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Flow of a Couple Stress Fluid Generated by a Circular Cylinder Subjected To Longitudinal and Torsional Oscillations

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Abstract

The flow generated by a circular cylinder, performing longitudinal and torsional oscillations, in an infinite expanse of couple stress fluid is studied. Analytical expressions for the velocity components are obtained under vanishing couple stresses of type A condition or super adherence condition of type B on the boundary. The effects of couple stress parameter, Reynolds number and the ratio of couple stress viscosities parameter on transverse and axial velocity components are presented graphically. The drag force acting on the wall of the cylinder is derived and effects of Couple stress parameters on drag are shown graphically.

Keywords: Couple-stress fluid, Longitudinal and Torsional Oscillations, Drag.

1. Introduction

The theory of couple stresses in fluids, developed by Stokes [17] in 1966, represents the simplest generalization of the classical theory and allows for polar effects such as the presence of couple stresses and body couples. The couple stress theory of fluids defines the rotation field in terms of the velocity field. In fact the rotation vector is equal to one-half the curl of the velocity vector as in the case with

Newtonian fluids. Second order gradient of the velocity vector, rather than the kinematically independent rotation vector of asymmetric hydromechanics, is introduced into the stress constitutive equations and consequently the theory yields only one vector equation to describe the velocity field.

The consideration of couple-stress, in addition to the classical Cauchy stress, has led to the recent development of theories of fluid microcontinua. This new branch of fluid mechanics has attracted a growing interest during past few decades mainly because it possesses the mechanism to describe such rheologically complex fluids as liquid crystals, polymeric suspensions and animal blood for which the Navier-Stokes' theory is inadequate.

The flow of a fluid due to a cylindrical rod oscillating with longitudinal and torsional motion has received considerable attention because of its relevance in many technical problems of practical importance such as mixing, oil drilling and towing operations.

The motion of a classical viscous fluid due to the rotation of an infinite cylindrical rod immersed in the fluid was first described by Stokes(1886). External flows generated due to longitudinal and torsional oscillations of a rod were found in the classical papers of Casarella, Laura(1969). Rajagopal(1983) studied the same problem for the case of a second grade fluid. The motion of a classical viscous fluid inside an infinite cylinder was studied by Ramkissoon(1990) and he derived an analytical expression for shear stresses, drag on the cylinder and velocity was depicted graphically. Camlet-Eluhu, Majumdar(1998) have investigated the same problem for a micropolar fluid and examined the effect of micropolar parameters on the two components of the velocity field through graphical curves by using Mathematica. Owen and Rahman(2006) studied the same type of flow with an Oldroyd-B liquid. Calmelet-Eluhu, Rosenhaus(2001) studied, a micropolar fluid inside a moving infinite circular cylinder due to its oscillations along and about its axis and they found analytical solutions by applying lie group methods. Using various types of fluids, the flow generated due to longitudinal and torsional oscillations of a circular cylinder was examined by few authors. Ramkissoon et al (1991) have examined a polar fluid by using transform methods and they have presented the effect of micropolar parameters on the microrotation and velocity fields graphically. Bandelli et al(1994) studied the flow of third grade fluid. Rajagopal and Bhatnagar (1995) presented two simple but elegant solutions for the flow of an Oldroyd-B fluid. In the first part, they considered the flow past an infinite porous plate and found that the problem admits an automatically decaying solution in the case of suction at the plate and that in the case of blowing it admits no such solution. In the second part, they studied longitudinal and torsional oscillations of an infinitely long rod of finite radius. Pontrelli(1997) has studied the axi-symmetric flow of a homogeneous Oldroyd-B fluid with suction or injection velocity applied at the surface. Akyildiz(1998) studied an Oldroyd-B fluid and he examined the effect of the elasticity on the velocity field and dynamic boundary layer. Fetacau and Corina

Fetacau (2006) have obtained the starting solutions corresponding to the motion of a second grade fluid by means of the finite Hankel transforms. Vieru et al (2007) investigated the exact solutions for the motion of a Maxwell fluid using Laplace Transforms. Karim Rahaman et al(2008) studied the motion of viscoelastic incompressible flow of the upper convected Maxwell fluid at different frequencies of oscillations of the cylinder along and about its axis and he presented velocity components graphically for particular values of the flow parameters. Mehrdad Massoudi and Tran X.Phouc (2008) have solved numerically the flow of a second grade fluid and they presented the results graphically for the shear stresses at the wall.

