Contemporary Engineering Sciences, Vol. 1, 2008, no. 2, 63 - 69

Soliton Perturbation Theory for the Generalized Fifth-Order Nonlinear Equation

Anjan Biswas

Center for Research and Education in Optical Sciences and Applications
Department of Applied Mathematics and Theoretical Physics
Delaware State University, Dover, DE 19901-2277, USA
biswas.anjan@gmail.com

Essaid Zerrad

Department of Physics and Pre-Engineering
Delaware State University, Dover, DE 19901-2277, USA

Abstract

The adiabatic parameter dynamics of 1-soliton solution of the generalized fifth-order nonlinear equation is obtained by virtue of the soliton perturbation theory. The adiabatic change of soliton velocity is also obtained in this paper.

1 INTRODUCTION

The dimensionless form of the generalized fifth-order nonlinear equation (gfNE) that is going to be studied in this paper is given by

$$q_t + aqq_{xxx} + bq_xq_{xx} + cq^2q_x + q_{xxxxx} = 0 \tag{1}$$

where a, b and c are real constants parameters. This equation is a generalized version of many equations. Some of these are given in the following table for specific values of a, b and c.

a	b	c	Equation
-15	$-75/2$	45	Kaup-Kupershmidt
-15	-15	45	Sawada-Kotera-I
5	5	5	Sawada-Kotera-II
10	20	30	Lax

These cases have already been studied for the past two decades and are therefore well understood [1-10]. The 1-soliton solution of (1) is given by [6]

$$q(x,t) = \frac{A}{\cosh^2 B\,(x-\bar{x})} \tag{2}$$

where

$$A = \frac{3a\sqrt{v}}{2c} \tag{3}$$

$$B = \frac{\sqrt[4]{v}}{2} \tag{4}$$

and the velocity of the soliton is defined as

$$v = \frac{d\bar{x}}{dt} \tag{5}$$

so that from (3) and (4)

$$B^2 = \frac{cA}{6a} \tag{6}$$

It is to be noted that this 1-soliton solution, for the generalized case, exists for

$$b = \frac{10c - a^2}{a} \tag{7}$$

2 MATHEMATICAL PROPERTIES

Equation (1) has at least two integrals of motion [9] that are known as linear momentum (M) and energy (E). These are respectively given by

$$M = \int_{-\infty}^{\infty} q dx = \frac{2A}{B} \tag{8}$$

and

$$E = \int_{-\infty}^{\infty} q^2 dx = \frac{4A^2}{3B} \tag{9}$$

These conserved quantities are calculated by using the 1-soliton solution given by (2). The center position of the soliton $\bar{x}$ is defined as

$$\bar{x} = \frac{\int_{-\infty}^{\infty} xq dx}{\int_{-\infty}^{\infty} q dx} = \frac{\int_{-\infty}^{\infty} xq dx}{M} \tag{10}$$

where M is defined in (8). Thus, the velocity of the soliton is given by

$$v = \frac{d\bar{x}}{dt} = \frac{\int_{-\infty}^{\infty} xq_t dx}{\int_{-\infty}^{\infty} q dx} = \frac{\int_{-\infty}^{\infty} xq_t dx}{M} \tag{11}$$

On using (1) and (8), the velocity of the soliton reduces to

$$v = \frac{4(b-3a)AB^2}{15} + \frac{8cA^2}{45} \tag{12}$$

3 PERTURBATION TERMS

The perturbed gfKdV equation that is going to be studied in this paper is given by

$$q_t + aqq_{xxx} + bq_x q_{xx} + cq^2 q_x + q_{xxxxx} = \epsilon R \tag{13}$$

where, in (13), ϵ is the perturbation parameter and $0 < \epsilon \ll 1$ [1, 3], while R gives the perturbation terms. In presence of perturbation terms, the momentum and the energy of the soliton do not stay conserved. Instead, they undergo adiabatic changes that lead to the adiabatic deformation of the soliton amplitude, width and a slow change in the velocity [1, 2]. Using (9), the law of adiabatic deformation of the soliton energy is given by [1, 2, 5]

$$\frac{dE}{dt} = 2\epsilon \int_{-\infty}^{\infty} qR dx \tag{14}$$

while the adiabatic law of change of the velocity of the soliton is given by [1, 2, 5]

$$v = \frac{4(b-3a)AB^2}{15} + \frac{8cA^2}{45} + \frac{\epsilon}{M} \int_{-\infty}^{\infty} xR dx \tag{15}$$

In this paper, the perturbation terms that are going to be considered are

$$\begin{aligned} R = {} & \alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q^m q_x + \lambda q_{xxx} + \nu q q_x q_{xx} \\ & + \sigma q_x^3 + \xi q_x q_{xxxx} + \eta q_{xx} q_{xxx} + \rho q_{xxxx} + \psi q_{xxxxx} + \kappa q q_{xxxx} \end{aligned} \tag{16}$$

In (16) for fluids, dissipation gives rise to the first two terms and so α and β are small dissipative coefficients [1, 2]. Also, δ or ψ represent the coefficient of higher order nonlinear dispersive term [4, 8] and m is a positive integer with $1 \leq m \leq 4$ [4, 8]. The coefficient of ρ provide a higher order stabilizing term and must therefore be taken into account [4, 8]. The perturbation term given by coefficient of η was recently considered [1, 2] while the remaining perturbation terms arise in the context of extended version of integrable equations [8].

