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An Investigation of Bspline and Bezier Methods for Interpolation of Data

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Abstract

Bspline and Bezier methods are two powerful methods for approximation of data in all branches of engineering problems. With some improvements and modifications they can be applied for interpolation of data. Although literature offers different approaches for the formulation of Bspline, there is a set of independent functions that defines the Bspline equation. The numbers of the coefficients of Bspline equation are equal to the numbers of pair data or control points. The advantage of Bspline method is that, the most of set functions diminishes at some control points. Therefore a simple tridiagonal linear system is obtained. The tridiagonal linear system can be easily solved by Thomas algorithm. Bezier curves can be obtained by drawing the governing parametric equations. The parametric equations are Bernstein polynomials. Bezier curve does not pass through all the data points but passes through end points. It is mainly applied for approximation approaches. By considering some complementary points between original points, the Bezier can be forced to pass through all control points, hence it can be applied for interpolation purposes. The drawn curves by this method are smoother and have less sinuosity forms. For the comparison of the interpolation properties, several problems are solved by these two methods. The results explain that both methods are powerful and robust for interpolation of highly non-uniform set of data. The Bezier model is superior to Bspline with regarding to accuracy, smoothness and less calculation operations.

Keywords: Bezier curve, Bspline curve, interpolation, approximation, tridiagonal system, control points

1 Introduction

The interpolation methods are applied a lot in all branches of engineering activities. Examples are the evaluation of surface area and volume of nonuniform and irregular objects. The determination of volumes of cutting, filling and embankment in earth work and engineering surveying. Volume of surface topography and morphology forms such as hills and valleys. Evaluation of ores bodies in mining exploration, volume of lake and reservoirs of large dams and the hydraulic properties of natural channels or rivers. The Bezier and Bspline methods are superior to the classical methods of interpolation such as: Newton divided difference, cubic spline and Hermit cubic spline because of presenting smooth curves with less sinusoidal form and little sensitivity to abrupt variations of data. These two methods can be applied for a lot of data or a huge volume of data.

Suppose there are n+1 pairs of data $\{(x_i, y_i), i = 0,1,2,\dots,n\}$ where n $\in [a,b]$. The data in general could be any measurement, laboratory test, field test or in form of statistical values. Consider a set of n+1 linearly independent function $\varphi_{i(x)}$ on interval $[a,b]$ and for f $\in [a,b]$ as a linear combination, [2], [6], and [8].

$$f(x) = \sum_{i=0}^{n} c_i \varphi_i(x) = c_0\varphi_0(x) + c_1\varphi_1(x) + \cdots + c_{n-1}\varphi_{n-1}(x) + c_n\varphi_n(x) \qquad (1)$$

If the n+l data points satisfy the Eq. (1), the following linear system of equation with dimension (n+1) obtains Eq. (2).

$$\begin{bmatrix} a_{00} & a_{01} & \cdots\cdots & a_{0n} \\ a_{10} & a_{11} & \cdots\cdots & a_{1n} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{Bmatrix} = \begin{Bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{Bmatrix} \Leftrightarrow [A]\{c\} = \{y\} \qquad (2)$$

where $a_{ii} = \varphi_i(x_i)$. In the linier system of equation (2) the matrix A is full and all its components are nonzero. If the functions $\varphi_i(x)$ be polynomials of degree i, the components of matrix A, in general, will be nonuniform with different order of magnitude. This causes the matrix A becomes ill-conditioned as its dimension increases. This difficulty can be removed by applying orthogonal functions for $\varphi_i(x)$. For this purpose the orthogonal functions of Legendre, Chebychev, Laguerre or Fourier series can be used. For these cases the matrix A is still full. If the $\varphi_i(x)$ function is substituted by Bspline function, the governing system of equations will be sparse and change to tridiagonal system of equations which decreases the calculation effort.

2. Bspline method

The Bspline function is constructed by summation or linear combination of series of Bsplie basic functions that are linearly independent in interval $[a,b]$, [5] and [9], as

$$S(x)=\sum_{i=0}^{n} c_i\,\beta_i\,[u_i(x)] = c_0\beta_0(u_0)+c_1\beta_1(u_1)+\cdots+c_n\beta_n(u_n) \qquad (3)$$

where $u_i(x)=\frac{x-x_i}{h}$, $u\in[-2,2]$ and $h=\frac{b-a}{n}$.

