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Three Simple Spline Methods for Approximation and Interpolation of Data

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Abstract

In this research, the method of interpolation of piecewise splines is used. One spline method of third order and two spline methods of forth order, with the usage of middle point and the approximation of the first derivative in given points are used. For second degree spline, there are continuity of function and its derivative at the points. In the cubic spline with the middle point there are continuity of function and the first derivative in each piece boundaries. Also the continuity of function and first derivative are satisfied in the third method. These three methods are usable for elements or pieces with different sizes $h_i = x_{i+1} - x_i$. The three mentioned methods are examined for three different functions. The results show in the situation where continuity C^1 is required, also with respect to the accuracy and the amount of calculations, these models are preferred to the classic cubic spline method for many problems.

Keywords: continuity, spline, derivative, interpolation, approximation, piecewise continuous

1 Introduction

There are different methods for approximation and interpolation of data, such as: Lagrange polynomial, Newton divided difference methods, Piecewise cubic spline methods, Bezier and Bspline methods, [1], [2], [3], [4], [5], and [6]. The cubic spline method has the most application, because of the smoothness of the curve pieces and continuity of C^2. In classic cubic spline method, the calculations of interpolation finally transfer to the solution of the governing linear system of equations according to the unknown coefficients of the forth order expressions of spline pieces. With solving this linear system of equations, it is possible to write the cubic spline equation for each piece. Limitations and difficulties which are the result of application of cubic spline can be mentioned as increase of linear system dimensions consequently increase of data. Therefore it causes an increase of the round off errors of system and the truncation errors. The mentioned difficulties for interpolation of data specially for two dimensional splines increase greatly. Such difficulties decrease the application of bicubic spline methods. Therefore, it is required to use easier and more useful methods for approximation and interpolation of data. The results show the presented models can be applied for great volume of data. They are also capable of extension in determination of two dimensional interpolations (bicubic splines).

2 Second degree spline with C^1 continuity

The $n+1$ numbers of pairs data $(x_i, f(x_i))$, $i = 0, 1, 2, \ldots, n$ are assumed. If the piecewise continuous splines for a set of data are shown in Fig. (1), then for each piece the spline function is defined as the following equation,

$$S_i = a_i + b_i(x - x_i) + c_i(x - x_i)^2 \quad (1)$$

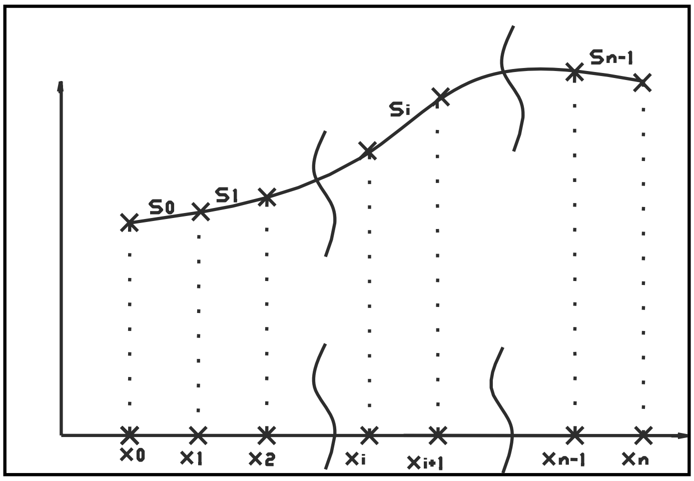

Figure 1: Data graphic presentation.

The parameters a_i, b_i, c_i are determined by satisfying the governing continuity conditions on each spline boundaries as the Eqs. (2).

$$\begin{cases} S_i(x_i) = f_i \\ S_i(x_i) = S_{i-1}(x_i) \quad i = 0, 1, 2, \ldots, n-1 \\ S_i^{'}(x_i) = S_{i-1}^{'}(x_i) \end{cases} \tag{2}$$

With respect to confirmation of continuity conditions and uniformity of elements (pieces), the spline parameters are derived by Eqs (3).

$$a_i = f_i \quad , \quad b_i = \frac{1}{h}(f_{i+1} - f_i) - h c_i \quad , \quad c_i = \frac{1}{h^2}(f_{i-1} - 2 f_i + f_{i+1}) - c_{i-1} \tag{3}$$

Where $h = \Delta x$ (element length) and c_0 is obtained from the existence of the first derivative or its approximation at point x_i, Eqs. (4).

$$c_0 = f^{'}(x_0) = f_0^{'} \quad , \quad c_0 = \frac{1}{2h^2}(f_0 - 2 f_1 + f_2) \tag{4}$$

