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Some Results on the Excess Wealth Order with Applications in Reliability Theory

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Abstract

In this paper, we establish some characterizations of the excess wealth order. As a consequence of our results, we provide some applications in reliability theory and characterize some well known classes of life distributions.

Keywords: Stochastic orders, proportional hazard (reversed hazard) model, DMRL, IMIT, NBUT

1. INTRODUCTION

Stochastic models are usually sufficiently complex in various fields of statistics, particularly in reliability theory. Obtaining bounds and approximations of their characteristics is of practical importance. That is, the approximation of a stochastic model either by a simpler model or by a model with simple constituent components might lead to convenient bounds and approximations for some particular and desired characteristics of the model. The study of changes in the properties of a model is also of great interest. Accordingly, since the stochastic components of models involve random variables, the topic of stochastic orders among random variables plays an important role in these areas. (see, Muller and Stoyan [22] and Shaked and Shanthikumar [24] for an exhaustive monograph on this topic).

Two such well-known stochastic orders are the excess wealth order, also known in literature as right spread order, and the total time on test transform order whose definitions are recalled here. Throughout this paper, X and Y are two random variables having distribution functions F and G, respectively, and denote by $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$ their respective survival functions, and by F^{-1} and G^{-1} their corresponding right continuous inverses. Moreover, we will use the term *increasing* in place of *non-decreasing*, and *decreasing* in place of *non-increasing.*

A non-negative random variable X is said to be smaller than Y in the excess wealth order (denoted by $X \leq_{ew} Y$) if, and only if,

$$\int_{F^{-1}(p)}^{\infty} \overline{F}(x)dx \leq \int_{G^{-1}(p)}^{\infty} \overline{G}(x)dx, \quad \text{for all } p \in (0,1),$$

provided the integrals are finite. This order was studied, for example, in Kochar, Li and Shaked [17], Fernandez-Ponce et al. [12] and Ahmad and Kayid [1].

Another ordering that has come to use in reliability and life testing lately is the following: A non-negative random variable X is said to be smaller than Y in the TTT-transform order (denoted by $X \leq_{ttt} Y$) if, and only if,

$$\int_{0}^{F^{-1}(p)} \overline{F}(x)\ dx \leq \int_{0}^{G^{-1}(p)} \overline{G}(x)\ dx, \quad p \in (0,1).$$

The TTT transform order has several applications in reliability theory. For example if $X \leq_{ttt} Y$, then the lifetime of a series system of n independent components whose lifetimes are distributed as X is less than that of a similar system with independent components whose lifetimes are distributed as Y, in the TTT transform order. Some other applications can be seen in Kochar, Li and Shaked [17], Ahmad, Kayid and Li [2] and Ahmad and Kayid [1].

For any random variable X, let

$$X_t \equiv [X - t \mid X > t], \quad t < t^*,$$

denote a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$, where $t^* = \sup\{x : F(x) < 1\}$. When the random variable X denotes the lifetime of a unit, X_t is known as the residual life (Barlow and Proschan [6]).

The comparison of the residual life at different times has been used to give definitions and characterizations of ageing classes. For instance, suppose F is the distribution function of the lifetime of a unit, which could be a living organism or a mechanical component or a system. To determine whether the component is ageing with time for the TTT transform associated with this distribution, Ahmad, Kayid and Li [3] proposed a new ageing notion related to the TTT transform order: A random variable X, or its distribution F, is said to be new better than used in the total time on test transform (NBUT) order if, and only if,

$$\int_{0}^{F_t^{-1}(p)} \overline{F}(u+t)du \leq \overline{F}(t) \int_{0}^{F^{-1}(p)} \overline{F}(u)du \quad \text{for all } p \in (0,1). \tag{1.1}$$

Recall also that, a random variable X is said to possess the decreasing mean residual life (DMRL) property if, and only if,

$$\mu_X(t) \equiv \int_t^\infty \frac{\overline{F}(x)}{\overline{F}(t)}\, dx,$$

is decreasing in t (see Alzaid [5]). For this ageing class it has been proved that (see Belzunce [7])

$$X \in \text{DMRL} \Leftrightarrow X_s \leq_{ew} X_t \ \text{ for all } t < s. \tag{1.2}$$

