Modeling and Mathematical Study of Sand Transport in the Desert

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Preface

In this book we will study the asymptotic analysis of sand transport model in the desert. The problem of wind transport of sand, which has given rise in recent years to numerous research works, represents a keen interest among the community of physicists in granular environments. The long-term objective is to arrive at a global understanding of the transport of sand by the wind, in order to apprehend the phenomena of formation and migration of dunes in desert environments. This represents a scientific challenge. On the one hand, the circulation of wind in the atmosphere is non-stationary, turbulent and influenced by the Earth's topography. The modeling of turbulent flows still raises many questions. The objective will be to study mathematically, the deformation and the propagation of sand dunes under the effect of air flow.
First, we will explain how to establish the models with the participation of the parameters and the framework. Then we plan to make a mathematical theoretical study of the proposed models. This mathematical study will focus on homogenization tools by considering a periodic environment.
In the first chapter we have given some useful notions on the observation of the movement of sand qualitative description of the movement transport mechanism aerodynamic forces and total flow of transported materials. These concepts have been well presented and detailed in [3, 7, 17, 21], [20] and [4]; they allow us to fully understand the phenomenon of sand movement in order to be able to model the transport of sand in the desert.
The second chapter is devoted to modeling the transport of sand in the desert.
We presented the Exner equation which models the transport of sand and the flow of Gerkerma, Bagnold, Komarova and Vittori for more details see [12] ,[3] et [25, 12].
By applying the Exner equation to the different models and after sizing we obtain a reference model.
In the third chapter under certain hypotheses we show the existence and the uniqueness of the solution of the sand transport model in the desert which is given by the following equation.

$$\begin{cases} \dfrac{\partial z^\varepsilon}{\partial t} - \dfrac{a}{\varepsilon^j}\nabla\cdot(g_a(|u^\varepsilon|)\nabla\cdot z^\epsilon) = \dfrac{b}{\varepsilon^i}\nabla\cdot(g_c(|u^\varepsilon|\ \dfrac{u^\epsilon}{|u^\epsilon|}), \\ z^{\epsilon,\nu}(0,x) = z_0(x) \end{cases} \tag{1}$$

with $i \in (0,1)$, $j \in (0,1,2)$ and the initial condition associated with (1) is in the form $z^\varepsilon(x,0) = z_0(x)$ with $x \in \mathbb{T}^2$; $z_0 \in H^1(\mathbb{T}^2)$
the functions g_a et g_c have increasing, regular, defined functions of $\mathbb{R}_+$ has values in $\mathbb{R}_+$, a and b are positive real constants,
$z^\varepsilon = z^\varepsilon(t,x)$ defined from $\mathbb{T}^2 \times [0,T] \to \mathbb{R}$ with $\mathbb{T}^2 \subset \mathbb{R}^2$, $t \in [0,T]$ et $T > 0$.

The position $x = (x_1, x_2) \in \mathbb{T}^2$ and the functions u: $[0, T] \times \mathbb{T}^2 \to \mathbb{R}_+$.
In (1), The functions g_a and g_c can cancel each other out so we have degenerate and singularly disturbed parabolic equations. For more details see [14] et Lions [16].
In the back chapter, using the theory of homogenization, we will study the behavior of z^ϵ, model solution (1) when $\epsilon \longrightarrow 0$.

Chapter 1

Preliminary on the transport of sand

In this chapter we give some useful notions on the observation of the sand movement, the qualitative description of the movement, the transport mechanisms, the aerodynamic forces and total flow of transported materials. These concepts have been well presented and detailed in [19, 3, 7, 17, 21] and they allow us to fully understand the phenomenon of sand movement in order to be able to model the transport of sand in the desert.

1.1 Observing the movement of sand

Wind transport is visible in nature during sandstorms. In strong winds, a thick cloud of grains of sand of the order of a meter in height appears above the surface, Bagnold observes that the majority of the grain transport takes place on a layer close to the ground relative to the cloud top. The size of the grains constituting this cloud is of the order of a few hundred microns, but grains having a millimeter in diameter can be found which reach heights of the order of a meter. How can grains of this size reach such heights? One can think that the turbulent fluctuations of the flow are able to entrain the grains to such heights.

1.2 Qualitative description of the movement

When the wind blows over an area of sand, it begins to move. There are three modes of grain transport see [19], [3] and [9]:
-*The suspension*: it is the transport of grains over a very long distance without contact with the ground (essentially restricted to particles of size less than 20 *mum* or to very violent and turbulent winds).
- *The saltation*: it is the movement followed by a grain which is dislodged from the ground by turbulence then driven by the wind but whose mass is large enough for it to fall back down. The grains which follow such a trajectory are called saltons. Upon impact with

the ground, thanks to its high kinetic energy, the salton sets in motion other grains.
- *The crawling*: this mode is similar to the previous one but the movement is slower, smaller because the energy of a repton is weaker than that of a salton.
The process takes place in the following way: when the wind begins to blow, it will roll a few grains which will eventually make leaps. We therefore define a static speed threshold: u_{*t}. If the wind is strong enough, during their leap, the grains will acquire enough energy to entrain other grains upon impact. These grains maintain the transport in such a way that even if the wind speed drops below u_{*t} the transport will continue. We therefore define a dynamic speed threshold u_{*i}.

1.3 Transport mechanism

1.3.1 The wind speed profile

We are interested in the flow of air over a surface of sand at low altitude (typically less than one meter), where wind transport takes place. In the atmospheric boundary layer, the flow is generally turbulent even at low speeds. Indeed, if we calculate the Reynolds number R for velocities V of the order of 5 to
10 m/s (for which wind transport occurs or even cite Fran), we find:

$$R = \frac{LV}{v} \approx 10^5 \gg 1.$$

L is a characteristic length of the flow that we have taken equal to the typical thickness of the layer of grains carried by the wind $(L \approx 1m)$ and v is the kinematic speed of air $(v = 1.4510^{-6} m^2/s)$. To describe a turbulent flow, we use the Reynolds equation, which is nothing other than the Navier-Stokes equation averaged over time (see Guyon et al. 2001) and [9] for more information). In its stationary form, it is written:

$$-\vec{\nabla}p + \mu\nabla^2\vec{V} + \vec{\nabla}\bar{\bar{\tau}} + \vec{F} = 0, \quad (1.1)$$

where $\vec{V}$ is the speed of the fluid, p the pressure, $\bar{\bar{\tau}}$ the tensor of the turbulent stresses and finally $vecF$ any volumetric forces that apply to the fluid. In the case of a turbulent flow (uniform stationary) parallel to a direction x along a plane $z = 0$, the Reynolds equation projected on the axis $0x$ gives :

$$\mu \frac{d^2u}{dz^2} + \frac{d\tau_t}{dz} + F_x = 0, \quad (1.2)$$

where u is the horizontal component of the velocity, τ_t the tangential turbulent stress and finally F_x the horizontal component of the volume forces. In addition, we neglected the pressure gradient in the direction $(0x)$. When we are far enough from the wall, we can neglect the transport of momentum by viscosity before the convective transport by turbulence. The Reynolds equation then simply reduces to:

$$\frac{d\tau_t}{dz} + F_x(z) = 0. \tag{1.3}$$

In the absence of grains in the flow, F_x has a zero value: the friction force exerted by the grains on the wind is zero. By integrating, we get:

$$\tau_t(z) = \tau_{\text{surface}} = Cte. \tag{1.4}$$

where τ_{surface} is the wall stress. The Reynolds equation simply means that the turbulent shear stress is independent of the height but it does not allow us to relate it to the properties of the mean flow. For that, we need an additional relation which is called closure equation. Within the framework of the theory of the mixing length (Prandtl, 1935), one can derive a relation between the turbulent shear stress and the gradient of the average flow, which is written:

$$\tau_t(z) = \rho_a l^2 \left(\frac{du}{dz}\right)^2 \tag{1.5}$$

where l is the mix length and ρ_a the air density.