In this paper we consider the flow of couple stress fluid generated by a circular cylinder subjected to longitudinal and torsional oscillations.

2. Description and Formation of the problem:

Consider a circular cylinder of radius '*a*' within an incompressible couple stress fluid. The cylinder is subjected to torsional oscillations $e^{i\omega_1 \tau}$ and longitudinal oscillations $e^{i\omega_2 \tau}$ with amplitudes $q_0 \sin\beta_0$, $q_0 \cos \beta_0$ along the respective directions with $\omega_1$as the frequency of the longitudinal oscillations, ω_2 as the frequency of the torsional oscillations, q_0 as the magnitude of oscillations and β_0 is the angle between the direction of torsional oscillation and the base vector $\mathbf{e}_\theta$.i.e The cylinder oscillates with velocity as given by the expression $\boldsymbol{Q}_\Gamma = q_0\left(Sin\,\beta_0\, e^{i\omega_1\tau}\, \boldsymbol{e}_\theta + Cos\,\beta_0\, e^{i\omega_2\tau}\, \boldsymbol{e}_z\right)$ and the flow of the couple stress fluid being generated due to these oscillations of the cylinder. Choose the cylindrical polar coordinate system(R,θ, Z) with the origin at the center of the cylinder and Z-axis along the axis of the cylinder. The physical model illustrating the problem under consideration is shown in figure 1.

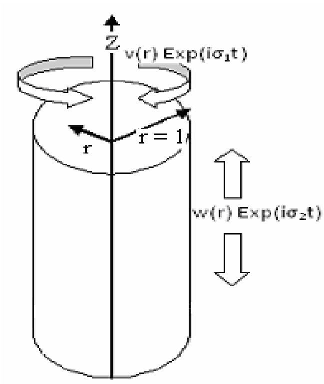

Fig 1 Geometry of the problem–non dimensional form

After neglecting body forces and body couples, the field equations governing the couple stress fluid dynamics as given by Stokes [1] are

$$\nabla_1 \cdot \boldsymbol{Q} = 0 \tag{1}$$

$$\rho\left(\frac{\partial \boldsymbol{Q}}{\partial \tau} + \boldsymbol{Q}\cdot\nabla_1 \boldsymbol{Q}\right) = -\nabla_1 P - \mu\,\nabla_1\times\nabla_1\times\boldsymbol{Q} - \eta\,\nabla_1\times\nabla_1\times\nabla_1\times\nabla_1\times\boldsymbol{Q} \tag{2}$$

where $\boldsymbol{Q}$ is velocity, P is pressure, ρ is density, τ is time, μ is viscosity and η is couple stress viscosity. By nature of the geometry and flow, the velocity components are assumed to be axially symmetric and depend only on radial distance and time. Hence the velocity is taken in the form

$$\boldsymbol{Q} = V(R,\tau)\boldsymbol{e}_\theta + W(R,\tau)\boldsymbol{e}_z \tag{3}$$

By introducing the following non-dimensional scheme

$$\boldsymbol{q} = \frac{\boldsymbol{Q}}{q_0},\ p = \frac{P}{\rho q_0^2},\ t = \frac{q_0}{a}\tau,\ r = \frac{R}{a} \text{ and } z = \frac{Z}{a} \tag{4}$$

The equations in (1), (2) and (3), for the flow take the following non-dimensional form

$$\nabla \cdot \boldsymbol{q} = 0 \tag{5}$$

$$Re\left(\frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{q}.\nabla \boldsymbol{q}\right) = -Re\ \nabla\ \boldsymbol{p} - \nabla\times\nabla\times\boldsymbol{q} - S\ \nabla\times\nabla\times\nabla\times\nabla\times\boldsymbol{q} \tag{6}$$

where $Re = \dfrac{\rho q_0 a}{\mu}$ = Reynolds number and $S = \dfrac{\eta}{\mu a^2}$ = couple stress parameter.