In many reliability problems it is of interest to consider variables of the kind $X_{(t)} = [t - X \mid X \leq t]$ for any $t > 0$, known in literature as inactivity time or idle time. Recently, some non-parametric classes of distributions have been defined based on the concept of inactivity time to describe reliability properties of random lifetimes. To provide reliability measure with the mission time, some authors proposed the increasing mean inactivity time (IMIT), defined as the class where $E[X_{(t)}]$ is increasing for all $t \geq 0$. The IMIT can used as a measure for improvement of reliability. Besides, a wide range of distributions happen to be IMIT. These include two-parameter Weibull, Makeham and linear failure rate distributions. (see, Ruiz and Navarro [22], Chandra and Roy [9], Kayid and Ahmad [16], Ahmad, Kayid and Pellerey [4], Ahmad and Kayid [2] and Li and Xu [19]).

For the IMIT class, there is a characterization similar to the one in (1.2), but which involves the inactivity time random variables instead of the residual life time random variables (Ahmad, Kayid and Pellerey [4])

$$X \in \text{IMIT} \Leftrightarrow X_{(s)} \leq_{ew} X_{(t)} \ \text{ for all } s < t. \tag{1.3}$$

The purpose of the current investigation is to provide some characterization results of the excess wealth order. The main results are given in Section 2 bellow. As consequences, in Section 3 we provide new characterizations of the IMIT class of life distributions under the proportional reversed hazard models. Similar conclusions of the NBUT and DMRL ageing notions under the proportional hazard models are presented.

2. **Characterization results**

Let $\mathcal{H}$ denote the set of all functions h such that $h(u) > 0$ for $u \in (0,1)$, and $h(u) = 0$ for $u \notin [0,1]$. For $h \in \mathcal{H}$, X is said to be smaller than Y in the generalized excess wealth order with respect to h (denoted by $X \leq_{ew}^{(h)} Y$) if, and only if,

$$\int_{F^{-1}(p)}^\infty h(F(x))\, dx \leq \int_{G^{-1}(p)}^\infty h(G(x))\, dx, \quad p \in (0,1), \tag{2.1}$$

or, equivalently,

$$\int_p^1 h(u)\ d\left[G^{-1}(u) - F^{-1}(u)\right] \geq 0, \quad p \in (0,1). \tag{2.2}$$

Example 2.1.

Let $h(u) = 1 - u$ for $u \in [0,1]$, and $h(u) = 0$ otherwise. Then $X \leq_{ew}^{(h)} Y$ if, and only if,

$$\int_{G^{-1}(p)}^{\infty} \overline{G}(x)\ dx \geq \int_{F^{-1}(p)}^{\infty} \overline{F}(x)\ dx, \quad p \in (0,1);$$

that is, $X \leq_{ew} Y$.

Next, we give some basic properties (Theorems 2.1 and 2.2, and Example 2.2) of the order $\leq_{ew}^{(h)}$. First, in order to obtain a relationship among the orders $\leq_{ew}^{(h)}$ for different h's, we recall the following lemma, which is essentially due to Barlow and Proschan ([6], p.120).

Lemma 2.1.

Let $\mathcal{L}$ be a measure on the interval (α, β), not necessarily non-negative. Let ϕ be a non-negative function defined on (α, β).

(a) If $\int_t^\beta d\mathcal{L}(x) \geq 0$ for all $t \in (\alpha, \beta)$, and if ϕ is increasing, then $\int_\alpha^\beta \phi(x) d\mathcal{L}(x) \geq 0$.

(b) If $\int_\alpha^t d\mathcal{L}(x) \geq 0$ for all $t \in (\alpha, \beta)$, and if ϕ is decreasing, then $\int_\alpha^\beta \phi(x) d\mathcal{L}(x) \geq 0$.