1.3.2 The turbulent boundary layer

For a turbulent flow on a flat ground, the speed of the flow at the altitude z is classically described by a logarthmic law see [5] , [3] and [9] :

$$u_z = \frac{V_*}{\kappa}\, ln\left(\frac{z}{z_o}\right) \tag{1.6}$$

Here

- κ is the Von Karman constant taken usually equal to 0.4
- z_o is the roughness of the soil, generally of the order of one thirtieth of the characteristic grain size. We consider that below z_o, $u_z = 0$.
- V_* is the shear speed, it characterizes the force that the wind exerts on the ground.

1.3.3 Aerodynamic forces

An isolated grain of sand will be subjected, during its flight in flow, to different aerodynamic forces see [7] and [9] :

- drag force This force represents the friction of the wind on the grain and tends to bring the speed of movement of the grain to that of the fluid:

$$\overrightarrow{F_D} = C_D \rho_a \, d^2 ||\vec{V_\tau}|| \vec{V_r} \tag{1.7}$$

where $\vec{V_\tau}$ denotes the relative speed of the grain compared to the air speed, ρ_a is the density of the air, d is the diameter grain. C_D is the drag coefficient which is a function of the particulate Reynolds number $R_p = \dfrac{dV_\tau}{v}$, with v is the viscosity of air ($v = 1.4510^{-6} m^2/s$) (Morsi and Alexander, 1972) see [17].

-The lifting force This lift force, generated by the shear of the flow, is linked to the pressure difference between the top and the bottom of the grain see [7]:

$$\overrightarrow{F_L} = C_L \rho_a d^2 V_\tau^2 \vec{n} \tag{1.8}$$

where $\vec{n}$ is the unit vector perpendicular to $\overrightarrow{V}_\tau$ is C_L the lift coefficient which is a function of the particulate Reynolds number. This force is negligible at a height of a few grain diameters above the bed, when the speed gradient decreases sharply, but it can be significant at takeoff.

- the strength of Magnus An additional lift force may appear following the proper rotation of the grain under the effect of the flow: this is the Magnus effect that is found in the "cut" or the "lift" given to the ball by table tennis players. This force can be taken into account using an additive term in the factor C_L, of the expression (1.8) of the lift force. Note that this effect is far from negligible in determining the real trajectory of a grain in saltation (White and Schulz, 1977) see [11] and [9].

1.3.4 Relevant flow parameters in flow estimation

The dominant factor responsible for the mobility of particles was pursued by focusing on six parameters: the flow of the flow q, the average speed of the flow U, the slope of the bed of the flow S, the friction of the fluid on the bottom τ, the powers of the flow U_τ according to Bagnold and S_τ according to Yang.

θ : Shields coefficient which is given by the relation

$\theta = \dfrac{\tau}{\rho g(s-1)D_G}$ with $\tau = \frac{|u^2|}{C^2}\frac{u}{|u|}$. Which give

$\theta = \dfrac{|u|^2}{\rho C^2 g(s-1)D_G}\frac{u}{|u|}$ wher ρ denotes the density of the fluid, D_G the diameter of a grain of sand and g gravity.

1.4 Total flow of materials transported

In this part we will give some approaches to the flow as a function of the speed of flow or friction. These approaches are based on the Saint-Venant equations and the Navier-Stokes equations.

1.4.1 Friction-related approaches

• Bagnold [4] defines the total flow q as being the sum of the flows transported in suspension q_s and by bed load q_r

$$q_s \simeq B_1\ q_r\ \ et\ \ q_r \simeq B_2\ (\tau - \tau_s)\ \tau^{1/2} \tag{1.9}$$

B_1 and B_2 are coefficients, function of s, g, v, d. For more information see [4] with τ shear stress.

• Engelung [8] gives a formula similar to that of Meyer-Peter et Muller :

$$q \simeq (\Theta - 0,047)^{3/2}, \tag{1.10}$$

in which the threshold $\Theta_s = \Theta_{s^o} = 0,047$ is constant.
The Meyer-Peter and Muller flow is given by:

$$q_r = \frac{8}{(s-1)gd}(\rho(\tau - \tau_s))^{3/2}$$

with the threshold of the number of Shields defined by $\Theta_s = \frac{\rho \tau_s}{(s-1)gd} \simeq 0,047$.

• For Richards [20], in turbulent conditions and for high shear, suspended water transport can be neglected. The total flow is then reduced to that transported by carting:

$$q = C_R \tau^b, \tag{1.11}$$

with τ the shear stress which is given by the relation $\tau = \rho \frac{|u|^2}{C^2}\frac{u}{|u|}$,ρ is the density of the area, $C = \frac{1}{k} ln(\frac{30\bar{z}}{D_G})$ where D_G being the grain of sand diameter, C_R is a constant depending on the properties of the fluid and the grains and $b = 3/2$.

• Sumer and Bakioglu in [23] make an extension of Richards' study in [20]. They are interested in the impact of viscosity on the formation of wrinkles and everywhere on the assumption that the formation of sand wrinkles at the bottom does not depend on the height of the fluid. The effect of gravity is included in the inclination of the bottom of the slope S. The flow considered is only that of the Bagnold chariage:

$$q = q_r = \frac{0.1 B_s \qquad (\tau - \tau_s)\tau - \tau_s)\tau^{1/2}}{(s-1)g\rho^{3/2}(c\tan\varphi + \tan S)cosS} \tag{1.12}$$

where B_s is a coefficient depending on the Reynolds number of the grains Re_p whose value tends to 8.5 when Re_p becomes high, $Re_p \approx 70$, $0 < c < 1$ and φ the friction angle

• For Blondeaux [6], Vitori and Blondeaux [24], the flow is due to the current and the agitation of the waves:

$$q \simeq |\tau - \Lambda \frac{\partial f}{\partial x}|^b sign\left(\tau - \Lambda\ \frac{\partial f}{\partial x}\right) \quad \text{with} \ \ b = 8.28 \tag{1.13}$$

• Sleath [22] in the case of an oscillating flow, on a horizontal background, estimating q according to its average value Q', established on a half-cycle of oscillation (of pulsation ω) according to:

$$q = \frac{8}{3} Q' cos(\omega t + \delta\varphi) sign(cos(\omega t + \delta\varphi)) \tag{1.14}$$

Q' is expressed by both Sleath and Nielson by:

$$\frac{Q'}{\sqrt{(s-1)d^3}} = \begin{cases} C_s(\Theta - \Theta_s)^{3/2}, & \Theta > \Theta_s \\ 0, & \Theta < \Theta_s \end{cases} \tag{1.15}$$

where C_s is an empirical constant with $C_s = 1.95$ for Sleath and $C_s = 3$ for Nielsen. $\delta\varphi$ is the phase between the transport and the speed of the fluid just above the boundary layer of the bottom (10 degree $< \delta\varphi <$ 20 degree), and Θ is the Shields coefficient.