The Bspline basic function $B_i(u_i)$ is constructed by using four cubic splines that have continuity of C^2 at boundaries of pieces, Fig. (1).

$$\beta_i(u)=\begin{cases}(2+u)^3 & , \quad -2\le u\le -1\\ 1+3(1+u)+3(1+u)^2-3(1+u)^3 & , \quad -1\le u\le 0\\ 1+3(1-u)+3(1-u)^2-3(1-u)^3 & , \quad 0\le u\le 1\\ (2-u)^3 & , \quad 1\le u\le 2\\ 0 & , \quad otherwise\end{cases} \qquad (4)$$

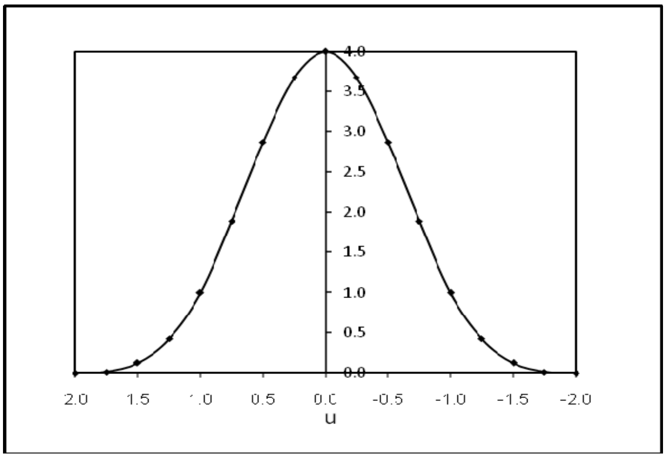

Figure 1: Bspline basic function.

In order to improve the applicability and continuity characteristics of Bspline basic function, the following functions are presented as follows:

$$y = a\,e^{-b\,x^2} - c \quad , a = 4.016478 \, , b = 1.3740615 \, , c = 0.016478 \qquad (5)$$

$$y = 4\cos^4\left(\frac{\pi}{4}x\right) \qquad (6)$$

$$y = 4 - 4\left(\sin\left(\frac{\pi}{8}x^2\right)\right)^{\alpha} \quad , \quad \alpha = \ell og(0.75) \big/ \ell og\left(\sin\frac{\pi}{8}\right) \qquad (7)$$

$$y = \frac{(4-x^2)^2}{5x^2+4} \quad (8)$$

The curves of Eqs. (5), (6), (7) and (8) are shown in Fig. (2). For the comparison of the interpolation characteristics of Bspline and Bezier methods, three different problems are solved and their results are compared.

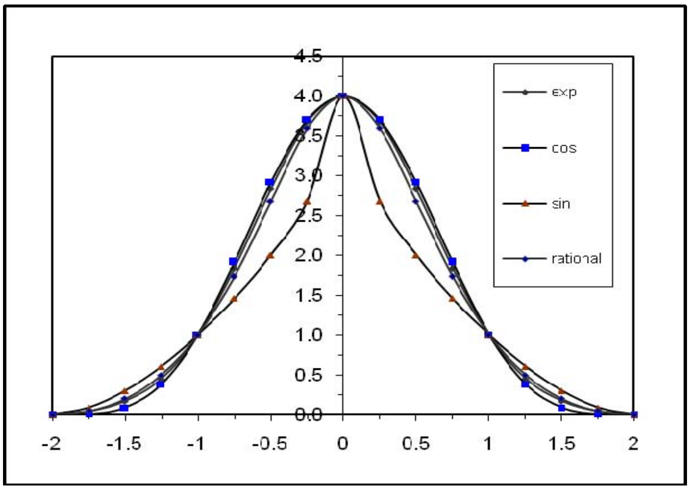

Figure 2: Different functions for Bspline basic function.