Theorem 2.1.

Let X and Y be two random variables with continuous distribution functions, and let $h_1, h_2 \in \mathcal{H}$. Suppose that

$$h_2(u)/h_1(u) \ \text{ is increasing in } (0,1).$$

Then

$$X \leq_{ew}^{(h_1)} Y \Rightarrow X \leq_{ew}^{(h_2)} Y. \tag{2.3}$$

Proof.

By (2.2), the assumption $X \leq_{ew}^{(h_1)} Y$ means

$$\int_p^1 h_1(u)\ d\left[G^{-1}(u) - F^{-1}(u)\right] \geq 0, \quad p \in (0,1). \tag{2.4}$$

Since $h_2(u)/h_1(u)$ is increasing on $(0,1)$, it follows from (2.4) and Lemma 2.1(a) that

$$\int_p^1 h_2(u)\; d\left[G(x) - F(x)\right] \geq 0, \quad p \in (0,1).$$

By (2.2), the latter inequality is equivalent to $X \leq_{ew}^{(h_2)} Y$. ∎

We will now give a result involving the preservation of the order $\leq_{ew}^{(h)}$ under a distortion ϕ of the lifetime distributions. Such results have applications in actuarial science, insurance, and in reliability theory (see, Kayid [15]). Let $\phi : [0,1] \to [0,1]$ be a continuous, strictly increasing function which satisfies $\phi(0) = 0$ and $\phi(1) = 1$, and let F and G be distribution functions. Then

$$F_\phi(\cdot) = \phi(F(\cdot)) \quad \text{and} \quad G_\phi(\cdot) = \phi(G(\cdot)),$$

are also distribution functions. We denote by $X^{(\phi)}$ and $Y^{(\phi)}$ any random variables that are distributed according to F_ϕ and G_ϕ, respectively.

Theorem 2.2.

Let $\phi : [0,1] \to [0,1]$ be a continuous, strictly increasing function which satisfies $\phi(0) = 0$ and $\phi(1) = 1$. Let X and Y be two any random variables with continuous distribution functions, and let $h \in \mathcal{H}$. Then

$$X \leq_{ew}^{(h)} Y \Leftrightarrow X^{(\phi)} \leq_{ew}^{(h(\phi^{-1}))} Y^{(\phi)}.$$

Proof.

By (2.2), $X \leq_{ew}^{(h)} Y$ is equivalent to

$$\int_p^1 h(u)\; d\left[G^{-1}(u) - F^{-1}(u)\right] \geq 0, \quad p \in (0,1).$$

Substituting $u = \phi^{-1}(v)$ in the above integral, we see that $X \leq_{ew}^{(h)} Y$ is equivalent to

$$\int_{\phi(p)}^1 h(\phi^{-1}(v))\; d\left[G^{-1}(\phi^{-1}(v)) - F^{-1}(\phi^{-1}(v))\right] \geq 0, \quad p \in (0,1).$$

Setting $q = \phi(p)$, the above inequality reduces to

$$\int_q^1 h(\phi^{-1}(v))\; d\left[G_\phi^{-1}(v) - F_\phi^{-1}(u)\right] \geq 0, \quad q \in (0,1).$$

Appealing to (2.2) the last inequality is equivalent to

$$X^{(\phi)} \leq_{ew}^{(h(\phi^{-1}))} Y^{(\phi)}.$$

This completes the proof. ∎

Example 2.2.