Let us choose the velocity vector $\boldsymbol{q}$ and pressure p in the form

$$\boldsymbol{q} = v(r)e^{i\sigma_1 t}\,\boldsymbol{e}_\theta + w(r)e^{i\sigma_2 t}\,\boldsymbol{e}_z \tag{7}$$

$$p = p_1(r)e^{2i\sigma_1 t} \tag{8}$$

Substituting (7) in (6), and comparing the coefficients of $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$ we get

$$\frac{dp_1}{dr} = \frac{v^2}{r} \tag{9}$$

$$i\sigma_1 Rev = D^2 v - S\,D^4 v \tag{10}$$

$$i\sigma_2 Rew = \left(w'' + \frac{1}{r}w'\right) - S\left(w^{iv} + \frac{2}{r}w''' - \frac{1}{r^2}w'' + \frac{1}{r^3}w'\right) \tag{11}$$

where $D^2 v = v'' + \dfrac{1}{r}v' - \dfrac{1}{r^2}v$ and $D^4 v = D^2(D^2 v)$

We express the equation (10) as

$$\left[D^4 - \frac{1}{S}D^2 + \frac{i\sigma_1 Re}{S}\right]v = 0 \tag{12}$$

This can be written as

$$\left(D^2 - \lambda_1^2\right)\left(D^2 - \lambda_2^2\right)v = 0 \tag{13}$$

where $\lambda_1^2 + \lambda_2^2 = \dfrac{1}{S}$ and $\lambda_1^2\,\lambda_2^2 = \dfrac{i\sigma_1 Re}{S}$

Similarly the equation (11) reduces to

$$\left[D^4 - \frac{1}{S}D^2 + \frac{i\sigma_2 Re}{S}\right]w' = 0 \tag{14}$$

which can be written as

$$\left(D^2-\alpha_1^2\right)\left(D^2-\alpha_2^2\right)w'=0 \tag{15}$$

where $\alpha_1^2+\alpha_2^2=\frac{1}{S}$ and $\alpha_1^2\,\alpha_2^2=\frac{i\sigma_2 \mathrm{Re}}{S}$

Now the equations (13) and (15) are solved for v and w under the no slip condition and type A condition or type B condition on the boundary. These conditions are given as follows.

No slip Condition

The velocity of the fluid on the boundary is equal to the velocity of the boundary. It is explicitly given by $\boldsymbol{Q}_{\Gamma}$ = Velocity of Γ = $q_0\left(Sin\beta_0\, e^{i\omega_1\tau}\,\boldsymbol{e}_\theta+Cos\beta_0\, e^{i\omega_2\tau}\,\boldsymbol{e}_z\right)$

It takes the following in non-dimensional form $\boldsymbol{q}\,\big|_{r=1} = cos\beta_0\, e^{i\sigma_1 t}\,\boldsymbol{e}_\theta+sin\beta_0\, e^{i\sigma_2 t}\,\boldsymbol{e}_z$

This condition can be explicitly written as in the following equations

$$v(1)=cos\beta_0 \quad \text{and} \quad w(1)=sin\beta_0 \tag{16}$$

Type A Condition

Type A-condition represents vanishing of couple stress tensor on the boundary. The constitutive equation for couple stress tensor $\boldsymbol{M}$ is given by

$$\boldsymbol{M}=mI+2\eta\,\nabla_1(\nabla_1\times\boldsymbol{Q})+2\eta'\left[\nabla_1(\nabla_1\times\boldsymbol{Q})\right]^T \tag{17}$$

Taking $\nabla\times\boldsymbol{q}=-\,w'\,e^{i\sigma_2 t}\,\boldsymbol{e}_\theta+\left(v'+\frac{v}{r}\right)\boldsymbol{e}_z$ in the equation for $\boldsymbol{M}$ above we get the expression for $\boldsymbol{M}$ as

$$\boldsymbol{M}=m(\boldsymbol{e}_r\boldsymbol{e}_r+\boldsymbol{e}_\theta\,\boldsymbol{e}_\theta+\boldsymbol{e}_z\,\boldsymbol{e}_z)+\left(-\frac{2\eta q_0}{a^2}w''+\frac{2\eta' q_0}{a^2}\frac{1}{r}w'\right)\boldsymbol{e}_r\,\boldsymbol{e}_\theta\,e^{i\sigma_2 t}+\left(\frac{2\eta q_0}{a^2}\frac{1}{r}w'-\frac{2\eta' q_0}{a^2}w''\right)\boldsymbol{e}_\theta\boldsymbol{e}_r\,e^{i\sigma_2 t}$$

$$+\frac{2\eta q_0}{a^2}D^2 v\,e^{i\sigma_1 t}\,\boldsymbol{e}_r\boldsymbol{e}_z+\frac{2\eta' q_0}{a^2}D^2 v\,e^{i\sigma_1 t}\,\boldsymbol{e}_z\boldsymbol{e}_r$$