• Yalin [26] breaks down the total flow into:

$$q = q_s + q_r \tag{1.16}$$

with q_r the flow coming from the thrusting and q_s that from the suspension. In each case, it establishes the values for a flat bottom and makes a correction when the bottom becomes wavy through a factor $\lambda_c = \frac{u_*}{\bar{u_*}}$ which is a dimensionless form of friction, by that of the flat bottom ($\lambda_o = 1$ for a flat bottom and $\lambda_c < 1$ for a wavy bottom). Considering the number of sediments entrained $\zeta^3 = \frac{Re_e^2}{3} = \frac{g(s_1)d^3}{v^2}$, and l relative intensity of the flow $\beta = \frac{\Theta}{\Theta_s}$ (ratio of the number of Shields and its threshold value), it obtains the formula of the flow in suspension (q_s) for a flat bottom:

$$q_s = 0.05\phi_1 u_b y_o\ \zeta^{-1}(\beta - 1) \ \ \text{et} \ \ q_r = \phi_1\ \frac{u_b}{\rho_p g}(\tau - \tau_s). \tag{1.17}$$

• Komarova and coll. [13] [15] consider that taking into account the condition of zero speed in the bottom bed, the speed of the fluid in the bottom bed is not the parameter which sets the grains in motion but rather the parietal friction. τ being the physical friction divided by the density of the fluid, they obtain:

$$q = \alpha|\tau|^b\left(\tau - \Lambda\ \frac{\partial f}{\partial x}\right), \quad b = 1/2. \tag{1.18}$$

Λ is a constant between 1/2 and 6. b is a parameter that translates nonlinearities and α is a proportionality coefficient.
• Yue and Mei [27], in the case of an oscillating flow on an inclined bottom proceed as Sleath [22], the total flux is given by:

$$q \simeq \left(\sqrt{(s-1)d^3 \Theta^{3/2}} \right) \tag{1.19}$$

with Θ a Shields coefficient defined by:

$$\Theta = \frac{1}{\rho g (s-1) d} \left(\tau - \Lambda \, \frac{\partial f}{\partial x} \right), \qquad \Lambda = \frac{\rho g (s-1) d \Theta_s}{tan \; \varphi_s} \tag{1.20}$$

where φ_s is the angle of rest, and Θ_s is the threshold of the Shields coefficient
• By Charru and Mouilleron-Arnold see [?], in the case of a continuous shearing of viscous fluid, the flux is given by the following equation:

$$q = V_s d C (\Theta - \Theta_s)^3, \qquad C = 7.5, \quad \Theta_s = 0.2 \tag{1.21}$$

where Θ is the number of Shields.

1.4.2 Speed approaches

• The Gerkerma flow is given as a function of the flow speed by the following expression:

$$q = \alpha |u|^3 \, \frac{u}{|u|} - \Lambda \nabla \cdot z \tag{1.22}$$

with z is the bottom shape, α proportional to the drag coefficient is taken equal to 10^{-4} m^{-1} s^2 and $1 < \Lambda < 3$ is constant. This expression only takes into account the flow carried. The first term of the flow is linked to the effect of the forcing of the grains on the granular bed under the effect of the fluid; the second term is the attenuation of the forcing by gravity.
• Vittori's expression is written a little differently by Roos and Blondeaux [21] in the form:

$$q = 1.23 \, \frac{s-1}{s} \, \Psi^{3.36} \, Re_P^{1.83} \, |\frac{2 \, V}{Re_p} - \frac{\mu}{\Psi} \, \nabla \, f|^{b-1} \, (\frac{2 \, V}{Re_p} - \frac{\mu}{\Psi} \nabla \cdot f), \tag{1.23}$$

with $V = (u, v)$, and Ψ grain mobility factor defined for an oscillating flow of amplitude A and frequency $f = \Omega / 2\pi$ (Ω his pulse). For numerical, statistical considerations, we retain that it is difficult to establish an analytical expression of the flow. However, in certain particular cases one can use the simplified expression due to Van Rijn [?] :

$$q_r \simeq U^{2.4} \; , \; U \; \text{ is the average speed of the fluid} \tag{1.24}$$

• Bagnold: According to Bagnold the flow of sand q can be expressed in terms of movement wind like this

$$q = 1.9 B \frac{\rho}{g} V_*^3. \tag{1.25}$$

where ρ is the air density, B the impact coefficient and V_* shear speed.

Chapter 2

Sand movement modeling and problem position

The Exner equation, which models the transport of sand is given by the following expression:

$$\frac{\partial z}{\partial t} + \frac{1}{1-p}\nabla \cdot q = 0 \tag{2.1}$$

where p denotes the porosity of the bottom, the percentage of void in the sediment, q the instantaneous flow of materials transported by the flow and $z = z(x,t)$ thickness of the sediment layer, see [3]. The main difficulty lies in estimating the flow $q(x,t)$. The flow $q(x,t)$ of materials transported therefore takes into account that due to the thrust noted q_r and the quantity which is transported in suspension q_s. There are two approaches to determine $q(x,t)$: one consists in separately estimating the flows q_r and q_s and adding them to obtain the total flow $q(x,t)$:

$$q = q_s + q_r \tag{2.2}$$

However in certain cases of viscous flow, the suspensions are negligible, and the knowledge of q_r is enough by itself to know $q(x,t)$. The second is to estimate directly in a single block the total flow q.

2.1 Formulation of the different models

In this section we will try to pose our problem. We will work with several flows; Bagnold, Gerkema, Komarava, coll and Vittori, we will present several models.

2.1.1 The model of Gerkerma

We get our model this time using the Gerkerma flux and the Exner equation given by (2.2).
Gerkerma, in his study of the linear stability of a sandy bottom gives a flow of the form

$$q = \alpha |u|^3 \ \frac{u}{|u|} - \Lambda \nabla \cdot z \tag{2.3}$$

It only takes into account the flow carried. The first term of the flow is linked to the effect of the forcing of the grains on the granular bed under the effect of the fluid, the second term is the attenuation of the forcing by gravity.
Using (2.3), the Exner equation becomes

$$\frac{\partial z}{\partial t} + \frac{1}{1-p} \nabla \cdot (\alpha |u|^3 \ \frac{u}{|u|} - \Lambda \nabla z) = 0, \tag{2.4}$$

which gives by developing the following equation

$$\frac{\partial z}{\partial t} + \frac{\alpha}{1-p} \nabla \cdot (|u|^3 \ \frac{u}{|u|}) - \frac{\Lambda}{1-p} \triangle z = 0 \tag{2.5}$$

where z is the bottom shape, α proportional to the drag coefficient is taken equal to $10^{-4}\ m^{-1}\ s^2$, and $1 < \Lambda < 3$ is constant, u is the speed of flow.

2.1.2 The model of Bagnold

We will now focus on the Bagnold flow to establish the associated model.
Bagnold's formula based on an energetic approach was modified by Godd et al [3] by including a critical speed of setting in motion. The flow of sediment is then determined by

$$q_{s_0} = \frac{B_G}{\varphi_s}(u(z) - u_c(z))^3. \tag{2.6}$$

Suppose in $z = 1$ below the bottom, the hypothesis of logarithmic profile is verified

$$u(z=1) = \frac{\sqrt{\tau_b}}{k\sqrt{\tau}} ln \frac{7,5}{d_g} \tag{2.7}$$

from where

$$q_{s_0} = \alpha(|\tau_b|^{1/2} - \tau_{bc}^{1/2})^3. \tag{2.8}$$

The total flow is given by

$$q_s = q_{s_0}(\frac{u}{\|u\|} - \lambda_s \nabla z) \tag{2.9}$$

where u indicates the speed of the fluid and

$$\tau_{bc} = \theta_{cg}(\rho_s - \rho_c)D_g, \tag{2.10}$$

$$\tau_b = \theta_g(\rho_s - \rho_c)D_g \tag{2.11}$$

with ρ_s the density of the sediment, ρ_c the density of the fluid. So using (2.10) and replacing in the Exner equation we get the following model

$$\frac{\partial z}{\partial t} + \frac{1}{1-p}\nabla.(\alpha(|\tau|_b^{1/2} - \tau_{bc}^{1/2})^3(\frac{u}{\|u\|} - \lambda_s \nabla z)) = 0. \tag{2.12}$$

2.1.3 The model of Komarova

First we will give the expression of the flow under the condition that the speed is zero at the bottom bed.
Komarova et al. [13] [15] consider that taking into account the condition of zero speed in the bottom bed, the speed of the fluid in the bottom bed is not the parameter which sets in motion the grains but rather the parietal friction . τ being the physical friction divided by the density of the fluid, they obtain:

$$q = \alpha|\tau|^b\left(\tau - \Lambda \bigtriangledown z\right) \quad b = 1/2. \tag{2.13}$$

The values of Λ are between 1/2 and 6. b is a parameter that translates non-linearities and α is a coefficient of proportionality.
By replacing the flow q given by (2.13) in the equation of Exner we obtain

$$\frac{\partial z}{\partial t} + \frac{1}{1-p}\nabla \cdot (\alpha|\tau|^b(\tau - \Lambda\nabla z)) = 0 \tag{2.14}$$

Which give

$$\frac{\partial z}{\partial t} + \frac{1}{1-p}\nabla \cdot (\alpha|\tau|^b\tau) - \frac{1}{1-p}\nabla \cdot (\Lambda\alpha|\tau|^b\nabla z) = 0 \tag{2.15}$$

with τ the shear stress given by the relation $\tau = \rho\frac{|u|^2}{C^2}\frac{u}{|u|}$ where ρ is the air density, $C = \frac{1}{k}ln(\frac{30\bar{z}}{D_G})$ et D_G being the grain of sand diameter.