In Theorem 2.2 take $h(u) = 1 - u$ (that is, by Example 2.1, $X \leq_{ew} Y$). Let ϕ be as in that theorem. Then, from Theorem 2.2, we obtain

$$X \leq_{ew} Y \Leftrightarrow X^{(\phi)} \leq_{ew}^{(1-\phi^{-1})} Y^{(\phi)}. \tag{2.5}$$

Suppose that ϕ is convex [concave]; that is, ϕ^{-1} is concave [convex]. It follows that $1 - \phi^{-1}(1 - v)$ in increasing and convex [concave] in $v \in [0, 1]$, and hence

$$\frac{v}{1 - \phi^{-1}(1 - v)} \text{ is decreasing [increasing] in } v \in [0, 1].$$

Thus,

$$\frac{h(v)}{h(\phi^{-1}(v))} = \frac{1 - v}{1 - \phi^{-1}(v)}$$

is increasing [decreasing] in $v \in [0, 1]$. From (2.5) and Theorem 2.1, we thus obtain that if ϕ is convex then $X \leq_{ew} Y \Rightarrow X^{(\phi)} \leq_{ew} Y^{(\phi)}$, and if ϕ is concave then $X^{(\phi)} \leq_{ew} Y^{(\phi)} \Rightarrow X \leq_{ew} Y$.

3. Reliability applications

Reliability theory is a body of ideas, mathematical models, and methods aimed at predicting, estimating, understanding, and optimizing the life span and failure distributions of systems and their components (Barlow and Proschan [6]). Also, it allows researchers to predict the age-related failure kinetics for a system of given architecture (reliability structure) and given reliability of its components. In this section we provide some applications of our results in reliability theory. For this purpose, we consider two kinds of models: the proportional reversed hazard and the proportional hazard models.

3.1. Proportional reversed hazard model

Let X be a non-negative random variable with distribution function F. For $\theta > 0$, let $X(\theta)$ denote a random variable with distribution function $F^{(\theta)}$. In reliability theory terminology, different $X(\theta)$'s are said to have proportional reversed hazard (PRH) model (see, for example, Di Crescenzo [11], Sengupta et al. [23] and Gupta and Gupta [14]).

If Y is another non-negative random variable with distribution function G, then denote by $Y(\theta)$ a random variable with distribution function $G^{(\theta)}$. Using Theorem 2.1, in the following theorem we obtain a result which points out the effect of applying the proportional reversed hazard model on random variables that are ordered with respect to the excess wealth order.

Theorem 3.1.

Let X and Y be two random variables with continuous distribution functions. Let $X(\theta)$ and $Y(\theta)$ have proportional reversed hazards as described above.

(*a*) If $X \leq_{ew} Y$, then $X(\theta) \leq_{ew} Y(\theta)$ for all $\theta > 1$.

(*b*) If $X(\theta) \leq_{ew} Y(\theta)$ for some $\theta \in (0,1)$, then $X \leq_{ew} Y$.

Proof.

Fix a $\theta > 0$. Define the function $h^{(\theta)}$ by $h^{(\theta)}(u) = 1 - u^{\theta}$ for $u \in (0,1)$. First we show that

$$X(\theta) \leq_{ew} Y(\theta) \Leftrightarrow X \leq_{ew}^{(h^{(\theta)})} Y. \tag{3.1}$$

To see it write $X(\theta) \leq_{ew} Y(\theta)$ as

$$\int_{F^{-1}(p^{1/\theta})}^{\infty} \left\{1 - [F(x)]^{\theta}\right\} dx \leq \int_{G^{-1}(p^{1/\theta})}^{\infty} \left\{1 - [G(x)]^{\theta}\right\} dx, \quad p \in (0,1). \tag{3.2}$$

Thus (3.2) is equivalent to

$$\int_{F^{-1}(q)}^{\infty} h^{(\theta)}(F(x))dx \leq \int_{G^{-1}(q)}^{\infty} h^{(\theta)}(G(x))dx, \quad q \in (0,1); \tag{3.3}$$

that is, $X \leq_{ew}^{(h^{(\theta)})} Y$.