From this, we get the conditions that

$$D^2 v=0 \text{ and } w''-e\frac{1}{r}w'=0 \text{ where } e=\frac{\eta'}{\eta} \text{ on } r=1 \tag{18}$$

Type B Condition

Type B-condition is the super adherence condition on the boundary. This condition requires that angular velocity on the boundary $=\boldsymbol{\omega}_\Gamma=\frac{1}{2}\mathrm{curl}\,\boldsymbol{q}_\Gamma$, which implies that

$$w'=-\,2\sigma_1 \quad \text{and} \quad v'+\frac{v}{r}=0 \text{ on } r=1 \tag{19}$$

The fluid is at rest as $r\to\infty$ the solutions of (13) and (15) can be written as

$$v=a_1K_1(\lambda_1 r)+a_2K_1(\lambda_2 r) \tag{20}$$

$$w=\frac{a_3}{\alpha_1}K_0(\alpha_1 r)+\frac{a_4}{\alpha_2}K_0(\alpha_2 r) \tag{21}$$

Now the constants a_1, a_2, a_3 and a_4 can be found out numerically for different values of couple stress parameters by using the boundary conditions (16),(18) and (19) in (20) and (21).

3. Calculation for Drag

The drag D acting on a cylinder of length L is given by

$$D = aL\int_0^{2\pi}\left(T_{21}\cos\beta_0 + T_{31}\sin\beta_0\right)d\theta \tag{22}$$

The stress component in (22) is defined by the following constitutive equation for couple stress fluids (Stokes [1]).

$$T_{ij} = -pI + \lambda_1(\nabla_1\cdot\boldsymbol{Q})I + \mu\left(\nabla_1\boldsymbol{Q} + (\nabla_1\boldsymbol{Q})^T\right) + \frac{1}{2}I\times(\nabla_1\cdot\boldsymbol{M}) \tag{23}$$

The stress components are calculated from (23) as

$$T_{31} = \left(\frac{\mu q_0}{a}w' + \frac{\eta q_0}{a^3}\left(w''' + \frac{1}{r}w'' - \frac{1}{r^2}w'\right)\right)e^{i\sigma_2 t} \quad \text{and}$$

$$T_{21} = \left(\frac{\mu q_0}{a}\left(v' - \frac{v}{r}\right) + \frac{\eta q_0}{a^3}\left(v''' + \frac{2}{r}v'' - \frac{1}{r^2}v' + \frac{1}{r^3}v\right)\right)e^{i\sigma_1 t}$$

The Non-dimensional form of stress components T_{31} and T_{21} on the cylinder can be calculated as

$$T_{31} = \frac{\mu q_0}{a}\left[a_3\left(1 + S\alpha_1^2\right)K_1(\alpha_1) + a_4\left(1 + S\alpha_2^2\right)K_1(\alpha_2)\right]e^{i\sigma_2 t} \tag{24}$$

$$T_{21} = \frac{\mu q_0}{a}\left[a_1\left(-\lambda_1\left(1 + S\lambda_1^2\right)K_2(\lambda_1) + 2S\lambda_1^2 K_1(\lambda_1)\right) + a_2\left(-\lambda_2\left(1 + S\lambda_2^2\right)K_2(\lambda_2) + 2S\lambda_2^2 K_1(\lambda_2)\right)\right]e^{i\sigma_1 t} \tag{25}$$

Now finally the non–dimensional drag D' is given by

$$D' = \left(T_{21}\cos\beta_0 + T_{31}\sin\beta_0\right) \quad \text{on } r = 1 \tag{26}$$

where $D' = \dfrac{D}{2\pi L\mu q_0}$

4. Numerical calculation and Results

The analytical expression for the non-dimensional velocity components v, w and drag are given by the equations (20), (21) and (26) respectively. For different values of parameters like couple stress parameter S, Reynolds number Re and the ratio of couple stress viscosities parameter e on velocity components v and w are computed numerically and results are graphically presented through figs 2–11. In figs 12–15, the velocity variations at different times are shown. The drag is calculated numerically at different times for fixed σ_1 and σ_2 and the results are presented through graphs 16–17.

