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Type-III LDPC Convolutional Codes

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Abstract

A class of geometrically structured quasi-cyclic low-density parity-check codes with a cylinder structure graph and girth 12 is considered. The parity-check matrix of such a code consists of blocks of zeros and circulant permutation matrices. A class of convolutional codes is assigned to these codes. The convolutional form generator matrix of these codes is obtained by elementary row and column operations. It is shown that the free distance of the obtained convolutional codes is equal to the minimum distance of their bock code counterparts.

Keywords: Quasi-cyclic LDPC code, Convolution code

1 Introduction

Low-density parity-check (LDPC) codes, introduced first by Gallager [4], were rediscovered in 1980[14] and 1996 [7][8]. An LDPC code $\mathcal{C}$ represented by a parity-check matrix H is called ultra-sparse if H has uniform column-weight 2. Though the distance properties of column-weight 2 binary LDPC codes are not as good as those for LDPC codes with column-weight $j \geq 3$, it has been shown in [5] that the non-binary version of these codes can achieve near-Shannon-limit performance. Also when compared with LDPC codes with column-weight $j \geq 3$, the column-weight 2 codes have their own advantages: their encoding and decoding operations is with lower computation and storage complexity;

they have better block error statistics and potential in such applications as partial response channels [9][10][11]. Quasi-cyclic (QC) LDPC codes are more attractive due to their linear-time encoding process and the small size of the required memory [3].

Relations between QC codes and convolution codes have been addressed in [2][6][12][13]. We apply the approach presented in [13] to assign a class of convolutional codes to the ultra-sparse girth-12 QC LDPC codes introduced in [1]. The general form of the generator matrix $G(D)$ of the associated convolutional codes is obtained by applying row and column operations on the parity-check matrix $H(D)$. Relationship between the minimum distance of these QC LDPC codes and the free distance of the corresponding convolutional codes is established.

Section 2 contains necessary backgrounds. The general form of the generator matrices of constructed convolutional codes is given in Section 3. It is shown in Section 4 that the minimum distance of the applied maximum-girth QC LDPC codes is equal to the free distance of the associated convolutional codes.

2 Preliminaries and Notations

Let m, s be positive integers with $0 \leq s \leq m-1$. The circulant permutation matrix $\mathcal{I}_m^s$, denoted $\mathcal{I}^s$ when m is known, is defined by $\mathcal{I}^s = \begin{pmatrix} 0 & \mathcal{I}_s \\ \mathcal{I}_{m-s} & 0 \end{pmatrix}$, where $\mathcal{I}_t$ is the $t \times t$ identity matrix. Let $l \geq 2$ be a positive integer. For $1 \leq i \leq l$ set $n_i := \lfloor \frac{l}{2} \rfloor$ when i is odd, else $n_i := \lceil \frac{l}{2} \rceil$, and suppose $r := n_1 + n_2 \geq 5$ and rl is even. Moreover, consider $A :=< A_1, A_2, ..., A_l >$ where $A_i = (s_{1,i}, s_{2,i}, ..., s_{n_i-1,i})$ for $1 \leq i \leq l-1$, and $A_l = (s_{1,l}, s_{2,l}, ..., s_{n_l,l})$, for some non-negative integers $0 \leq s_{j,k} \leq m-1$.

The Type-III width-m length-l code with regularity r and slop set A, denoted $T3(m, l, r, A)$, introduced in [1], is the QC LDPC code represented by the following parity-check matrix H:

$$H = \begin{pmatrix} H_1' & & & & H_l'' \\ H_1'' & H_2' & & & \\ & H_2'' & & & \\ & & \ddots & \ddots & \\ & & & H_{l-1}'' & H_l' \end{pmatrix} \quad (1)$$