If the element lengths be nonuniform $h_i = \Delta x_i$, the parameters b_i and c_i are determined from the following Eqs. (5) and (6).

$$b_i = \frac{1}{h_i}(f_{i+1} - f_i) - h_i c_i \tag{5}$$

$$c_i = -\frac{h_{i-1}}{h_i} c_{i-1} + \frac{h_i f_{i-1} - (h_{i-1} + h_i) f_i + h_{i-1} f_{i+1}}{h_{i-1} h_i^2} \tag{6}$$

2.1 Example 1

The second degree spline model is examined by the data obtained from Eq. (7). Hear the elements or sections lengths are uniform and homogenous ($h = \Delta x_i$=0.2). The values for $f(x_i)$ are obtained from the Eq. (7).

$$f(x) = \frac{100}{x^2} Sin\left(\frac{10}{x}\right) \tag{7}$$

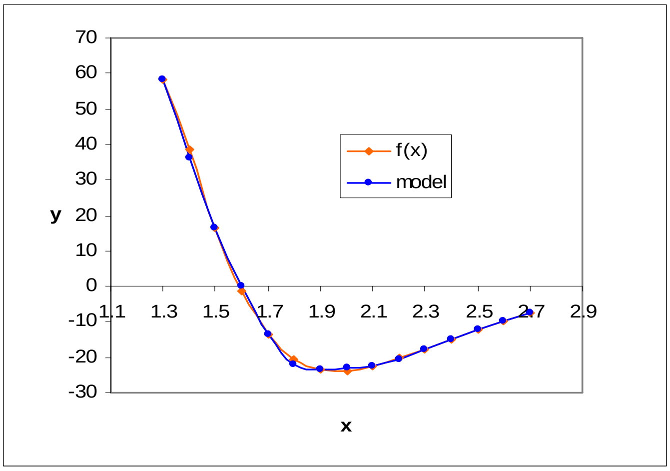

Figure 2: The comparison between the data and presented model.

The comparison of results which are derived by this model and real data is shown in Fig. (2). As it seems in the figure, there is a reasonable and close relationship between the interpolated data and real values.

2.2 Example 2

The mentioned model was studied by the data obtained from the Eq. (8). The $f(x)$ values are derived from the following equation.

$$f(x) = 3e^{0.7x^2 Sin\, x} + Sin\sqrt{0.5x} \quad (8)$$

Fig. (3) shows the comparison between the derived data for the midpoints for the model and the above function. As it can be seen from the figure there is a close relation between the presented model and the real data in the elements middle points.

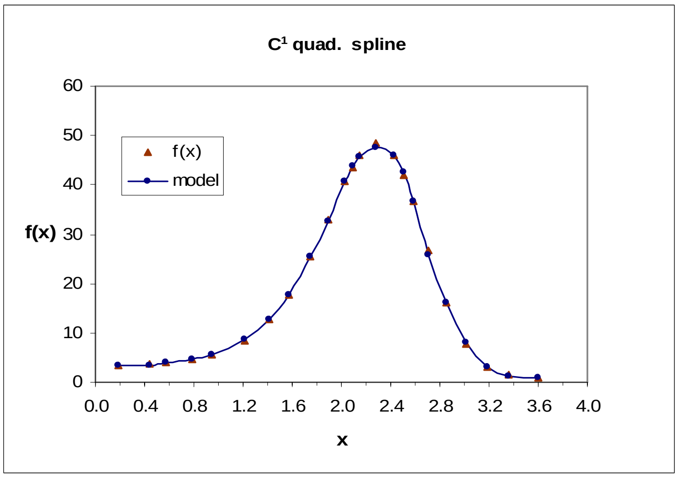

Figure 3: The comparison between the data and presented model**.**

3 Cubic spline method with continuity C^1 and using middle point

Assume there are $2n$ pairs of data (x_i , y_i) where y_i is derived with respect to x_i by experimental tests. The data are shown on the coordinate system such as Fig. (4). It is assumed the Δx_i to be uniform and constant due to simplicity of calculations for each element, $\Delta x_i = x_{i+1} - x_i = h$. In the figure each element (piece) consists of three nodes. One cubic spline is defined on each element such as Eq. (9).

$$S_i(x) = a_i + b_i(x - x_{2i-2}) + c_i(x - x_{2i-2})(x - x_{2i-1}) + d_i(x - x_{2i-2})^2(x - x_{2i-1}) \qquad (9)$$