When the angle $\beta_0=0$ the problem reduces to rotary oscillations about the axis of the cylinder. When $\beta_0=\pi/2$, the problem becomes the special case of longitudinal oscillations along the axis of the cylinder. When $\sigma_1 = \sigma_2$ and oscillations are periodic, our results are similar to the results of Calmelet-Eluhu and Mazumdar [7].

The numerical results for velocities and drag are computed at S=10, σ_1=1.5, e=2.5, σ_2=2.5, β_0=0.7, $t = \pi$, Re=0.7 by fixing one of these values in the graphs. The figures for type A boundary condition are shown on left column and the figures for type B conditions are on the right column. The graphs 2,3,6 and 7 indicate that the condition type A and the condition type B show no much difference for transverse velocity v. But from figures 4, 5, 8 and 9, in type B condition, the axial velocity w takes more negative vales i.e. fluid particles are pushed downwards near to the cylinder. It can be seen from Figs 2–5 that as the couple stress parameter S increases; the values of transverse velocity v increase for type A and type B conditions whereas the values of axial velocity w increase for type A condition, but the values of w decrease for type B condition. We observe that from Figs 6–9 that as the Reynolds number Re increases; the values of transverse velocity v decrease for type A and type B conditions whereas the values of axial velocity w decrease for type A condition, but these values of w increases for type B condition. Type B boundary condition does not involve the parameter e which is the ratio of couple stress viscosity coefficients η and η'. In type A condition, as it can be seen from Figs 10–11 the parameter e increases the axial velocity w increases, but the effect of e on transverse velocity v is not very significant. i.e., the variation in the values of e does not result in much variation in the values of v. i.e. there is no significant effect on v due to the parameter e. The figs 12–15 indicate the case of viscous fluids for the velocities v and w at different times and our results are in good agreement with the observations of Ramkissoon [6].

The non-dimensional drag is calculated numerically for different values of non-dimensional time in multiples of π/σ_2 at fixed values of σ_1, σ_2 and the results are shown in the Figs 16–17 and it is observed that as S increases, the amplitude of oscillation for the drag also increases for both type A condition and type B conditions. i.e., the amplitude of oscillations for drag in the case of viscous fluids is less than that of the couple stress fluids. Hence the couple stress fluids offer more drag than that of viscous fluids. We observe, from fig 18, that as the parameter e increases drag increases.