where

$$\begin{cases} \begin{pmatrix} H_1' \\ H_1'' \end{pmatrix} = \begin{pmatrix} \mathcal{I} & \mathcal{I} & \cdots & \mathcal{I} \\ \mathcal{I} & \mathcal{I}^{s_{1,1}} & \cdots & \mathcal{I}^{s_{1,n_1-1}} \end{pmatrix} \\ \begin{pmatrix} H_2' \\ H_2'' \end{pmatrix} = \begin{pmatrix} \mathcal{I} & \mathcal{I} & \cdots & \mathcal{I} \\ \mathcal{I} & \mathcal{I}^{s_{2,1}} & \cdots & \mathcal{I}^{s_{2,n_2-1}} \end{pmatrix} \\ H_{l-1}'' = \begin{pmatrix} \mathcal{I} & \mathcal{I}^{s_{l-1,1}} & \cdots & \mathcal{I}^{s_{l-1,n_{l-1}-1}} \end{pmatrix} \\ \begin{pmatrix} H_l'' \\ H_l' \end{pmatrix} = \begin{pmatrix} \mathcal{I}^{s_{l,1}} & \cdots & \mathcal{I}^{s_{l,n_l}} \\ \mathcal{I} & \cdots & \mathcal{I} \end{pmatrix} \end{cases}$$

In [13] a convolutional code has been defined by a syndrome former transfer matrix $H^T(D)$ whose entries are polynomials in the indeterminate delay operator D. Following the approach used in [13], we construct LDPC convolutional codes associated with the Type-III codes. In the context of convolutional codes a circulant block matrix $\mathcal{I}^s,\ s \geq 0$, is denoted by D^s if $s > 0$ and by 1 when $s = 0$, where D stands for the delay operation. Thus the matrix H given by (1) is changed to the following $l \times \frac{rl}{2}$ matrix $H(D)$.

$$H(D) = \begin{pmatrix} H_1'(D) & & & & H_l''(D) \\ H_1''(D) & H_2'(D) & & & \\ & H_2''(D) & & & \\ & & \ddots & \ddots & \\ & & & H_{l-1}''(D) & H_l'(D) \end{pmatrix} \qquad (2)$$

where

$$\begin{cases} \begin{pmatrix} H_1'(D) \\ H_1''(D) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & D^{s_{1,1}} & \cdots & D^{s_{1,n_1-1}} \end{pmatrix} \\ \begin{pmatrix} H_2'(D) \\ H_2''(D) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & D^{s_{2,1}} & \cdots & D^{s_{2,n_2-1}} \end{pmatrix} \\ H_{l-1}''(D) = \begin{pmatrix} 1 & D^{s_{l-1,1}} & \cdots & D^{s_{l-1,n_{l-1}-1}} \end{pmatrix} \\ \begin{pmatrix} H_l''(D) \\ H_l'(D) \end{pmatrix} = \begin{pmatrix} D^{s_{l,1}} & \cdots & D^{s_{l,n_l}} \\ 1 & \cdots & 1 \end{pmatrix} \end{cases}$$

It is worth mentioning that though the matrix H given by (1) is not full-rank, with one redundant row, its convolution form, $H(D)$, given by (2) is a full-rank matrix.

3 Generators of Type-III LDPC Convolutional Codes

Obtaining the generator matrix $G(D)$ associated with $H(D)$ is not in general an easy task. However, due to the structure of Type-III codes, determination of the generator $G(D)$ of the associated LDPC convolutional codes is straightforward.

Example 1 Set $l = 4$, $r = 5$. The corresponding LDPC convolutional cod is represented by the following parity-check matrix $H(D)$.

$$H(D) = \begin{pmatrix} 1 & 1 & & & & & & D^{s_{4,1}} & D^{s_{4,2}} & D^{s_{4,3}} \\ 1 & D^{s_{1,1}} & 1 & 1 & 1 & & & & & \\ & & 1 & D^{s_{2,1}} & D^{s_{2,2}} & 1 & 1 & & & \\ & & & & & 1 & D^{s_{3,1}} & 1 & 1 & 1 \end{pmatrix}$$

By elementary row and column operations we obtain the following matrix $H'(D)$ wherein $\Delta(D) = 1+D^{s_{1,1}}, a_1(D) = 1+D^{s_{2,1}}, a_2(D) = 1+D^{s_{2,2}}, a_3(D) = 1 + D^{s_{3,1}}, a_4(D) = 1 + D^{s_{4,1}}, a_5(D) = 1 + D^{s_{4,2}}, a_6(D) = 1 + D^{s_{4,3}}, b_1(D) = 1 + D^{s_{4,1}+s_{1,1}}, b_2(D) = 1 + D^{s_{4,2}+s_{1,1}}, b_3(D) = 1 + D^{s_{4,3}+s_{1,1}}$.