2.1.4 The model of Vittori

Vittori's expression is written a little differently by Roos and Blondeaux [21] in the form :

$$q = 1.23\ \frac{s-1}{s}\ \Psi^{3.36}\ Re_P^{1.83}\ |\frac{2\ V}{Re_p} - \frac{\mu}{\Psi}\ \nabla\ f|^{b-1}\ (\frac{2\ V}{Re_p} - \frac{\mu}{\Psi}\ \nabla \cdot f), \tag{2.16}$$

Where $V = (u, v)$, et Ψ is the grain mobility factor defined for an oscillating flow of amplitude A and frequency $f = \Omega/2\pi$ (Ω his pulse). So, by replacing q given by (2.16) in the Exner equation, we get

$$\frac{\partial f}{\partial t} + \frac{1}{1-p}\nabla \cdot (1.23\ \frac{s-1}{s}\ \Psi^{3.36}\ Re_P^{1.83}\ |\frac{2\ V}{Re_p} - \frac{\mu}{\Psi}\ \nabla\ f|^{b-1}\ (\frac{2\ V}{Re_p} - \frac{\mu}{\Psi}\ \nabla \cdot f)) = 0 \quad (2.17)$$

which ultimately gives

$$\frac{\partial f}{\partial t} + \frac{1}{1-p}\nabla(1.23\ \frac{s-1}{s}\ \Psi^{3.36}\ Re_P^{1.83}\ |\frac{2\ V}{Re_p} - \frac{\mu}{\Psi}\ \nabla\ f|^{b-1}\ \frac{2\ V}{Re_p}) = \quad (2.18)$$
$$\frac{1}{1-p}\nabla(-1.23\ \frac{s-1}{s}\ \Psi^{3.36}\ Re_P^{1.83}\ |\frac{2\ V}{Re_p} - \frac{\mu}{\Psi}\ \nabla\ f|^{b-1}\ \frac{\mu}{\Psi}\ \nabla \cdot f)$$

avec $\mu = 0,15$, s = 2,65 et p=0,45.

2.2 Scaling and parameterized models

In this section we construct characteristic variables of time and length to make the equations dimensionless. We consider the characteristic variables of time $\bar{t}$ and of length $\bar{L}$, and the variables without dimensions $t^{'}$ et $x^{'}$. We define the dimensionless unitary variables t' and x' by:

$$t = \bar{t}t^{'} \quad et \quad x = \bar{L}x^{'} \quad (2.19)$$

. We also define $\bar{z}$, by the characteristic height of the dunes and $z^{'}(t^{'}, x^{'})$ the dimensionless height of the dunes by the following relation:

$$z^{'}(t^{'}, x^{'}) = \frac{1}{\bar{z}}z(\bar{t}t^{'}, \bar{L}x^{'}) \quad (2.20)$$

.

We finally introduce the characteristic speed $\bar{u}$ and the dimensionless speed $u^{'}$ by

$$u^{'}(t^{'}, x^{'}) = \frac{1}{\bar{u}}u(\bar{t}t^{'}, \bar{L}x^{'}) \text{ et } \nabla\cdot = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} = \frac{1}{L}\nabla^{'}\cdot \quad (2.21)$$

So therefore, we have the following expressions:

$$\frac{\partial z}{\partial t} = \frac{\bar{z}}{\bar{t}}\ \frac{\partial z^{'}}{\partial t^{'}} \text{ et } \nabla z = \frac{\bar{z}}{L}\nabla^{'} z^{'} \quad (2.22)$$

So, with The different variables, we get the following dimensionless models.

2.2.1 Scaling and parameterized of Gerkema model

Using the relations (2.19),(2.20),(2.21),(2.22) et (2.5) is rewritten as follows

$$\frac{\partial(\bar{z}z^{'})}{\partial t}+\frac{\alpha}{1-p}\nabla\cdot(|\bar{u}u^{'}|^{3}\ \frac{\bar{u}u^{'}}{|\bar{u}u^{'}|})-\frac{\Lambda}{1-p}(\nabla\cdot(\nabla\cdot\bar{z}z^{'}))=0 \qquad (2.23)$$

Using the equations (2.19),(2.20) et (2.21) we have the following relationships:

$$\frac{\partial(\bar{z}z^{'})}{\partial t}=\frac{\bar{z}}{\bar{t}}\ \frac{\partial z^{'}}{\partial t^{'}} \qquad (2.24)$$

$$\nabla\cdot(|\bar{u}u^{'}|^{3}\ \frac{\bar{u}u^{'}}{|\bar{u}u^{'}|})=\frac{\bar{u}^{3}}{\bar{L}}\nabla^{'}\cdot(|u^{'}|^{3}\ \frac{u^{'}}{|u^{'}|}) \qquad (2.25)$$

$$\nabla\cdot(\nabla\bar{z}z^{'})=\frac{\bar{z}}{\bar{L}^{2}}\Delta z^{'} \qquad (2.26)$$

By replacing in equation (2.23) we get our dimensioned model:

$$\frac{\partial z^{'}}{\partial t^{'}}-\frac{\bar{t}\Lambda}{(1-p)\bar{L}^{2}}\Delta z^{'}=-\frac{\bar{t}}{\bar{z}}\frac{\alpha\bar{u}^{3}}{(1-p)\bar{L}}\nabla^{'}\cdot(|u^{'}|^{3}\ \frac{u^{'}}{|u^{'}|}) \qquad (2.27)$$

2.2.2 Scaling and parameterized of Bagnold model

Using the following relationships: (2.19), (2.20), (2.21) and (2.12)and the shear stress τ which is given by the relation $\tau_b=\rho\frac{|u|^2}{C^2}\frac{u}{|u|}$,$\tau_{bc}=\rho\frac{u^2}{C^2}$ where ρ is air density,$C=\frac{1}{k}ln(\frac{30\bar{z}}{D_G})$,D_G being the grain of sand diameter. more the speed of the fluid u is given by $u=u^{'}\bar{u}$ we obtain the model:

$$\frac{\partial z^{'}}{\partial t^{'}}+\frac{\alpha\rho^{3/2}\bar{u}^{3}\bar{t}}{L(1-p)C^{3}\bar{z}}\nabla^{'}\cdot((\|u\|-u_c)^{3}\frac{u}{\|u\|})-\frac{\lambda\alpha\rho^{3/2}\bar{u}^{3}\bar{t}}{L^{2}(1-p)C^{3}}\nabla^{'}\cdot((\|u\|-u_c)^{3}\nabla z)=0 \qquad (2.28)$$

2.2.3 Scaling and parameterized of Komarova model

Using the relations (2.19),(2.20) , (2.21)et(2.15) we get

$$\frac{\partial(\bar{z}z^{'})}{\partial t}+\frac{\alpha\bar{\tau}|\bar{\tau}|^{b}}{1-p}\nabla\cdot(|\tau^{'}|^{b}\tau^{'})-\frac{\Lambda\alpha|\bar{\tau}|^{b}}{1-p}\nabla\cdot(|\tau^{'}|^{b}\bigtriangledown(\bar{z}z^{'}))=0 \qquad (2.29)$$

$$\frac{\partial z^{'}}{\partial t^{'}}+\frac{\alpha\bar{\tau}|\bar{\tau}|^{b}\bar{t}}{\bar{z}(1-p)}\nabla\cdot(|\tau^{'}|^{b}\tau^{'})-\frac{\bar{t}\Lambda\alpha|\bar{\tau^{'}}|^{b}}{1-p}\nabla\cdot(|\tau^{'}|^{b}\nabla z^{'})=0 \qquad (2.30)$$

The shear stress τ is given by the relation $\tau=\rho\frac{|u|^2}{C^2}\frac{u}{|u|}$ where $\rho=1.225$ is the density of the area, ,$C=\frac{1}{k}ln(\frac{30\bar{z}}{D_G})$, D_G being the diameter of grain of sand. So we have $\bar{\tau}=\rho\frac{|\bar{u}u'|}{C^2}\frac{u'}{|u|}$.