2.1 Problem 1

The data consist of eleven control points. They obtained from the Eq. (9) at interval $[1.5, 2.5]$, [1], the elements lengths are uniform and equal to 0.1 $(\Delta x_i = 0.1)$.

$$f(x) = \frac{100}{x^2}\sin(\frac{10}{x}) \quad (9)$$

The linear system of equations governing to this problem is

$$\begin{bmatrix} 4 & 2 & 0 & \vdots & \vdots & 0 \\ 1 & 4 & 1 & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_{10} \\ c_{11} \end{Bmatrix} = \begin{Bmatrix} y_1 + \dfrac{y_1' eh}{2ab} \\ y_2 \\ \vdots \\ \vdots \\ y_{10} \\ y_{11} - \dfrac{y_{11}' eh}{2ab} \end{Bmatrix} = \begin{Bmatrix} 11.51022 \\ -1.29606 \\ \vdots \\ \vdots \\ -14.8395 \\ -12.7701 \end{Bmatrix} \quad (10)$$

For the above linear system of equations the applied boundary conditions consist of first derivative at end control points. The parameters a and b are in Eq. (5) and e is Napier number. With considering the applied boundary condition coefficients c_0 and c_{12} are obtained as:

$$c_0 = c_2 - \frac{y_1' eh}{2ab} \quad , \quad c_{12} = c_{10} + \frac{y_{11}' eh}{2ab} \tag{11}$$

Because of unavailability of the first derivative at end points, they can be approximated as

$$\begin{cases} y_1' = \dfrac{1}{2h}(-3y_1 + 4y_2 - y_3) \\ y_n' = \dfrac{1}{2h}(y_{n-2} - 4y_{n-1} + 3y_n) \end{cases} \tag{12}$$

Fig. (3) Shows the relationship between the Bspline curve and the values are obtained from Eq. (9) at midpoint elements.

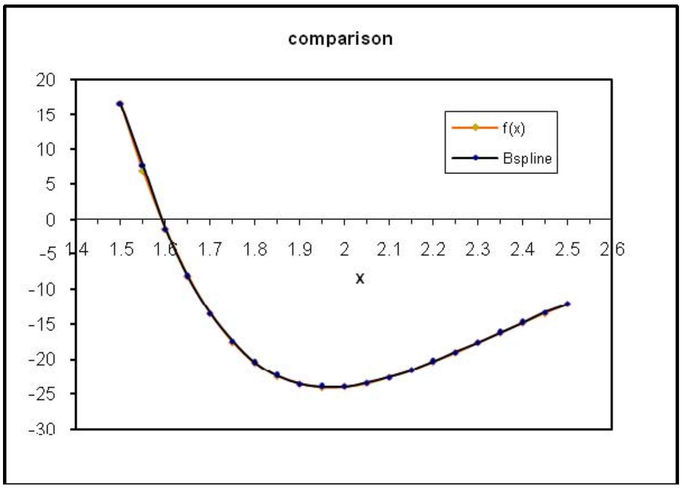

Figure 3: The comparison of Bspline curve and function f(x).

2.2 Problem 2

In this problem 15 pairs of data (x_i, y_i) are obtained from Eq. (9) for interval [1,3]. Because of high variability of data the nonuniform elements are considered; $h_i = x_{i+1} - x_i$. Bspline model was applied for these elements. For this purpose the following transfer equation is used.

$$t_i = \varphi_i + \phi_i x + \gamma_i x^2 \quad , \quad t_i \in [-2, 2] \quad , \quad x \in [x_{i-2}, x_{i+2}] \tag{13}$$

where $i = 1, 2, 3, \ldots, n$. The coefficient of Eq. (13) and M_i can obtain from the Eqs. (14) and (15).

$$\begin{Bmatrix} \varphi_i \\ \phi_i \\ \gamma_i \end{Bmatrix} = \frac{1}{M_i} \begin{Bmatrix} 2x_i(x_{i-2}x_i + x_i x_{i+2} - x_{i-2}^2 - x_{i+2}^2) \\ 2(x_{i-2}^2 - 2x_i^2 + x_{i+2}^2) \\ -2(x_{i-2} - 2x_i + x_{i+2}) \end{Bmatrix} \tag{14}$$

$$M_i = (x_i x_{i+2}^2 + x_{i-2}^2 x_{i+2} + x_{i-2} x_i^2) - (x_i^2 x_{i+2} + x_{i-2} x_{i+2}^2 + x_{i-2}^2 x_i) \tag{15}$$