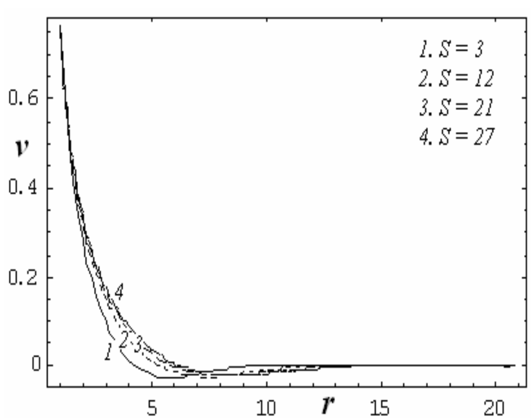

Fig 2 Type A variation of v with r

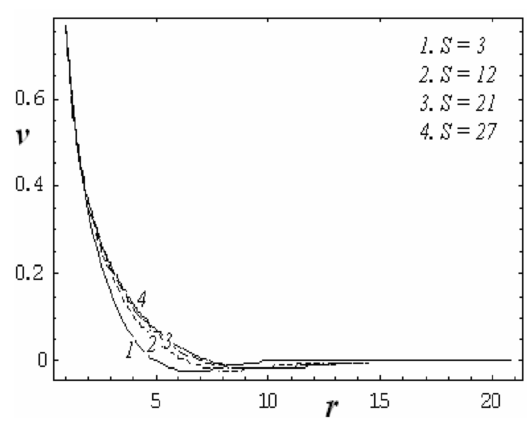

Fig 3 Type B variation of v with r

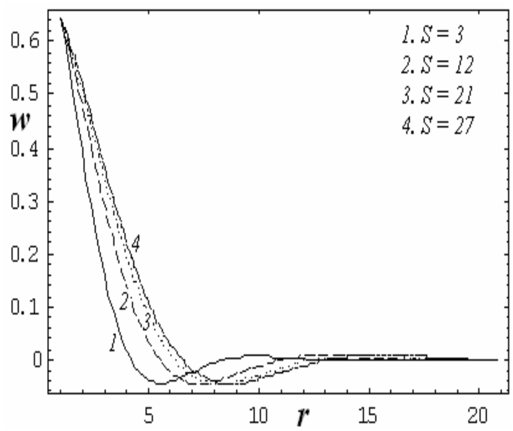

Fig 4 Type A variation of *w* with *r*

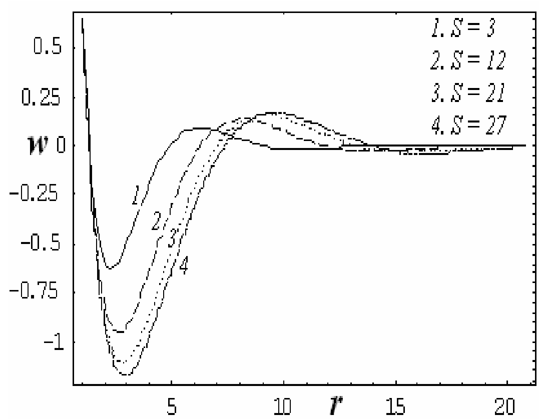

Fig 5 Type B variation of *w* with *r*

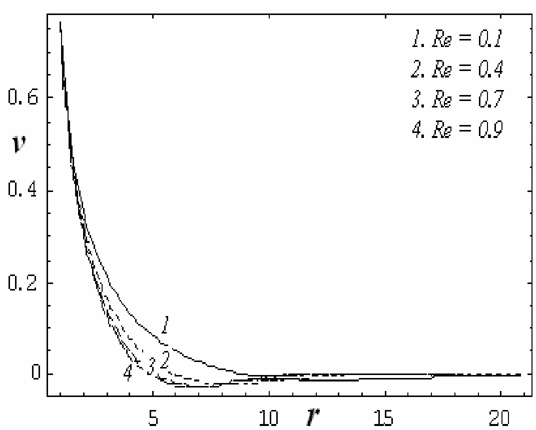

Fig 6 Type A variation of *v* with *r*

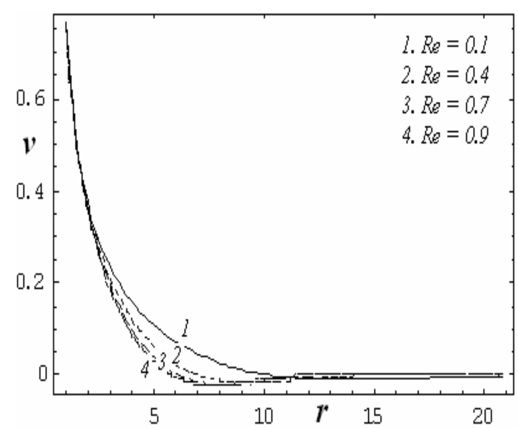

Fig 7 Type B variation of *v* with *r*