So we get the dimensionless equation of the Komarova model

$$\frac{\partial z'}{\partial t'} + \frac{\alpha \bar{u}^3 \rho^{3/2} \bar{t}}{\bar{z}(1-p)C^3 \bar{L}} \nabla' \cdot (|u'|^2 u') - \frac{\bar{u} \rho^{1/2} \bar{t} \Lambda \alpha}{(1-p)C^3 \bar{L}^2} \nabla' \cdot (|u'| \nabla z') = 0, \qquad (2.31)$$

2.2.4 Scaling and parameterized of Vittori model

Using the equations (2.19),(2.20) , (2.22),(2.15) and the Vittori flow we get:

$$\frac{\partial(\bar{z}z')}{\partial t} + \frac{1}{1-p} \nabla \cdot (1.23 \; \frac{s-1}{s} \; \Psi^{3.36} \; Re_P^{1.83} \; |\frac{2\bar{V}V'}{Re_p} - \frac{\mu}{\Psi} \; \nabla \; (\bar{z}z')|^{b-1} \; \frac{2\bar{V}V'}{Re_p}) = \qquad (2.32)$$
$$\frac{1}{1-p} \nabla \cdot (-1.23 \; \frac{s-1}{s} \; \Psi^{3.36} \; Re_P^{1.83} \; |\frac{2\bar{V}V'}{Re_p} - \frac{\mu}{\Psi} \; \nabla \; \bar{z}z'|^{b-1} \; \frac{\mu}{\Psi} \; \nabla \bar{z}z')$$

After simplification we have the equation:

$$\frac{\partial(\bar{z}z')}{\partial t} + \frac{Re_P^{1.83} 1.23(s-1)}{Re_p s(1-p)} \nabla \cdot (\; \Psi^{3.36} \; |\frac{2\bar{V}V'}{Re_p} - \frac{\mu}{\Psi} \; \nabla \; (\bar{z}z')|^{b-1} \; 2\bar{V}V') = \qquad (2.33)$$
$$-\frac{Re_P^{1.83} 1.23(s-1)}{s(1-p)} \nabla \cdot (\; \Psi^{3.36} \; |\frac{2\bar{V}V'}{Re_p} - \frac{\mu}{\Psi} \; \nabla \; \bar{z}z'|^{b-1} \; \frac{\mu}{\Psi} \; \nabla \bar{z}z')$$

So,

$$\frac{\partial z'}{\partial t'} + \frac{\bar{V}\bar{t} \; Re_P^{0.83} 2.46(s-1)}{\bar{z}s(1-p)} \nabla \cdot (\; \Psi^{3.36} \; |\frac{2\bar{V}V'}{Re_p} - \frac{\mu}{\Psi} \; \nabla \; (\bar{z}z')|^{b-1} \; V') = \qquad (2.34)$$
$$-\frac{\bar{t} \; Re_P^{1.83} 1.23(s-1)}{s(1-p)} \nabla \cdot (\; \Psi^{3.36} \; |\frac{2\bar{V}V'}{Re_p} - \frac{\mu}{\Psi} \; \nabla \bar{z}z'|^{b-1} \; \frac{\mu}{\Psi} \; \nabla z')$$

2.3 Characteristic values of the dunes

A barkhane is a dune in the shape of a crescent elongated in the direction of the wind. It is born where the sand supply is weak and under unidirectional winds.
After obtaining a dimensionless model, we will now examine several positions of the dunes, with the aim of obtaining characteristic values ??characterizing the different positions of the evolution of the dunes (short, medium and long term). First, we set the characteristic dimensions which are common to different situations. We can now use the scale invariance, which stipulates that, at the given form, the speed of the wind at the crest of the dune does not depend on the height z. In reality, the dunes sufficiently large to present an avalanche face have fairly similar morphologies. In particular, the width of the dunes is around 15 z and their length is around 12 z. Consequently, the speed u is inversely

proportional to the size of the dune.
Consider the case of the barkhans of the Atlantic Sahara for which, on average, the peak flow is worth $300m^2/year$. That means that a small dune of height $z = 1m$ traverses $300m/year$, a large dune of height $z = 10m$ traverses $30m/year$ and a mega-barkhane of height $z = 50m$ traverses $6m/year$. The average residence time of the grains in these three dunes is respectively 1 month, 8 years and 2 centuries or even [25].
We consider that $\bar{t}$ is an observation period of time. We take as $\bar{t}$ the order of the smallest magnitudes of the time during which the dunes undergo a significant evaluation formed by the wind in an environment.
The spectral analysis of the wind speed in the turbulent boundary layer makes it possible to highlight several time scales of fluctuation.

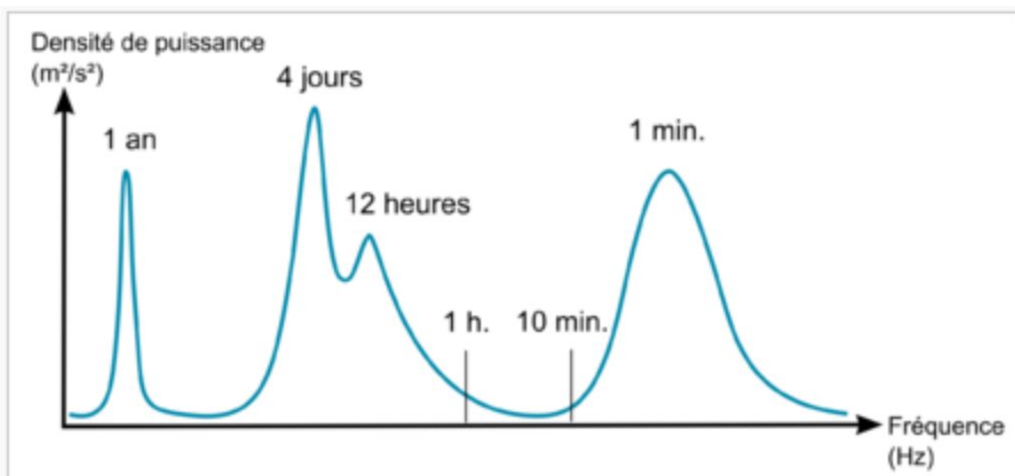

Figure 2.1:

The figure opposite shows the shape of a power density spectrum representative of the horizontal wind speed at 100 meters above the ground according to Van der Hoven. It is a statistical representation of the repetitiveness of the wind power fluctuations at this point: ?Atmospheric turbulence can be illustrated by the existence of vortices within a flow. Turbulence is thus made up of perfectly random movements sweeping across a broad spectrum of spatial and temporal scales. ?
The left part of the graph relates to systems on a planetary scale which have a periodicity between 1 day and a year, which corresponds to a return period of different types of synoptic weather systems. Thus, a year represents the annual winds like the trade winds, four days the winds associated with the average period between two meteorological depressions and 12 hours the alternating day and night winds, see [12] and [10]. We introduce $\bar{w}$ the wavelength frequency of grain movement in the desert, which leads to the definition of a parameter ε very small $\frac{1}{\bar{t}\bar{w}} = \varepsilon$.
We consider several types of grain residences.