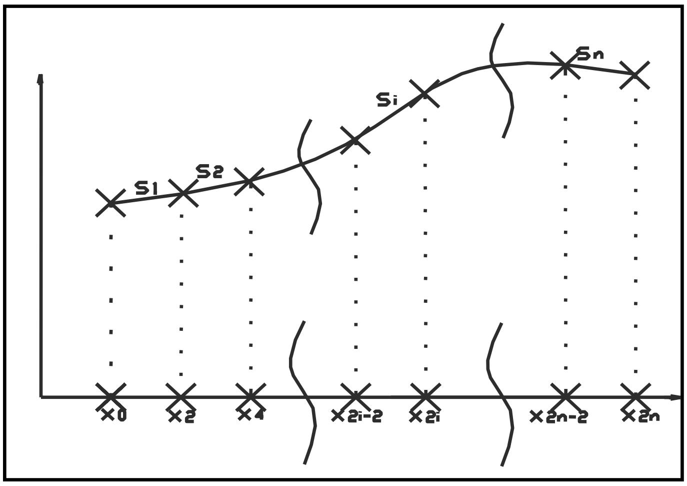

Figure 4: Graphical representation of data.

So the points x_{2i-2} , x_{2i-1} and x_{2i} associate in the governing cubic spline function, Eq. (9). The parameters a_i , b_i , c_i and d_i are determined by the following continuity conditions.

$$\begin{cases} S_i(x_{2i-2}) = f_{2i-2} \\ S_i(x_{2i-1}) = f_{2i-1} \\ S_i(x_{2i-2}) = S_{i-1}(x_{2i-2}) \\ S_i'(x_{2i-2}) = S_{i-1}'(x_{2i-2}) \end{cases} \tag{10}$$

After confirmation of the above conditions in piecewise continuous cubic splines, the parameters for each spline are obtained by the equations below.

$$a_i = f_{2i-2} \quad i = 1,2,3,\ldots,n \quad , \quad b_i = \frac{1}{h}(f_{2i-1} - f_{2i-2}) \tag{11}$$

$$c_i = c_{i-1} - \frac{1}{h^2}\left[3\left(f_{2i-2} - f_{2i-3}\right) + \left(f_{2i-4} - f_{2i-1}\right)\right] \tag{12}$$

$$c_{i-1} + 2h\,d_{i-1} = \frac{1}{2h^2}\left(f_{2i-4} - 2f_{2i-3} + f_{2i-2}\right) \tag{13}$$

With respect to the forward condition of this method, the boundary condition in point x_0 is derived by Eq. (14) with considering the availability of its first derivative S_1' in this point.

$$c_1 = \frac{1}{2h^2}\left(f_0 - 2f_1 + f_2\right) \tag{14}$$

3.1 Example 3

The C^1 spline model with middle point is used for data of this problem. The Eq. (15) is used for the values of $f(x_i)$.

$$f(x) = 2.3\,Sin\left(\frac{\pi}{2.3}\sqrt{1.33x}\right) + 1.45\sqrt{2}\;e^{-1.37(x-4.201)^2} \tag{15}$$

Fig. (5) shows the visual relation between real and interpolated values of data by this model. As it seems from figure, there is a close and logical relationship between the data and the approximated values.

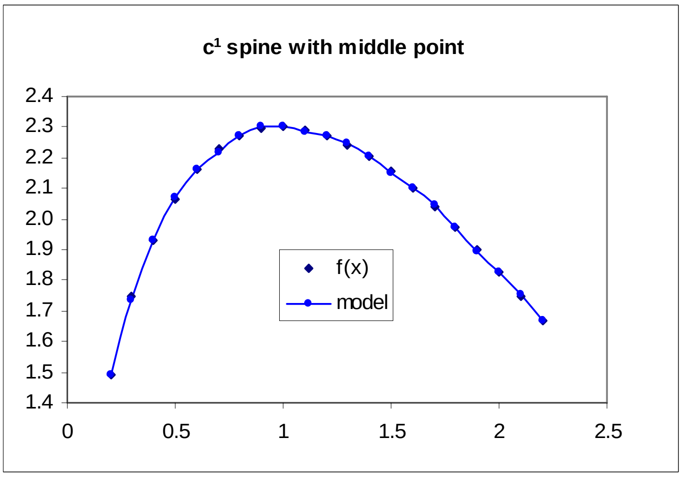

Figure 5: The comparison between the data and presented model.