$$H'(D) := \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{a_2(D)}{\Delta(D)} & \frac{a_1(D)}{\Delta(D)} & \frac{a_3(D)}{\Delta(D)} & \frac{b_1(D)}{\Delta(D)} & \frac{b_2(D)}{\Delta(D)} & \frac{b_3(D)}{\Delta(D)} \\ 0 & 1 & 0 & 0 & \frac{a_2(D)}{\Delta(D)} & \frac{a_1(D)}{\Delta(D)} & \frac{a_3(D)}{\Delta(D)} & \frac{a_4(D)}{\Delta(D)} & \frac{a_5(D)}{\Delta(D)} & \frac{a_6(D)}{\Delta(D)} \\ 0 & 0 & 1 & 0 & D^{s_{2,2}} & D^{s_{2,1}} & a_3(D) & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & D^{s_{3,1}} & 1 & 1 & 1 \end{pmatrix}$$

A generator matrix $[\mathcal{I}_k \ A]$ of a given binary linear code $\mathcal{C}$ introduces the parity-check matrix $[A^T \ \mathcal{I}_{n-k}]$ for $\mathcal{C}$. Applying this and elementary row and column operations we obtain the following generator matrix $G(D)$: It is a full-rank matrix with rows orthogonal to the rows of $H(D)$.

$$G(D) = \begin{pmatrix} \frac{1+D^{s_{2,1}}}{\Delta(D)} & \frac{1+D^{s_{2,1}}}{\Delta(D)} & D^{s_{2,1}} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1+D^{s_{2,2}}}{\Delta(D)} & \frac{1+D^{s_{2,2}}}{\Delta(D)} & D^{s_{2,2}} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1+D^{s_{3,1}}}{\Delta(D)} & \frac{1+D^{s_{3,1}}}{\Delta(D)} & 1+D^{s_{3,1}} & 0 & 0 & D^{s_{3,1}} & 1 & 0 & 0 & 0 \\ \frac{1+D^{s_{4,1}+s_{1,1}}}{\Delta(D)} & \frac{1+D^{s_{4,1}}}{\Delta(D)} & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1+D^{s_{4,2}+s_{1,1}}}{\Delta(D)} & \frac{1+D^{s_{4,2}}}{\Delta(D)} & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \frac{1+D^{s_{4,3}+s_{1,1}}}{\Delta(D)} & \frac{1+D^{s_{4,3}}}{\Delta(D)} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition 1 Suppose $n = (r-2)l/2 = \sum_{h=1}^{l}(n_h - 1)$ is the length of a $T3(m, l, r, A)$ code. Then the *φ-function* for $1 \le i \le n$ is defined as follows:

$$\varphi(i) := \begin{cases} 1, i+1, \text{ if } 1 \le i \le (n_1 - 1) - 1; \\ 2, i+1-(n_1-1), \text{ if } n_1 - 1 \le i \le (n_1-1)+(n_2-1)-1; \\ \cdots \\ t+1, i+1-\sum_{h=1}^{t}(n_h-1), \text{ if } \sum_{h=1}^{t}(n_h-1) \le i \\ \qquad \le \sum_{h=1}^{t+1}(n_h-1)-1; \\ \cdots \\ l-1, i+1-\sum_{h=1}^{l-2}(n_h-1), \text{ if } \sum_{h=1}^{l-2}(n_h-1) \le i \\ \qquad \le \sum_{h=1}^{l-1}(n_h-1)-1; \\ l, i+1-\sum_{h=1}^{l-1}(n_h-1), \text{ if } \sum_{h=1}^{l-1}(n_h-1) \le i \le \sum_{h=1}^{l}(n_h-1). \end{cases}$$

Theorem 1 The generator matrix $G(D)$ of the LDPC convolutional code associated with the parity-check matrix $H(D)$ given by (2), is the $\frac{(r-2)l}{2} \times \frac{rl}{2}$ matrix $G(D) = (g_{i,j}(D))$ where $g_{i,j}(D)$ is defined below.