2.4 Finalization of some models of sand dunes

In this section we finalize the different models of dunes obtained:

2.4.1 The model of Gerkerma

The parameterized model of Gekerma is given by:

$$\frac{\partial z^{'}}{\partial t^{'}} - \frac{\bar{t}\Lambda}{(1-p)\bar{L}^2}\Delta z^{'} = -\frac{\bar{t}}{\bar{z}}\frac{\alpha\bar{u}^3}{(1-p)\bar{L}}\nabla^{'}\cdot(|u^{'}|^3\ \frac{u^{'}}{|u^{'}|}), \tag{2.35}$$

With $\alpha = 10^{-4}m^{-1}s^2$ and we take $\Lambda = 3$,$\bar{u} = 1m/s$ .Furthermore, the order of magnitude of the coefficient $\frac{\lambda}{1-p}$ is 1, so we get $\frac{\lambda}{1-p} = 1$, $\frac{1}{1-p} = 2$.

A)Study of dune dynamics in the short term

Let us consider a small dune of height $z = 1m$ traversing $300m/year$. The average grain residence time in these dune cases is 100 days which will be compared to the average period of alternating day and night winds which is around $\frac{1}{\bar{w}} = 13hours$, $\bar{L} = 300m$, $\bar{z} = 1m$ and $\bar{t} = 1month$

By estimating the different coefficients of the equation (2.35), we have

$\frac{1}{\bar{w}\bar{t}} \simeq \frac{1}{200} \simeq \varepsilon$,

$\dfrac{\bar{t}\Lambda}{(1-p)\bar{L}^2} \simeq \frac{3}{\varepsilon}$,

$\dfrac{\bar{t}\alpha\bar{u}^3}{\bar{z}(1-p)\bar{L}} \simeq 6$.

So finally we get the following model:

$$\frac{\partial z^{'}}{\partial t^{'}} - \frac{3}{\varepsilon}\Delta^{'} z^{'} = 6\nabla^{'}\cdot(|u^{'}|^3\ \frac{u^{'}}{|u^{'}|}). \tag{2.36}$$

B)Study of dune movement in the case of medium-term residence

The average residence time of the grains of the dunes is 8 years and in this case we have $\frac{1}{\bar{w}} = 4days$ the average period between two meteorological depressions. $\bar{L} = 30m$, $\bar{z} = 10m$, $\bar{t} = 8years$.

By estimating the different coefficients of the equation (2.35), we have

$\frac{1}{\bar{w}\bar{t}} = \dfrac{4jours}{8\times 365jours} \simeq 10^{-3} = \varepsilon$,

$\dfrac{\bar{t}\Lambda}{(1-p)\bar{L}^2} \simeq \frac{16}{\varepsilon}$,

$\dfrac{\bar{t}\alpha\bar{u}^3}{\bar{z}(1-p)\bar{L}} \simeq \frac{1}{4\varepsilon}$.

By replacing in (2.35) we get the following model:

$$\frac{\partial z^{'}}{\partial t^{'}} - \frac{16}{\varepsilon}\Delta^{'} z^{'} = -\frac{1}{4\varepsilon}\nabla^{'}\cdot(|u^{'}|^3\ \frac{u^{'}}{|u^{'}|}). \tag{2.37}$$

C)Study of dune movement in the long term case
Let us consider that a dune of height $\bar{z} = 50m$ travels $300m/year$. The average residence time of the grains in this case of dunes is worth 2 centuries which will be compared with the average period of the annual cyclical winds which is around $\frac{1}{\bar{w}} = 1year$ and we take $\bar{L} = 300m$ and $\bar{z} = 50m$.
By estimating the different coefficients of the equation (2.35), we have
$\frac{1}{\bar{w}\bar{t}} \simeq \frac{1}{200} \simeq \varepsilon$
$\dfrac{\bar{t}\Lambda}{(1-p)\bar{L}^2} \simeq \frac{5}{\varepsilon^2}$
$\dfrac{\bar{t}\alpha\bar{u}^3}{\bar{z}(1-p)\bar{L}} \simeq \frac{1}{2\varepsilon}$
So finally we get the following model:

$$\frac{\partial z'}{\partial t'} - \frac{5}{\varepsilon^2}\Delta' z' = \frac{1}{2\varepsilon}\nabla' \cdot (|u'|^3 \frac{u'}{|u'|}) \tag{2.38}$$

2.4.2 Le model of Komarova

The parameterized model of Komarova is given by:

$$\frac{\partial z'}{\partial t'} + \frac{\alpha\bar{u}^3\rho^{3/2}\bar{t}}{\bar{z}(1-p)C^3\bar{L}}\nabla \cdot (|u'|^2 u') - \frac{\bar{u}\rho^{1/2}\bar{t}\Lambda\alpha}{(1-p)C^3\bar{L}^2}\nabla \cdot (|u'|\nabla z') = 0, \tag{2.39}$$

Where $C = \frac{1}{k}ln(\frac{30\bar{z}}{D_G})$, $\Lambda = 6$, $\bar{u} = 1m/s$, $\alpha = 100$ and the average diameter of the grains of sand $D_G = 3\,10^{-4}mm$ and $k = 0,4$ See [?]. Furthermore, the order of magnitude of the coefficient $\frac{\lambda}{1-p}$ is 1, so we get $\frac{\lambda}{1-p} = 1$ et $\frac{1}{1-p} = 2$.
A)Study of dune dynamics in the short term
The average grain residence time in these dune cases is 100 days which will be compared to the average period of alternating day and night winds which is around $\frac{1}{\bar{w}} = 1p.m.$. By estimating the different coefficients of the equation (2.39), we have
$\frac{1}{\bar{w}\bar{t}} \simeq \frac{1}{200} = \varepsilon$
$C = \frac{1}{k}ln(\frac{30\bar{z}}{D_G}) \simeq 33,5$
$\dfrac{\alpha\bar{u}^3\rho^{3/2}\bar{t}}{\bar{z}(1-p)C^3\bar{L}} \simeq \frac{3}{2\varepsilon}$
$\dfrac{\bar{u}\rho^{1/2}\bar{t}\Lambda\alpha}{(1-p)C\bar{L}^2} \simeq \frac{3}{\varepsilon}$
with($\bar{L} = 300m$, $\bar{z} = 1m$, $\bar{t} = 100jours$). So finally we get the following model:

$$\frac{\partial z'}{\partial t'} + \frac{3}{2\varepsilon}\nabla'.(|u'|^2 u') - \frac{3}{\varepsilon}\nabla \cdot (|u'|\nabla' z') = 0. \tag{2.40}$$

B)Study of dune movement in the case of medium-term residence
. The average grain residence time in these dune cases is 8 years so we have $\frac{1}{\bar{w}} = 4days$

the average period between two meteorological depressions.
By estimating the different coefficients of the equation (2.39) in the case where $\bar{L} = 100m$, $\bar{t} = 8years$ and $\bar{z} = 10m$, we have
$\frac{1}{\bar{w}\bar{t}} = \dfrac{4days}{8 \times 365days} \simeq 10^{-3} = \varepsilon$,
$C = \frac{1}{k} ln(\frac{30\bar{z}}{D_G} = 34, 5$,
$\dfrac{\alpha \bar{u}^3 \rho^{3/2} \bar{t}}{\bar{z}(1-p)C^3\bar{L}} = \frac{1}{10\varepsilon}$,
$\dfrac{\bar{u}\rho^{1/2}\bar{t}\Lambda\alpha}{(1-p)C\bar{L}^2} = \frac{39}{\varepsilon}$,
After determining the coefficients we get:

$$\frac{\partial z^{'}}{\partial t'} + \frac{1}{10\varepsilon}\nabla^{'} \cdot (|u'|^2 u') - \frac{39}{\varepsilon}\nabla^{'} \cdot (|u'|\nabla^{'} z^{'}) = 0. \tag{2.41}$$