The governing system of equation for this example in schematic form is

$$\begin{bmatrix} 0 & 4 & \beta_2(x_1) - \dfrac{\beta_2'(x_1)}{\beta_0'(x_1)} \\ \beta_1(x_2) & 4 & \beta_3(x_2) \\ \beta_2(x_3) & 4 & \beta_4(x_3) \\ \vdots & \vdots & \vdots \\ \beta_{n-2}(x_{n-1}) & 4 & \beta_n(x_{n-1}) \\ \beta_{n-1}(x_n) - \dfrac{\beta_{n-1}'(x_n)}{\beta_{n+1}'(x_n)} & 4 & 0 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-1} \\ c_n \end{Bmatrix} = \begin{Bmatrix} y_1 + \dfrac{h_0 e^b}{2ab} y_1' \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n - \dfrac{h_n e^b}{2ab} y_n' \end{Bmatrix} \tag{16}$$

The boundary conditions in this problem are similar to problem 1, except the mesh is not uniform. The first derivative at the end points can be obtained from the Eqs. (17) and (18) as:

$$y_1' = \begin{bmatrix} -\dfrac{2h_1 + h_2}{h_1(h_1 + h_2)} & \dfrac{h_1 + h_2}{h_1 h_2} & \dfrac{h_1}{h_2(h_1 + h_2)} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \tag{17}$$

$$y_n' = \begin{bmatrix} \dfrac{h_{n-1}}{h_{n-2}(h_{n-2} + h_{n-1})} & -\dfrac{h_{n-1} + h_{n-2}}{h_{n-1} h_{n-2}} & \dfrac{h_{n-2} + 2h_{n-1}}{h_{n-1}(h_{n-2} + h_{n-1})} \end{bmatrix} \begin{Bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{Bmatrix} \tag{18}$$

The complementary coefficients c_0 and c_{n+1} are determined from the following equations.

$$\begin{cases} c_0 = y_1 - 4c_1 - \beta_2(x_1) c_2 \\ c_{n+1} = y_n - 4c_n - \beta_{n-1}(x_n) c_{n-1} \end{cases} \tag{19}$$

The tridiagonal linear system of equations for this problem in the schematic form is

$$\begin{bmatrix} - & 4 & 0.07 \\ 1.234 & 4 & 0.869 \\ 0.954 & 4 & 0.905 \\ 0.857 & 4 & 0.948 \\ \vdots & 4 & \vdots \\ 0.92 & 4 & 1 \\ 1 & 4 & 1 \\ 2 & 4 & - \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \\ c_{13} \\ c_{14} \\ c_{15} \end{Bmatrix} = \begin{Bmatrix} -40.214 \\ -17.516 \\ 20.861 \\ 50.378 \\ \vdots \\ -12.109 \\ -6.279 \\ -3.308 \end{Bmatrix} \quad (20)$$

Fig. (4) shows the comparison behaviors of Bspline model for this problem and exact values, Eq. (9) for points at the centers of elements. As it illustrates there is a close relationship between Bspline model and real data. For satisfactory results the criterion $0.4 \le {(x_{i+2} - x_i)}/{(h_i - h_{i-2})} \le 1$ should be adapted for the case of nonuniform elements.

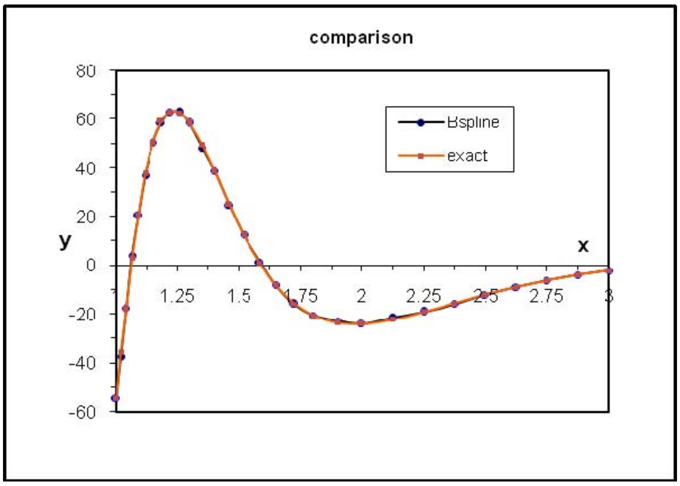

Figure 4: The comparison behavior of Bsplie model and data.