$$g_{i,j}(D) = \begin{cases} \frac{D^{s_{1,1}}+D^{s_{\varphi(i)}}}{1+D^{s_{1,1}}}, & \text{if } j=1, 1 \leq i \leq n_1 - 2; \\ \frac{1+D^{s_{1,1}+s_{\varphi(i)}}}{1+D^{s_{1,1}}}, & \text{if } j=1, (r-2)l/2 - n_l + 1 \leq i \leq (r-2)l/2; \\ \frac{1+D^{s_{\varphi(i)}}}{1+D^{s_{1,1}}}, & \text{if } j=2 \quad \text{or} \quad j=1, n_1 - 1 \leq i \leq (r-2)l/2 - n_l; \\ 1+D^{s_{\varphi(i)}}, & \text{if } \sum_{h=1}^{t+1}(n_h - 1) \leq i \leq (r-2)l/2 - n_l, \\ & \quad j = \sum_{h=1}^{t} n_h + 1, 1 \leq t \leq l-2; \\ D^{s_{\varphi(i)}}, & \text{if } \sum_{h=1}^{t}(n_h - 1) \leq i \leq \sum_{h=1}^{t+1}(n_h-1) - 1, \\ & \quad j = \sum_{h=1}^{t} n_h + 1, 1 \leq t \leq l-1; \\ 1, & \text{if } (r-2)l/2 - n_l + 1 \leq i \leq (r-2)l/2, \\ & \quad j = \sum_{h=1}^{t} n_h + 1, 1 \leq t \leq l-1, \\ & \text{or } \ 3 \leq j \leq n_1, i = j-2, \\ & \text{or} \sum_{h=1}^{t} n_h + 2 \leq j \leq \sum_{h=1}^{t+1} n_h, \\ & \quad i = j - t - 2, 1 \leq t \leq l-2, \\ & \text{or } rl/2 - n_l + 1 \leq j \leq rl/2, i = j - l. \end{cases}$$

In other words, the $\frac{(r-2)l}{2} \times \frac{rl}{2}$ matrix $G(D)$ is in the following form wherein $D' := 1 + D^{s_{l-1,n_{l-1}-1}}$.

$$G(D) = \begin{pmatrix} \frac{D^{s_{1,1}}+D^{s_{1,2}}}{1+D^{s_{1,1}}} & \frac{1+D^{s_{1,2}}}{1+D^{s_{1,1}}} & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \frac{D^{s_{1,1}}+D^{s_{1,3}}}{1+D^{s_{1,1}}} & \frac{1+D^{s_{1,3}}}{1+D^{s_{1,1}}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{D^{s_{1,1}}+D^{s_{1,n_1-1}}}{1+D^{s_{1,1}}} & \frac{1+D^{s_{1,n_1-1}}}{1+D^{s_{1,1}}} & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \frac{1+D^{s_{2,1}}}{1+D^{s_{1,1}}} & \frac{1+D^{s_{2,1}}}{1+D^{s_{1,1}}} & 0 & \cdots & 0 & D^{s_{2,1}} & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1+D^{s_{2,n_2-1}}}{1+D^{s_{1,1}}} & \frac{1+D^{s_{2,n_2-1}}}{1+D^{s_{1,1}}} & 0 & \cdots & 0 & D^{s_{2,n_2-1}} & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \frac{1+D^{s_{3,1}}}{1+D^{s_{1,1}}} & \frac{1+D^{s_{3,1}}}{1+D^{s_{1,1}}} & 0 & \cdots & 0 & 1+D^{s_{3,1}} & 0 & \cdots & 0 & D^{s_{3,1}} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1+D^{s_{3,n_3-1}}}{1+D^{s_{1,1}}} & \frac{1+D^{s_{3,n_3-1}}}{1+D^{s_{1,1}}} & 0 & \cdots & 0 & 1+D^{s_{3,n_3-1}} & 0 & \cdots & 0 & D^{s_{3,n_3-1}} & 0 & \cdots & 0 \\ \frac{1+D^{s_{4,1}}}{1+D^{s_{1,1}}} & \frac{1+D^{s_{4,1}}}{1+D^{s_{1,1}}} & 0 & \cdots & 0 & 1+D^{s_{4,1}} & 0 & \cdots & 0 & 1+D^{s_{4,1}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{D'}{1+D^{s_{1,1}}} & \frac{D'}{1+D^{s_{1,1}}} & 0 & \cdots & 0 & D' & 0 & \cdots & 0 & D' & 0 & \cdots & 0 \\ \frac{1+D^{s_{1,1}+s_{l,1}}}{1+D^{s_{1,1}}} & \frac{1+D^{s_{l,1}}}{1+D^{s_{1,1}}} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1+D^{s_{1,1}+s_{l,n_l}}}{1+D^{s_{1,1}}} & \frac{1+D^{s_{l,n_l}}}{1+D^{s_{1,1}}} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 1 \end{pmatrix}$$