Let us observe that

$$X \leq_{ew} Y \Leftrightarrow X \leq_{ew}^{(h^{(1)})} Y. \tag{3.4}$$

Now, if $\theta > 1$ then $h^{(\theta)}/h^{(1)}$ is increasing on $(0,1)$, and part (a) follows from Theorem 2.1, (3.1) and (3.4). On the other hand, if $0 < \theta < 1$ then $h^{(1)}/h^{(\theta)}$ is increasing on $(0,1)$, and part (*b*) follows, again, from Theorem 2.1, (3.1) and (3.4). ∎

If $\theta = n$, a positive integer, then $(F)^n$ is the distribution function of $\max\{X_1, X_2, .$
$.., X_n\}$, where $X_1, X_2, \ldots, X_n$ are independent copies of X; that is, $(F)^n$ is the distribution function of a parallel system of size n where the component lifetimes are independent copies of X. If we take $\theta = n$ in Theorem 3.1(a), we obtain Theorem 5.1(b) of Kochar, Li and Shaked [17]; that is, if the lifetimes

of the (identical) components of one parallel system, are comparable to the lifetimes of the (identical) components of another parallel system with respect to the excess wealth order, then the two system lifetimes are also comparable with respect to this order.

For the proportional reversed hazard models, one problem of interest is to determine if certain ageing properties are preserved under the transformation $X \to X(\theta)$. Theorem 2.1 of Gupta et al. [13] proved that if X is increasing failure rate (IFR) and $\theta > 1$ then $X(\theta)$ is IFR. Di Crescenzo [11] proved similar preservation results for the new better than used (NBU) and increasing in likelihood ratio (ILR) ageing notions. For the definitions of the IFR, NBU and ILR, we refer the readers to Gupta et al. [13] and Di Crescenzo [11]. Much of the earlier literature is cited in those papers where definitions, inter-relations and discussion of above classes are presented. In the following, we provide a similar preservation results for the IMIT class.

Theorem 3.2.

Let X be a random variable with continuous distribution function, and let $X(\theta)$ have proportional reversed hazards.

(*a*) If $X(\theta)$ is IMIT for some $\theta > 1$, then X is IMIT.

(*b*) If X is IMIT, then $X(\theta)$ is IMIT for any $0 < \theta < 1$.

Proof.

Fix a $\theta > 0$. Define the function $h^{(\theta)}$ by $h^{(\theta)}(u) = (1-u)^{\theta}$ for $u \in (0,1)$. First we show that

$$X(\theta) \text{ is IMIT} \Leftrightarrow X_{(s)} \leq_{ew}^{(h^{(\theta)})} X_{(t)}, \text{ for all } s < t. \tag{3.5}$$

To see it, from (1.3) it follows that $X(\theta) \in \text{IMIT}$ is equivalent to

$$(X(\theta))_{(s)} \leq_{ew} (X(\theta))_{(t)}, \text{ for all } s < t,$$

which, by a straightforward computation, is equivalent to

$$\int_{F_{(s)}^{-1}(q)}^{\infty} [\overline{F}_{(s)}(x)]^{\theta} dx \leq \int_{F_{(t)}^{-1}(q)}^{\infty} [\overline{F}_{(t)}(x)]^{\theta} dx, \quad p \in (0,1), \tag{3.6}$$

where $q = 1-(1-p)^{1/\theta}$. As p varies from 0 to 1, so does $q = 1-(1-p)^{1/\theta}$ and hence (3.6) can be written as

$$\int_{F_{(s)}^{-1}(p)}^{\infty} h^{(\theta)}(F_{(s)}(x))\, dx \leq \int_{F_{(t)}^{-1}(p)}^{\infty} h^{(\theta)}(F_{(t)}(x))\, dx, \quad p \in (0,1);$$

that is, $X_{(s)} \leq_{ew}^{(h^{(\theta)})} X_{(t)}$ whenever $s < t$.