2.3 Problem 3

The problem consists of 35 pairs of data, [4]. They are obtained from soil-water saturation laboratory test, where, S_w is soil saturation and h_c is capillary pressure head. The data are related to weld salty clay loam soil. For analysis of unsaturated flow through vadose zone the $S_w - h_c$ relationship is required. About fifteen control points from the data are considered with uniform elements $(\Delta h_c = 15\, cm)$. The Bspline interpolation curve is developed from the governing data. The boundary condition at fully saturation (h_c=0) is $S_w'(0) = 0$. Fig. (5) shows the comparison between Bspline curve and the data. The Bspline model shows a close relationship with the laboratory test data.

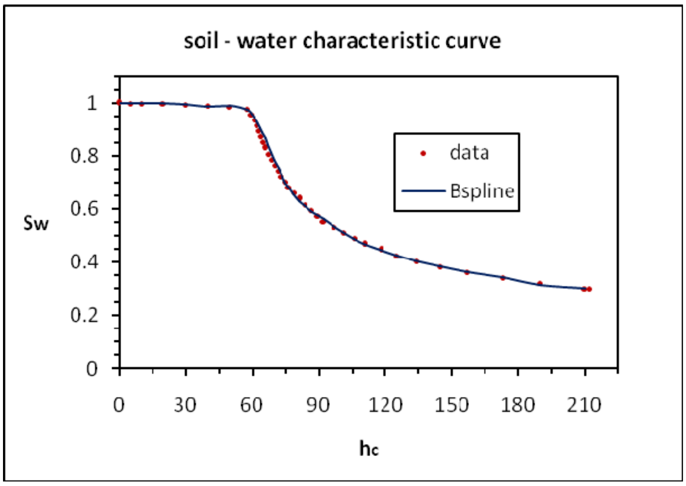

Figure 5: The Bspline curve for silty clay soil data.

3 Bezier Method

It is a method of approximation, which is basis to computer aided geometric design (CAGD). Bezier curves have a lot of applications in computer graphics and engineering problems for drawing smooth curves. The curves will not undulate or wiggle widely or exhibit extra bends. The curves are within its convex hull of data. Bezier curves are affine invariant, therefore rescaling or reorienting them is not difficult. The curves are invariants under parametric transformations, [3]. Bezier curves were developed by P. Bezier around 1962. He used them for the design of body of the Renault car in 1970, [7]. The Bezier function P(t) based on the n+1 points $p_0, p_1, \ldots, and\ p_n$ is given by

$$p_n(t) = \sum_{i=0}^{n} p_i\, b_{i,n}(t) \quad , \quad t \in [0,1] \tag{21}$$

Where $B_i(t)$ are known as Bernstein polynomials or blending functions. The i-th degree Bernstein polynomial is defined, [7], as

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad , \quad i = 0, 1, \cdots, n \tag{22}$$

For n equals 2 and 3 the Bezier function $p_2(t)$ and $p_3(t)$ are

$$p_2(t) = \lfloor t \rfloor [B] \{p\} = \lfloor 1 \quad t \quad t^2 \rfloor \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{Bmatrix} p_0 \\ p_1 \\ p_2 \end{Bmatrix} \tag{23}$$

$$p_3(t) = \lfloor 1 \quad t \quad t^2 \quad t^3 \rfloor \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{Bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{Bmatrix} \tag{24}$$

Interpolation of data of problem 2 is done by the 3rd degree Bezier method with the same nonuniform elements. Each element is defined by its end points x_i and x_{i+1} and two complementary points x_{i1} and x_{i2} with the aid of Eqs. (25) and (26).

$$\begin{cases} x_{i1} = x_i + \dfrac{h_i}{5} \\ x_{i2} = x_{i+1} - \dfrac{h_i}{5} \end{cases} \tag{25}$$

$$\begin{cases} y_{i1} = y_i + y_i' \dfrac{h_i}{5} \\ y_{i2} = y_{i+1} - y_{i+1}' \dfrac{h_i}{5} \end{cases} \tag{26}$$