4 The cubic spline method with continuity of C^1

There are $n+1$ numbers of pair data corresponding to Fig. (1). The governing cubic spline equation for piece i is according to the Eq. (16).

$$S_i = a_i (x - x_i)^3 + b_i (x - x_i)^2 + c_i (x - x_i) + d_i \quad (16)$$

The continuity characteristics for the above equation are,

$$S_i(x_i) = f_i \ , \ S_{i-1}(x_i) = f \quad , \quad S_i^{'}(x_i) = f_i^{'} \quad , \quad S_{i-1}^{'}(x_i) = f_i^{'} \quad (17)$$

By confirmation of the continuity conditions in Eq. (16), the parameters a_i , b_i , c_i and d_i are determined by the Eqs. (18), (19) and (20) as,

$$d_i = f_i \quad , \quad c_i = f_i^{'} \quad (18)$$

$$b_{i-1} = \frac{3}{h_{i-1}^2}(f_i - f_{i-1}) - \frac{1}{h_{i-1}}(2 f_{i-1}^{'} + f_i^{'}) \quad (19)$$

$$a_{i-1} = \frac{1}{3h_{i-1}^2}(3 f_{i-1}^{'} + 3 f_i^{'}) - \frac{2}{h_{i-1}^3}(f_i - f_{i-1}) \quad (20)$$

Where in the above equations h_{i-1} and $f_1^{'}$ are,

$$h_{i-1} = x_i - x_{i-1} \quad , \quad f_i' = f'(x_i) = \left\{ \frac{df(x)}{dx} \right\}_{x=x_i} \tag{21}$$

In this method the boundary conditions correspond to Eq. (21) with the possibility of determining numerical derivative at the beginning and end points are,

$$f_0' = \frac{1}{6h}(-11 f_0 + 18 f_1 - 9 f_2 + 2 f_3) \tag{22}$$

$$f_n' = \frac{1}{6h}(-2 f_{n-3} + 9 f_{n-2} - 18 f_{n-1} + 11 f_n) \tag{23}$$

For determining the first derivative in the other points, the following approximated equations are used.

$$f_1' = \frac{1}{6h}(-2 f_0 - 3 f_1 + 6 f_2 - f_3) \tag{24}$$

$$f_i' = \frac{1}{6h}(f_{i-2} - 8 f_{i-1} + 8 f_{i+1} - f_{i+2}) \quad \text{for} \quad i = 2,3,4,\cdots,n-2 \tag{25}$$

$$f_{n-1}' = \frac{1}{6h}(f_{n-3} - 6 f_{n-2} + 3 f_{n-1} + 2 f_n) \tag{26}$$

4.1 Example 4

The above cubic spline model C^1 is used for the data relating to Eq. (7). In this table the values of $f(x_i)$ are obtained from the governing equation. The Fig. (6) shows the comparison between the real data and its approximated values by this model. As it is shown from the figure there is a close relationship between the presented model and the real values data at the mid elements points.

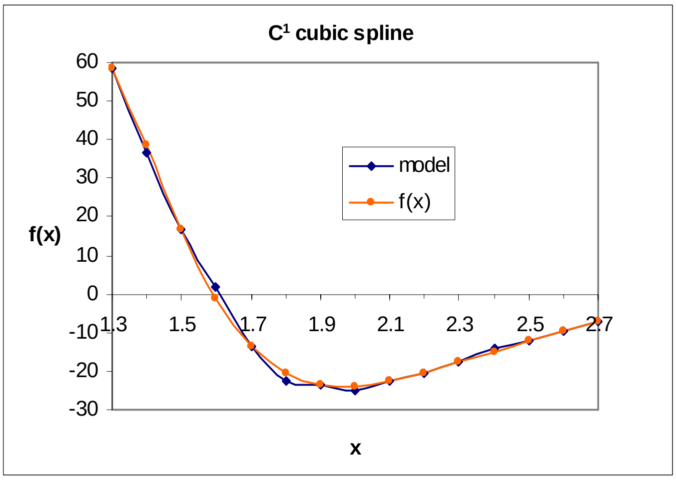

Figure 6: The comparison between the data and presented model.

5 Results and suggestions

The usage of the above mentioned methods in the four examples, shows three presented models can be used in many interpolation and approximation problems. The preference of the above models to the classic and current models is the simplicity of their relationship and their equations also is the reduction in volume of calculation operations for large volume of data. This reduction in calculation operations seems to be more specified and distinguished for interpolation and approximation of two dimensional problems. It is suggested to use these models for approximation and determination of graphs and shells. Also it is recommended to verify and to compare the amount of the reduction in volume of calculation operations of presented models with the available and classic methods.

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