Let us observe that

$$X \text{ is } IMIT \Leftrightarrow X_{(s)} \leq_{ew}^{(h^{(1)})} X_{(t)}, \text{ for all } s < t. \tag{3.7}$$

Now, if $\theta > 1$, then $h^{(1)}/h^{(\theta)}$ is increasing on $(0,1)$, and part (a) follows from Theorem 2.1, (3.5) and (3.7). On the other hand, if $0 < \theta < 1$, then $h^{(\theta)}/h^{(1)}$ is increasing on $(0,1)$, and part (b) follows, again, from Theorem 2.1, (3.5) and (3.7). ∎

3.2. Proportional hazard model

For $\theta > 0$, let $X(\theta)$ denote a random variable with survival function $(\overline{F})^{\theta}$. In reliability theory terminology, different $X(\theta)$'s are said to have PH model (see, Cox [10]). Denote by $F^{(\theta)}$ the distribution function of $X(\theta)$; that is, $F^{(\theta)} = 1 - (\overline{F})^{\theta}$. Similarly, if Y is a non-negative random variable with distribution function G, then denote by $Y(\theta)$ a random variable with distribution function $G^{(\theta)} = 1 - (\overline{G})^{\theta}$. The proportional hazard analog of Theorem 3.1 is the following.

Theorem 3.3.

Let X and Y be two random variables with continuous distribution functions. Let $X(\theta)$ and $Y(\theta)$ have proportional hazards as described above.

(a) If $X \leq_{ew} Y$, then $X(\theta) \leq_{ew} Y(\theta)$ for all $\theta \in (0,1)$.

(b) If $X(\theta) \leq_{ew} Y(\theta)$ for some $\theta > 1$, then $X \leq_{ew} Y$.

Proof.

Fix a $\theta > 0$. Define $h^{(\theta)}(u) = (1-u)^{\theta}$ for $u \in (0,1)$. A similar argument to that in the proof of Theorem 3.1 can establish that

$$X(\theta) \leq_{ew} Y(\theta) \Leftrightarrow X \leq_{ew}^{(h^{(\theta)})} Y. \tag{3.8}$$

Note also that

$$X \leq_{ew} Y \Leftrightarrow X \leq_{ew}^{(h^{(1)})} Y. \tag{3.9}$$

Now, if $0 < \theta < 1$ then $h^{(\theta)}/h^{(1)}$ is increasing on $(0,1)$, and part (a) follows from Theorem 2.1, (3.8) and (3.9). On the other hand, if $\theta > 1$ then $h^{(1)}/h^{(\theta)}$ is increasing on $(0,1)$, and part (b) follows, again, from Theorem 2.1, (3.8) and (3.9). ∎

If $\theta = n$, where n is a positive integer, then $(\overline{F})^n$ is the survival function of $\min\{X_1, X_2, \ldots, X_n\}$, where $X_1, X_2, \ldots, X_n$ are independent copies of X; that is, $(\overline{F})^n$ is the survival function of a series system of size n where the component lifetimes are independent copies of X. If we take $\theta = n$ in Theorem 3.3(b), we obtain Theorem 3.2(a) of Li and Yam [20]; that is, if the lifetime of one series system with identical components is comparable to the lifetime of another series system with identical components with respect to the excess wealth order, then the lifetimes of components in two systems are also comparable with respect to the excess wealth order.

If $\theta < 1$, then $(\overline{F})^\theta$ is the survival function of the lifetime of a component with lifetime X which is subjected to imperfect repair procedure, where θ is the probability of minimal (rather than perfect) repair (see Brown and Proschan [8]). Theorem 3.3(a) states that the excess wealth order is preserved under imperfect repair.

For the proportional hazard models, Brown and Proschan [8] proved that if X is IFR, NBU and DMRL then so is $X(\theta)$ for $0 < \theta < 1$. The next result is the special case of Theorem 2.2 in Brown and Proschan [8]. We state it here and give a new proof.

Theorem 3.4.

Let X be a random variable with continuous distribution function, and let $X(\theta)$ have proportional hazards.

(a) If $X(\theta)$ is DMRL for some $\theta > 1$, then X is DMRL.

(b) If X is DMRL, then $X(\theta)$ is DMRL for any $0 < \theta < 1$.

Proof.