Proof. As mentioned before, the rows of $G(D)$ are independent, since $G(D)$ contains the identity matrix of order $\frac{(r-2)l}{2}$. Therefore, it suffices to show that $H(D) * G^T(D) = \mathbf{0}$. For $H(D) = (h_{i,j}(D))_{l\times \frac{rl}{2}}$, we have:

$$h_{i,j}(D) = \begin{cases} 1, & \text{if} \quad \sum_{h=1}^{i} n_h - n_1 + 1 \le j \le \sum_{h=1}^{i} n_h \quad \text{or} \\ & \quad i \ge 2, j = \sum_{h=1}^{i-1} n_h - n_{i-1} + 1; \\ D^{s_{l,t}}, & \text{if} \quad i = 1, 1 \le t = j - \sum_{h=1}^{l-1} n_h \le n_l; \\ D^{s_{i-1,t-1}}, & \text{if } i \ge 2, 2 \le t = j - \sum_{h=1}^{i-1} n_h + n_{i-1} \le n_{i-1}. \end{cases}$$

Setting $P(D) := (p_{i,k}(D))_{l\times \frac{(r-2)l}{2}} = H(D) * G^T(D)$ we get

$$p_{i,k}(D) = \sum_{j=1}^{rl/2} h_{i,j}(D) g_{k,j}(D).$$

To complete the proof, the following cases are considered.

a. If $i = 1$ and $t = \sum_{h=1}^{l-1} n_h$, then it follows from $h_{1,j}(D) = 0, \ n_1+1 \le j \le t$, that

$$\begin{aligned} p_{1,k}(D) &= \sum_{j=1}^{rl/2} h_{1,j}(D) g_{k,j}(D) \\ &= \sum_{j=1}^{n_1} h_{1,j}(D) g_{k,j}(D) + \sum_{j=n_1+1}^{t} h_{1,j}(D) g_{k,j}(D) + \sum_{j=t+1}^{t+n_l} h_{1,j}(D) g_{k,j}(D) \\ &= \sum_{j=1}^{n_1} g_{k,j}(D) + \sum_{j=t+1}^{t+n_l} D^{s_{l,j-t}} g_{k,j}(D). \end{aligned}$$

The following three subcases are examined.

a.1. If $j \ge t+1 > n_1$ or $j \ge 3$ and $j \ne k+2$ then $g_{k,j}(D) = 0$. Therefore, if $1 \le k \le n_1 - 2$ then

$$\begin{aligned} p_{1,k}(D) &= g_{k,1}(D) + g_{k,2}(D) + \sum_{j=3}^{n_1} g_{k,j}(D) + \sum_{j=t+1}^{t+n_l} D^{s_{l,j-t}} g_{k,j}(D) \\ &= \frac{D^{s_{1,1}} + D^{s_{\varphi(k)}}}{1+D^{s_{1,1}}} + \frac{1+D^{s_{\varphi(k)}}}{1+D^{s_{1,1}}} + g_{k,k+2}(D) = 1 + 1 = 0. \end{aligned}$$

a.2. If $3 \le j \le n_1$ or $j \ge t+1$ then $g_{k,j}(D) = 0$. Thus, if $n_1 - 1 \le k \le \sum_{h=1}^{l-1} (n_h - 1) - 1 = \frac{(r-2)l}{2} - n_l$ then $p_{1,k}(D) = \sum_{j=1}^{2} g_{k,j}(D) + \sum_{j=3}^{n_1} g_{k,j}(D) + \sum_{j=t+1}^{t+n_l} D^{s_{l,j-t}} g_{k,j}(D) = \sum_{j=1}^{2} \frac{1+D^{s_{\varphi(k)}}}{1+D^{s_{1,1}}} + 0 + 0 = 0$.