Where h_i is the element length and y_i' is the derivative of data distribution at x_i which can be calculated from the Eq. (27).

$$y_i' = \left\lfloor -\frac{h_i}{h_{i-1}(h_{i-1}+h_i)} \quad \frac{h_i - h_{i-1}}{h_{i-1}h_i} \quad \frac{h_{i-1}}{h_i(h_{i-1}+h_i)} \right\rfloor \begin{Bmatrix} y_{i-1} \\ y_i \\ y_{i+1} \end{Bmatrix} \tag{27}$$

Fig. (6) Shows the Bezier curve for problem 2 with the same boundary condition. As it can be seen from the figure the interpolation by Bezier method gives a close fit to the data, even at center of each element. The Bezier curve of order 4 for data of problem 3 is shown in Fig. (7). The control points and boundary conditions are the same for both Bezier and Bspline curves. The results on Figs. (4) to (7) show that using interpolation by Bezier curve is more realistic than Bspline curve.

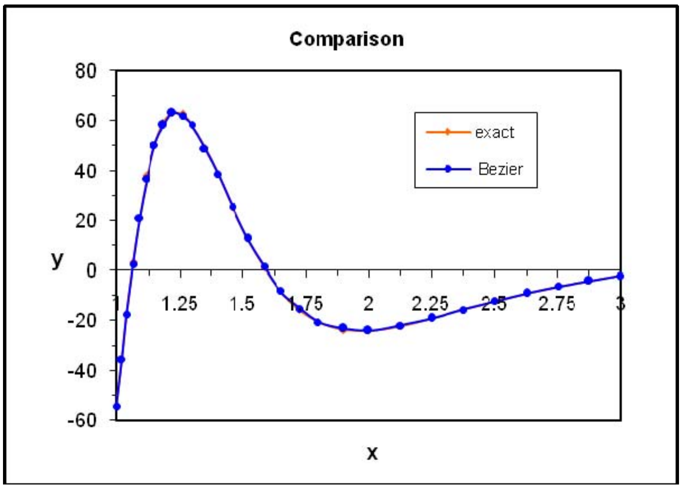

Figure 6: The comparison behavior of Bezier model and data.

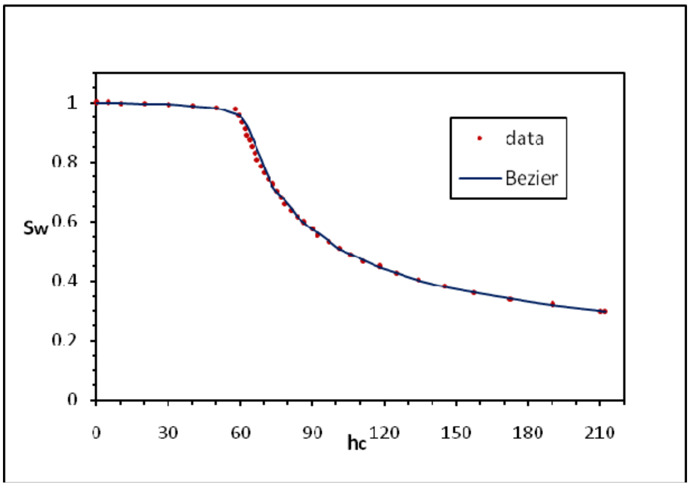

Figure 7: The Bezier curve for silty clay soil data.

4 Conclusions and Suggestions

The Bezier and Bspline methods are powerful for interpolations and approximation of data, especially for data with irregular or nonuniform distribution. Both methods can be applied for variable element size. The Bspline method is more sensitive to abrupt change in element size and sudden jump in data trend. As the element size departs from uniformity the Bsplie curves lose its smoother properties. It is recommended that the above results are investigated and checked for two dimensional Bspline and Bezier interpolation methods. If there is a bend or break in slope of data it is highly recommended to use Bezier curve. The Bspline model gives more wiggles, unsmoothed or fluctuated curve. While there is need to solve a linear system of equations for Bspline method the formulation and obtaining the

Bezier coefficient is straight forward, therefore the calculation operations are less specially for a large volume of data.

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