Fix a $\theta > 0$. Define $h^{(\theta)}(u) = (1-u)^\theta$ for $u \in (0,1)$. A similar argument to that in the proof of Theorem 3.2 can establish the desired result by using the facts that

$$X(\theta) \text{ is DMRL} \Leftrightarrow X_t \leq_{ew}^{(h^{(\theta)})} X_s, \quad \text{for all } s < t,$$

and that

$$X \text{ is DMRL} \Leftrightarrow X_t \leq_{ew}^{(h^{(1)})} X_s, \quad \text{for all } s < t.$$

This completes the proof. ∎

If we take $\theta = n$, a positive integer, in Theorem 3.4 (a), we obtain Theorem 2.2 (ii) of Li and Yam [20], which concerns the reversed preserved property of DMRL under the formation of series systems. Next, in Theorem 3.5 below, we obtain a preservation result of the NBUT class under the proportional hazards model. First, we recall the following lemma, which is due to Li and Shaked [18].

Lemma 3.1.

Let X and Y be two random variables with continuous distribution functions, having 0 as the common left endpoint of their supports. Let h_1, $h_2 \in \mathcal{H}$. Suppose that

$$h_2(u)/h_1(u) \text{ is decreasing in } (0,1).$$

Then

$$X \leq_{ttt}^{(h_1)} Y \Rightarrow X \leq_{ttt}^{(h_2)} Y.$$

Theorem 3.5.

Let X be a random variable with continuous distribution function, and let $X(\theta)$ have proportional hazards.

(*a*) If X is NBUT, then $X(\theta)$ is NBUT for each $\theta > 1$.

(*b*) If $X(\theta)$ is NBUT for some $\theta \in (0,1)$, then X is NBUT.

Proof.

Fix a $\theta > 0$. Define the function $h^{(\theta)}$ by $h^{(\theta)}(u) = (1-u)^\theta$ for $u \in (0,1)$. First we show that

$$X(\theta) \text{ is NBUT} \Leftrightarrow X_t \leq_{ttt}^{(h^{(\theta)})} X. \tag{3.10}$$

To see it, write the NBUT property of $X(\theta)$ as

$$\int_0^{\left(F^{(\theta)}\right)^{-1}(p)} [\overline{F}(x)]^\theta dx \geq \int_0^{\left(F_t^{(\theta)}\right)^{-1}(p)} [\overline{F}_t(x)]^\theta dx, \quad p \in (0,1), \tag{3.11}$$

where $\overline{F}_t$ is the survival function of X_t. A straightforward computation yields that

$$\left(F^{(\theta)}\right)^{-1}(p) = F^{-1}\left(1-(1-p)^{1/\theta}\right) \quad \text{and} \quad \left(F_t^{(\theta)}\right)^{-1}(p) = F_t^{-1}\left(1-(1-p)^{1/\theta}\right).$$

Thus (3.11) reduces to

$$\int_0^{F^{-1}(q)} h^{(\theta)}(F(x))dx \geq \int_0^{F_t^{-1}(q)} h^{(\theta)}(F_t(x))dx, \quad p \in (0,1), \tag{3.12}$$

where $q = F^{-1}\left(1-(1-p)^{1/\theta}\right)$. As p varies from 0 to 1, so does $q = q(p)$. So (3.12) is equivalent to $X_t \leq_{ttt}^{(h^{(\theta)})} X$.

Let us observe that

$$X \text{ is NBUT} \Leftrightarrow X_t \leq_{ttt}^{(h^{(1)})} X. \tag{3.13}$$

Now, if $\theta > 1$, then $h^{(\theta)}/h^{(1)}$ is decreasing on $(0,1)$, and part (a) follows from Lemma 3.1, (3.10) and (3.13). On the other hand, if $\theta < 1$, then $h^{(1)}/h^{(\theta)}$ is decreasing on $(0,1)$, and part (b) follows, again, from Lemma 3.1, (3.10) and (3.13). ∎

Corollary 2.1 of Ahmad, Kayid and Li [3] states that NBUT is closed under the formation of series systems with i.i.d. components. This is a special consequence of Theorem 3.3(a) with $\theta = n$. Part (b) of Theorem 3.5 gives the reversed preservation property of NBUT under the formation of imperfect repair of a component.

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