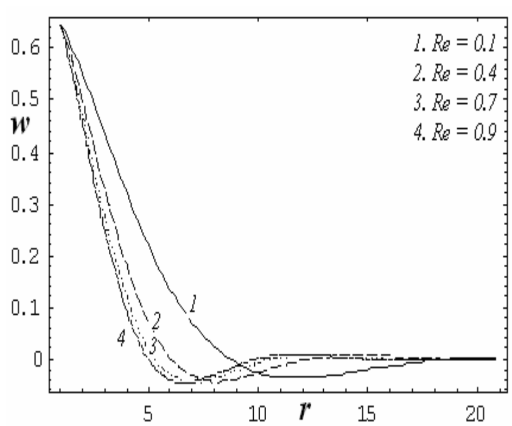

Fig 8 Type A variation of *w* with *r*

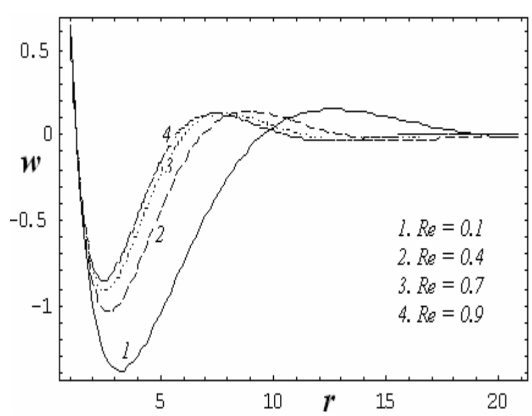

Fig 9 Type B variation of *w* with *r*

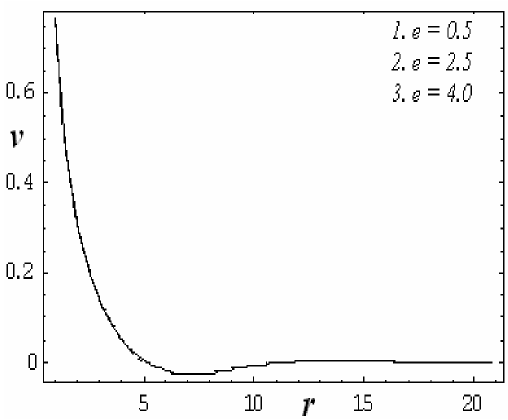

Fig 10 Type A variation of v with r

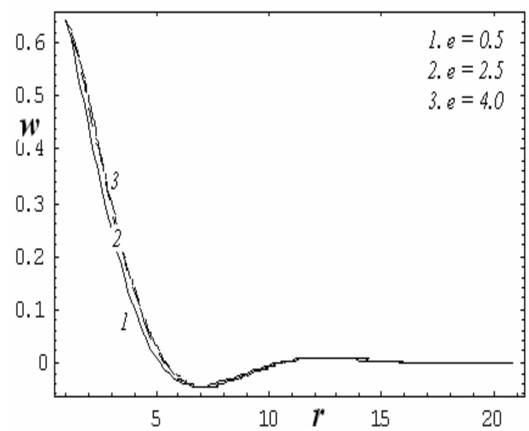

Fig 11 Type A variation of w with r

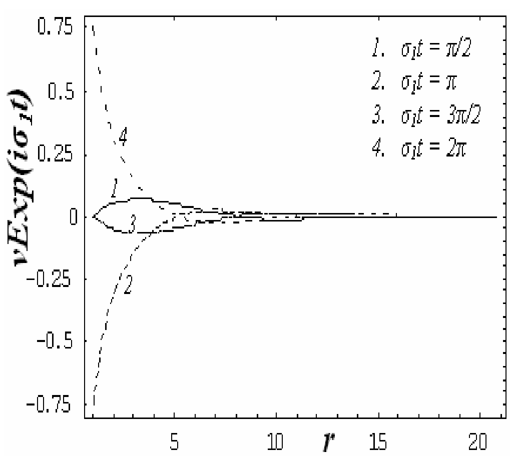

Fig 12 Type A variation of vExp(iσ_1t) with r

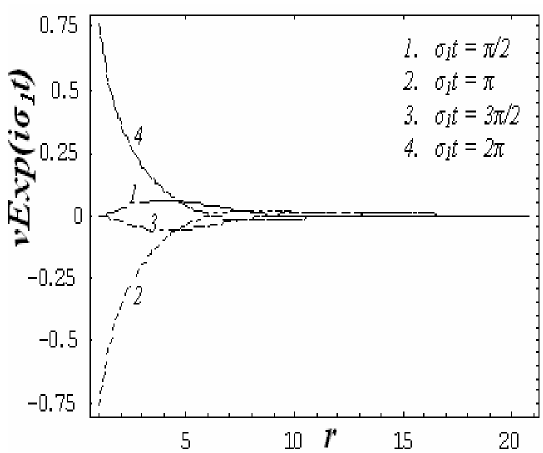

Fig 13 Type B variation of vExp(iσ_1t) with r