a.3. We have $g_{k,k+1}(D) = 1$, and if $3 \le j \le n_1$ or $\frac{rl}{2} - n_l + 1 = t+1 \le j \le t + n_l = \frac{rl}{2}$, then $g_{k,j}(D) = 0$. Therefore, if $\frac{(r-2)l}{2} - n_l + 1 \le k \le \frac{(r-2)l}{2}$ then

$$\begin{aligned} p_{1,k}(D) &= g_{k,1}(D) + g_{k,2}(D) + \sum_{j=3}^{n_1} g_{k,j}(D) + \sum_{j=t+1}^{t+n_l} D^{s_{l,j-t}} g_{k,j}(D) \\ &= \frac{1+D^{s_{1,1}+s_{\varphi(k)}}}{1+D^{s_{1,1}}} + \frac{1+D^{s_{\varphi(k)}}}{1+D^{s_{1,1}}} + 0 + D^{s_{l,k+1-t}} = D^{s_{\varphi(k)}} + D^{s_{l,k+l-t}} = 0. \end{aligned}$$

Note that in this case we have $\varphi(k) = l, k + 1 - \sum_{h=1}^{l-1} (n_h - 1) = l, k + l - t$.

b. If $i > 1$ and $t = \sum_{h=1}^{i-1} n_h$, then

$$\begin{aligned} p_{i,k}(D) &= \sum_{j=1}^{rl/2} h_{i,j}(D) g_{k,j}(D) \\ &= \sum_{j=1}^{t-n_{i-1}} h_{i,j}(D) g_{k,j}(D) + h_{i,t-n_{i-1}+1}(D) g_{k,t-n_{i-1}+1}(D) \\ &+ \sum_{j=t-n_{i-1}+2}^{t} h_{1,j}(D) g_{k,j}(D) + \sum_{j=t+1}^{t+n_i} h_{1,j}(D) g_{k,j}(D) \\ &= g_{k,t-n_{i-1}+1}(D) + \sum_{j=t-n_{i-1}+2}^{t} D^{s_{i-1,j-t+n_{i-1}-1}} g_{k,j}(D) + \sum_{j=t+1}^{t+n_i} g_{k,j}(D). \end{aligned}$$

This is divided into subcases **b.1** with $i = 2$ and **b.2** specified by the constraint $l \geq i > 2$. The proof of these cases is essentially the same as the arguments given above for the case **a.**, and hence is omitted. ■

4 Free distance of Type-III LDPC Convolutional Codes

The following theorem establishes a minimum distance relationship between Type-III QC LDPC codes and the associated Type-III LDPC convolutional codes.

Theorem 2 Let $\mathcal{C}$ be a Type-III QC LDPC code code with maximum girth 12 and minimum distance d. Also, let $\mathcal{C}(D)$ be the corresponding Type-III LDPC convolutional code with free distance d_{free}. Then $d = d_{free}$.

Proof. It can be easily verified that the minimum distance of a column-weight 2 code is equal to half of its girth. Thus $d = 6$. On the other hand, in any convolutional code with free distance d_{free} associated to a QC code with minimum distance d we have $d \leq d_{free}$ (Theorem 1 in [13]). Therefore, it suffices to show that $d_{free} \leq d = 6$. Encoding the length-l vector $U(D) = [u_0(D), u_1(D), ..., u_{l-1}(D)] = [1+D^{s_{1,1}}, 0, ..., 0]$, we obtain the length-$\frac{rl}{2}$ vector $V(D) = U(D)G(D) = [D^{s_{1,1}} + D^{s_{1,2}}, 1 + D^{s_{1,2}}, 1 + D^{s_{1,1}}, 0, ..., 0]$ having the following representation:

$$\begin{aligned} v(D) &= V_0(D^{rl/2}) + D \times V_1(D^{rl/2}) + D^2 \times V_2(D^{rl/2}) \\ &+ \cdots + D^{rl/2-1} \times V_{rl/2-1}(D^{rl/2}) \\ &= D^{rls_{1,1}/2} + D^{rls_{1,2}/2} + D(1 + D^{rls_{1,2}/2}) + D^2(1 + D^{rls_{1,1}/2}) \\ &= D + D^2 + D^{rls_{1,1}/2} + D^{rls_{1,2}/2} + D^{1+rls_{1,2}/2} + D^{2+rls_{1,1}/2}. \end{aligned}$$

We show $w(v(D)) = 6$. To do this, we prove that the powers of D in $v(D)$ are distinct.