3.1 APPLICATIONS

In presence of these perturbation terms, the adiabatic variation of the energy of the soliton is given by

$$\frac{dE}{dt} = \frac{16\epsilon A^2}{105} (35\alpha - 7\beta + 5\rho) \tag{17}$$

Using (9), one can integrate equation (17) to yield

$$A(t) = \left[\frac{105\sqrt{6aA_0}}{105\sqrt{6a} - 4\epsilon t\sqrt{cA_0}\,(35\alpha - 7\beta + 5\rho)} \right]^2 \tag{18}$$

where A_0 is the initial amplitude of the soliton. This leads to the long term behaviour of the soliton amplitude as

$$\lim_{t\to\infty} A(t) = \begin{cases} A_0, & : \quad 7\beta = 35\alpha + 5\rho \\ 0, & : \quad 7\beta \neq 35\alpha + 5\rho \end{cases} \tag{19}$$

The law of the change of velocity for the given perturbation terms in (16) is given by

$$\begin{aligned} v = {} & \frac{4(b-3a)AB^2}{15} + \frac{8cA^2}{45} \\ & -\epsilon\left[\frac{m\delta A^m}{(m+1)(2m+1)} B\left(m, \frac{1}{2}\right)\right. \\ & \left. + \frac{A}{315}\left\{3\left(7\gamma - 14\lambda - 15\xi + 5\eta + 25\kappa\right) + 2\nu A\right\}\right] \end{aligned} \tag{20}$$

4 CONCLUSIONS

In this paper, soliton perturbation theory is used to study the peturbed gfNE. This theory is used to establish the adiabatic parameter dynamics of the soliton energy. This relation is then further exploited to obtain the adiabatic variation of the soliton amplitude. The constraints on the perturbation parameters are then determined under which the soliton amplitude vanishes or stays constant. Also, it is shown that the velocity undergoes a slow change due to these perturbation terms. The quasi-stationary aspects of the perturbed soliton in presence of such perturbation terms will be studied and reported in future publications.

ACKNOWLEDGEMENT

This research of the first author (AB) was fully supported by NSF Grant No: HRD-0630388 and the support is genuinely and sincerely appreciated.

The research work of the second author (EZ) is financially supported by the Army Research Office (ARO) under the award number W911NF-05-1-0451 and this support is thankfully appreciated.

References

[1] A. Biswas & S. Konar. "Soliton perturbation theory for the compound KdV equation". *International Journal of Theoretical Physics*. Vol 46, Number 2, 237-243. (2007).

[2] A. Biswas, S. Konar & E. Zerrad. "Soliton perturbation theory for the generalized modified Degasperis-Procesi Camassa-Holm equation". *International Journal of Modern Mathematics*. Vol 2, Issue 1, 35-40. (2007).

[3] E. Fan & Y. C. Hon. "Applications of the extended tanh method to 'special' types of nonlinear equations". *Applied Mathematics and Computations*. Vol 141, Issues 2-3 351-358. (2003).

[4] Y. Kivshar & B. A. Malomed. "Dynamics of solitons in nearly integrable systems" *Reviews of Modern Physics* Vol 61, No 4, 763-915. (1989).

[5] Y. Kodama & M. J. Ablowitz. "Perturbations of solitons and solitary waves" *Studies in Applied Mathematics* Vol 64, 225-245. (1981).

[6] Q. M. Al-Mdallal & M. I. Syam. "Sine-Cosine method for finding the soliton solutions of the generalized fifth-order nonlinear equation". em Chaos, Solitons & Fractals. Vol 33, 1610-1617. (2007).

[7] X. Niu & Z. Pan. "New approximate solution of the perturbed KdV equation". *Physics Letters A*. Vol 349, 192-197. (2006).

[8] A. R. Osborne. "Approximate asymptotic integration of a higher order water-wave equation using the inverse scattering transform. *Nonlinear Processes in Geophysics*. Vol 4, Number 1, 29-53. (1997).

[9] Y. Ruo-Xia & L. Zhi-Bin. "Conservation laws and new exact solutions for the generalized seventh order KdV equation". *Chaos, Solitons & Fractals*. Vol 20, 259-266. (2004).

[10] P. E. Zhidkov. *Korteweg-de Vries and Nonlinear Schrödinger's Equations: Qualitative Theory*. Springer Verlag. New York, NY. (2001).

Received: December 3, 2007