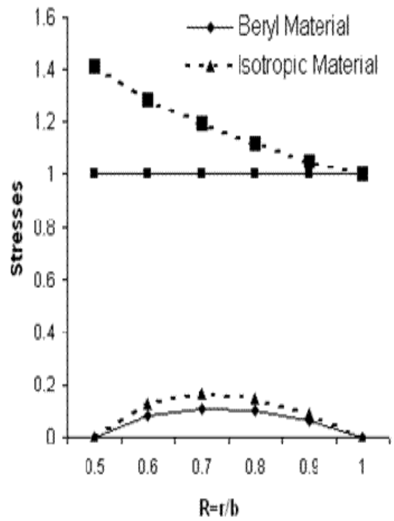

Fig. 2. Stresses for fully plastic state with respect to radii ratio R = r/b

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Received: February, 2009