In the case where $\bar{L} = 10m$,$\bar{z} = 10m$ et $\bar{t} = 8ans$ we get the following parameters:
$\dfrac{\alpha \bar{u}^3 \rho^{3/2} \bar{t}}{\bar{z}(1-p)C^3\bar{L}} = \frac{1}{\varepsilon}$
$\dfrac{\bar{u}\rho^{1/2}\bar{t}\Lambda\alpha}{(1-p)C\bar{L}^2} = 4$
Then we have the following model:

$$\frac{\partial z^{'}}{\partial t'} + \frac{1}{\varepsilon}\nabla^{'} \cdot (|u'|^2 u') - 4\nabla^{'} \cdot (|u'|\nabla^{'} z^{'}) = 0 \tag{2.42}$$

C)**Study of dune movement in the long term case**
In this case, the characteristic time $\bar{t}$ is worth 2 centuries which will be compared to the average period of the annual cyclical winds which is of the order of $\frac{1}{\bar{w}} = 1year$, with $\bar{L} = 300m$, $\bar{z} = 50m$.
Which give
$\frac{1}{\bar{w}\bar{t}} = \dfrac{1}{200ans} \simeq 0,005 \simeq \varepsilon$,
$C = \frac{1}{k} ln(\frac{30\bar{z}}{D_G}) = 39$,
$\dfrac{\alpha \bar{u}^3 \rho^{3/2} \bar{t}}{\bar{z}(1-p)C^3\bar{L}} = \frac{9}{\varepsilon}$,
$\dfrac{\bar{u}\rho^{1/2}\bar{t}\Lambda\alpha}{(1-p)C\bar{L}^2} = \frac{5}{8\varepsilon^2}$.
By replacing in the equation (2.39) we get:

$$\frac{\partial z^{'}}{\partial t'} + \frac{9}{\varepsilon}\nabla^{'} \cdot (|u'|^2 u') - \frac{5}{8\varepsilon^2}\nabla^{'} \cdot (|u'|\nabla z^{'}) = 0. \tag{2.43}$$

Chapter 3

Mathematical study of dune transport models

In this chapter, we are interested in the existence and uniqueness of sand transport models.

3.1 Sand transport modeling results

We recall the transport models from transport flows from Gekerma and Komarova in the case of dune dynamics in the short, medium and long term
model of Gerkema
In the case of short-term sand dune dynamics:

$$\frac{\partial z^\epsilon}{\partial t} - \frac{3}{\varepsilon}\Delta z^\epsilon = 6\nabla \cdot (|u^\epsilon|^3 \ \frac{u^\epsilon}{|u^\epsilon|}). \tag{3.1}$$

In the case of the dynamics of sand dunes in the medium term:

$$\frac{\partial z^\epsilon}{\partial t} - \frac{16}{\varepsilon}\Delta z^\epsilon = -\frac{1}{4\varepsilon}\nabla \cdot (|u^\epsilon|^3 \ \frac{u^\epsilon}{|u^\epsilon|}). \tag{3.2}$$

In the case of long-term sand dune dynamics

$$\frac{\partial z^\epsilon}{\partial t} - \frac{5}{\varepsilon^2}\Delta z^\epsilon = \frac{1}{2\varepsilon}\nabla' \cdot (|u^\epsilon|^3 \ \frac{u^\epsilon}{|u^\epsilon|}). \tag{3.3}$$

Model of Komarova
In the case of short-term sand dune dynamics:

$$\frac{\partial z^\epsilon}{\partial t} + \frac{3}{2\varepsilon}\nabla \cdot (|u^\epsilon|^2 u^\epsilon) - \frac{3}{\varepsilon}\nabla \cdot (|u^\epsilon|\nabla z^\epsilon) = 0. \tag{3.4}$$

In the case of the dynamics of sand dunes in the medium term:

$$\frac{\partial z^\epsilon}{\partial t} + \frac{1}{\varepsilon}\nabla \cdot (|u^\epsilon|^2 u^\epsilon) - 4\nabla \cdot (|u^\epsilon|\nabla z^\epsilon) = 0. \tag{3.5}$$

In the case of long-term sand dune dynamics

$$\frac{\partial z^\epsilon}{\partial t} + \frac{9}{\varepsilon}\nabla \cdot (|u^\epsilon|^2 u^\epsilon) - \frac{5}{8\varepsilon^2}\nabla \cdot (|u^\epsilon|\nabla z^\epsilon) = 0. \tag{3.6}$$

3.2 Mathematical formulation of the problem

We note after the exploitation of the different models that the short and medium term models characterizing the movement of the dunes and sandbanks can be rewritten as follows;
A generic form of all problems (3.1), (3.2), (3.3), (3.4, (3.5) and (3.6) is given in the form

$$\frac{\partial z^\varepsilon}{\partial t} - \frac{a}{\varepsilon^j}\nabla \cdot (g_a(|u^\varepsilon|)\nabla z^\epsilon) = \frac{b}{\varepsilon^i}\nabla \cdot (g_c(|u^\varepsilon| \ \frac{u^\epsilon}{|u^\epsilon|}), \tag{3.7}$$

with $i = 0, 1, \ j = 0, 1, 2$.
To this equation (3.7), we associate the initial condition

$$z^\varepsilon(0, x) = z_0(x), \ x \in \mathbb{T}^2. \tag{3.8}$$

The functions g_a and g_c are increasing, regular, defined functions of $\mathbb{R}_+$ with values in $\mathbb{R}_+$ and checking

$$\begin{cases} g_a \geq g_c \geq 0, \ g_c(0) = g_c'(0) = 0, \\ \exists\, d \geq 0, \ \sup_{u\in\mathbb{R}^+} |g_a(u)| + \sup_{u\in\mathbb{R}^+} |g_a'(u)| \leq d, \ \sup_{u\in\mathbb{R}^+} |g_c(u)| + \sup_{u\in\mathbb{R}^+} |g_c'(u)| \leq d, \\ \exists U_{thr} \geq 0, \ \exists G_{thr} > 0, \text{ such as } u \geq U_{thr} \Longrightarrow g_a(u) \geq G_{thr}. \end{cases} \tag{3.9}$$

$z^\varepsilon = z^\varepsilon(t, x)$ is defined from $[0, T] \times \mathbb{T}^2 \to \mathbb{R}$ with $\mathbb{T}^2 \subset \mathbb{R}^2, \ t \in [0, T]$ et $T > 0$.
The position $x = (x_1, x_2) \in \mathbb{T}^2$ and the function $u : [0, T] \times \mathbb{T}^2 \to \mathbb{R}_+$.
In(3.7), The functions g_a and g_c can cancel each other out so we have degenerate and singularly disturbed parabolic equations.