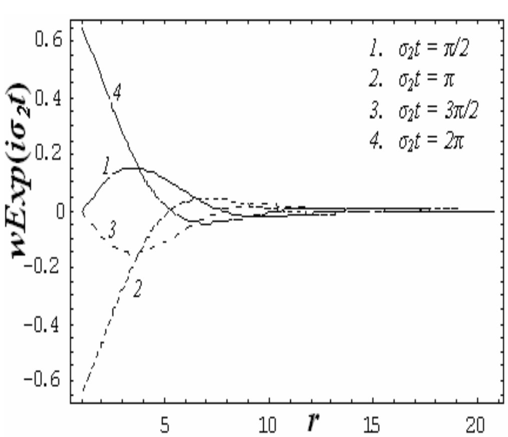

Fig 14 Type A variation of wExp($i\sigma_2 t$) with r

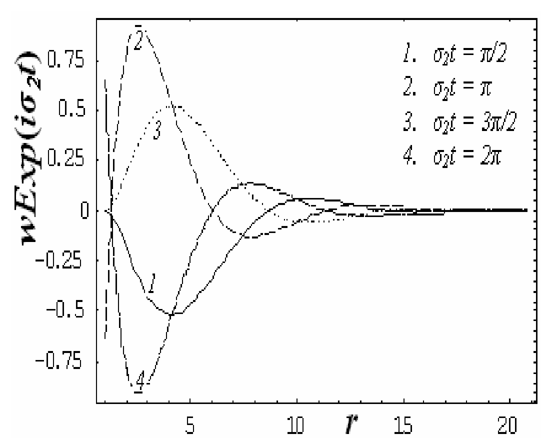

Fig 15 Type B variation of wExp($i\sigma_2 t$) with r

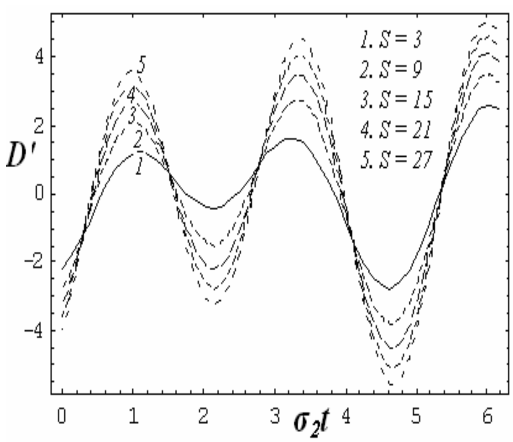

Fig 16 Type A variation of D' with $\sigma_2 t$

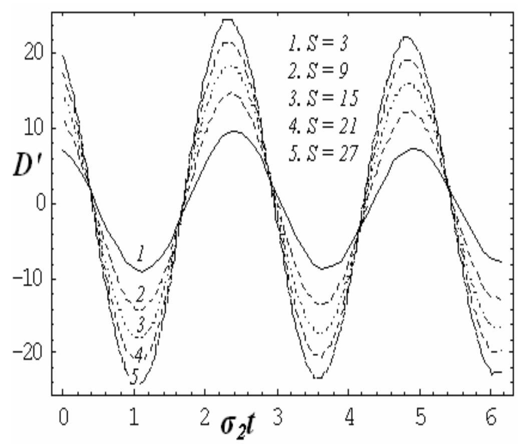

Fig 17 Type B variation of D' with $\sigma_2 t$

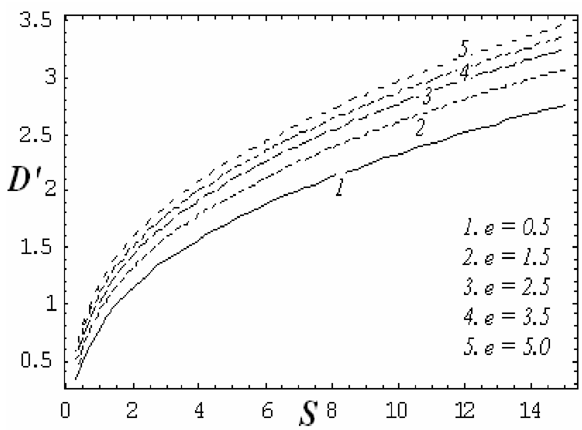

Fig 18 Type A variation of D' with S

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