If $s_{1,1} = 0$ or $s_{1,2} = 0$ then the girth of $\mathcal{C}$ will be 4. Thus we may assume $l \geq 2$, $r \geq 5$ and $s_{1,1}, s_{1,2} \geq 1$. It follows that $\frac{rls_{1,1}}{2}, \frac{rls_{1,2}}{2} \geq 5$. Not also that if $\frac{rls_{1,1}}{2} = \frac{rls_{1,2}}{2}$ then $s_{1,1} = s_{1,2}$ and hence the girth will be 4, a contradiction. Thus the first four elements of $v(D)$ are distinct. We show that $\frac{rls_{1,1}}{2} + 2 \neq \frac{rls_{1,2}}{2}$ and $\frac{rls_{1,1}}{2} \neq \frac{rls_{1,2}}{2} + 1$.

If $\frac{rls_{1,1}}{2} + 2 = \frac{rls_{1,2}}{2}$ then $rl(s_{1,2} - s_{1,1}) = 4$. However, $s_{1,1} \neq s_{1,2}$, $l \geq 2$ and $r \geq 5$, and hence $|rl(s_{1,2} - s_{1,1})| \geq 10$ which is in contradiction with $rl(s_{1,2} - s_{1,1}) = 4$. Similarly, if $\frac{rls_{1,1}}{2} = \frac{rls_{1,2}}{2} + 1$ then $rl(s_{1,1} - s_{1,2}) = 2$, a contradiction. Therefore, the first six elements of $v(D)$ are distinct and hence $d_{free} \leq w(v(D)) = 6$. ■

5 Conclusion

A class of ultra-sparse QC LDPC codes was applied to generate a class of LDPC convolutional codes. The general form of generator matrices $G(D)$ of the constructed LDPC convolutional codes was formulated. It was shown that the minimum distance of the considered girth-12 QC LDPC codes is equal to the free distance of the associated convolutional codes.

References

[1] M. Esmaeili and M. Gholami, Maximum-girth slope-based quasi-cyclic $(2, k \geq 5)$-LDPC Codes,*IET Communications*, doi:10.1049/iet-com:20080013.

[2] M. Esmaeili, T.A. Gulliver, N.P. Secord and S.A. Mahmoud, A link between quasi-cyclic codes and convolutional codes, *IEEE Trans. Inform. Theory*, 44 (1998), 431–435.

[3] M.P.C. Fossorier, Quasi-cyclic low-density parity-check codes from circulant permutation matrices, *IEEE Trans. Inform. Theory*, 50 (2004), 1788–1794.

[4] R.G. Gallager, Low density parity-check codes,*IEEE Trans. Inform. Theory*, 8 (1962), 21-28.

[5] X.-Y. Hu and E. Eleftheriou, Binary representation of cycle Tanner-graph GF(2b) codes, *Proc. of ICC*, June 2004.

[6] Y. Levy and J. Costello, Jr., An algebraic approach to constructing convolutional codes from quasi-cyclic codes, *DIMACS Series in Discr. Math. and Theor. Comp. Sci.*, 14 (1993), 189–198.

[7] D.J.C. MacKay and R.M. Neal, Near Shannon limit performance of low density parity-check codes, *Electron. Lett.*, 32 (1996), 1645-1646.

[8] D.J.C. MacKay, Good error-correcting codes based on very sparse matrices, *IEEE Trans. Inform. Theory*, 45 (1999), 399-432.

[9] H. Song, J. Liu and B.V.K.V. Kumar, Low complexity LDPC codes for magnetic recording, *IEEE Globecom 2002*, Taipei, Taiwan.

[10] H. Song, J. Liu and B.V.K.V. Kumar, Low complexity LDPC codes for partial response channels, *IEEE Globecom 2002*, vol. 2, pp. 1294–1299, Taipei, Taiwan.

[11] H. Song H., J. Liu and B.V.K.V. Kumar, Large girth cycle codes for partial response channels, *IEEE Trans. on Magnetics*, 40 (2004), 3084–3086.

[12] R.M. Tanner, D. Sridhara, A. Sridharan, T. Fuja and D. Costello Jr., LDPC block and convolutional codes based on circulant matricies, *IEEE Trans. Inform. Theory*, 50 (2004), 2966-2984.

[13] R.M. Tanner, Convolutional codes from quasi-cyclic codes: A link between the theories of block and convolutional codes, *Computer Research Laboratory*, Technical Report, USC-CRL-87-21, Nov. 1987.

[14] R.M. Tanner, A recursive approach to low complexity codes, *IEEE Trans. Inform. Theory*, 27 (1981), 533–547.

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