3.3 Study of the existence and uniqueness of the model solution

In this section under certain hypotheses we study the existence and the uniqueness of the solution of the sand transport model in the desert, which is given by the following equation:

$$
\begin{cases} \dfrac{\partial z^{\varepsilon}}{\partial t} - \dfrac{a}{\varepsilon^j}\nabla \cdot (g^{\varepsilon}(x,t)\nabla z^{\varepsilon}) = \dfrac{b}{\varepsilon^i}\nabla \cdot (f^{\epsilon}(t,x)) \qquad (x,t) \in \mathbb{T}^2 \times [0.T] \\ z^{\epsilon}(x,0) = z_0(x) \qquad x \in \mathbb{T}^2. \end{cases} \tag{3.10}
$$

with $i = 0, 1,\ \ j = 0, 1, 2,\ z_0 \in L^2(\mathbb{T}^2)$, and the functions $f^{\epsilon}(t,x)$ and $g^{\epsilon}(t,x)$ are given by the following relations

$$
f^{\epsilon}(t,x) = g_c(|\mathbf{u}^{\epsilon}(t,x)|)\frac{\mathbf{u}^{\epsilon}(t,x)}{|\mathbf{u}^{\epsilon}(t,x)|} \text{ and } g^{\epsilon}(t,x) = g_a(|\mathbf{u}^{\epsilon}(t,x)|), \tag{3.11}
$$

where the functions g_a and g_c check (3.9). We will first establish the existence and uniqueness of $z^{\epsilon}(t,x)$, solution of (3.10). And in the second step we prove that the standard of $z^{\epsilon}(t,x)$ as well as its derivatives are bounded independently of ϵ.

3.3.1 Existence and uniqueness for a weak solution

We are going to study the existence and the uniqueness of the solution of (3.10) under the following condition:

$$
\big|f^{\epsilon}(t,x)\big|, \big|\nabla \cdot f^{\epsilon}(t,x)\big| \le \epsilon^i \gamma,\ i = 0, 1. \tag{3.12}
$$

with γ an independent constant of ϵ.

Theorem 3.3.1. *let $\epsilon > 0$, $a > 0$, b and c real ones. Under the assumptions (3.9), (3.11) and (3.12), si $z_0 \in H^1(\mathbb{T}^2)$, for everything $T > 0$, there is only one solution $z^{\epsilon} \in L^2((0,T], L^2(\mathbb{T}^2))$ of the equation (3.10) satisfactory*

$$
\int_{\mathbb{T}^2} \frac{\partial z^{\epsilon}}{\partial t} dx = 0, \tag{3.13}
$$

and

$$
\|z^{\epsilon}\|_{L^2((0,T)\times\mathbb{T}^2)} \le \tilde{\gamma}, \tag{3.14}
$$

for a constant $\tilde{\gamma}$ independent of ϵ.

As the coefficients of the equation (3.10) can cancel each other out, we will start with the regularization. Is $\nu > 0$, we consider the following problem

$$
\begin{cases} \frac{\partial z^{\epsilon,\nu}}{\partial t} - \frac{a}{\epsilon^j}\nabla \cdot \Big((g^{\epsilon}(t,x) + \nu)\nabla z^{\epsilon,\nu}\Big) = \frac{b}{\epsilon^i}\big(\nabla \cdot f^{\epsilon}(t,x)\big)\ [0,T) \times \mathbb{T}^2, \\ z^{\epsilon,\nu}(x,0) = z_0(x)\ \ \mathbb{T}^2. \end{cases} \tag{3.15}
$$

Proposition 3.3.2. *Under the same assumptions as in the theorem 3.3.1, for everything $\epsilon > 0$, for everything $\nu > 0$, there is a unique solution $z^{\epsilon} \in L^2((0,T], L^2(\mathbb{T}^2))$ solution of (3.15).*

Proof Let $z_1^{\epsilon,\nu}$ et $z_2^{\epsilon,\nu}$ two solutions to the equation (3.15). Therefore $z_1^{\epsilon,\nu} - z_2^{\epsilon,\nu}$ is solution of the equation:

$$\begin{cases} \frac{\partial(z_1^{\epsilon,\nu} - z_2^{\epsilon,\nu})}{\partial t} - \frac{a}{\epsilon^j}\nabla \cdot \Big((g^\epsilon(t,x) + \nu)\nabla(z_1^{\epsilon,\nu} - z_2^{\epsilon,\nu})\Big) = 0, \ [0,T) \times \mathbb{T}^2, \\ z_1^{\epsilon,\nu}(x,0) - z_2^{\epsilon,\nu} = 0 \ \ \mathbb{T}^2. \end{cases} \tag{3.16}$$

By multiplying (3.16) by $(z_1^{\epsilon,\nu} - z_2^{\epsilon,\nu})$ and integrating on $\mathbb{T}^2$ we obtain:

$$\frac{1}{2}\frac{d}{dt}\|z_1^{\epsilon,\nu} - z_2^{\epsilon,\nu}\|_2^2 + \frac{a}{\epsilon^j}\int_{\mathbb{T}^2}(g^\epsilon + \nu)(\nabla|z_1^{\epsilon,\nu} - z_2^{\epsilon,\nu}|)^2 dx = 0$$

As the second term is positive we get the following inequality:

$$\frac{1}{2}\frac{d}{dt}\|z_1^{\epsilon,\nu} - z_2^{\epsilon,\nu}\|_2^2 \leq 0. \tag{3.17}$$

By integrating this last 'inequality from 0 to t, $t \in [0,T)$, we obtain the following inequality

$$\|z_1^{\epsilon,\nu}(t) - z_2^{\epsilon,\nu}(t)\|_2^2 \leq 0. \tag{3.18}$$

Which shows that $z_1^{\epsilon,\nu}(t) = z_2^{\epsilon,\nu}(t)$, $\forall t \in [0,T)$, which proves the uniqueness of the solution for the equation(3.15).
The existence of $z^{\epsilon,\nu}$ solution of the equation(3.15)is a consequence of the work of Ladyzenskaja, Solonnikov , Ural'Ceva [14] and Lions [16].

Proposition 3.3.3. *Under the same assumptions as in the theorem 3.3.1, for everything $\nu > 0$,the solution suite $z^{\epsilon,\nu}$ of (3.15) satisfies the following inequalities:*

$$\|z^{\epsilon,\nu}\|_{L^\infty((0,T),L^2(\mathbb{T}^2))} \leq \sqrt{\|z_0\|_2^2 + b\gamma T}, \tag{3.19}$$

$$\|\nabla z^{\epsilon,\nu}\|_{L^\infty((0,T),L^2(\mathbb{T}^2))} \leq \frac{\epsilon^j}{aG_{thr}}\Big(\|z_0\|_2^2 + 2b\gamma\Big), j = 0,1 \tag{3.20}$$

$$\Big\|\frac{\partial z^{\epsilon,\nu}}{\partial t}\Big\|_{L^2([0,T),L^2(\mathbb{T}^2))} \leq \epsilon^j \frac{b\gamma}{G_{thr}}, j = 0,1. \tag{3.21}$$

Proof By multiplying the equation (3.15) by $z^{\epsilon,\nu}$ and by integrating in $\mathbb{T}^2$, we get

$$\frac{1}{2}\frac{d\|z^{\epsilon,\nu}\|_2^2}{dt} + \int_{\mathbb{T}^2}\frac{a}{\epsilon^j}((g^\epsilon + \nu)|\nabla z^{\epsilon,\nu}|^2 \leq \int_{\mathbb{T}^2}\frac{b}{\epsilon^i}|f^\epsilon||\nabla z^{\epsilon,\nu}|. \tag{3.22}$$

Integrating (3.22) from 0 to T, we have

$$\frac{1}{2}\|z^{\epsilon,\nu}(T)\|_2^2 + \int_0^T\int_{\mathbb{T}^2}\frac{a}{\epsilon^j}(g^\epsilon + \nu)|\nabla z^{\epsilon,\nu}|^2 dx dt \leq \frac{b\gamma}{\epsilon^i}\int_0^T(\int_{\mathbb{T}^2}|\nabla z^{\epsilon,\nu}|^2 dx)^{\frac{1}{2}} dt + \frac{1}{2}\|z_0\|_2^2. \tag{3.23}$$

According to (3.23), using (3.9), we have

$$G_{thr}\int_0^T\int_{\mathbb{T}^2}|\nabla z^{\epsilon,\nu}|^2dxdt \leq \frac{b\gamma\epsilon^{j-i}}{a}\int_0^T(\int_{\mathbb{T}^2}|\nabla z^{\epsilon,\nu}|^2dx)^{\frac{1}{2}}dt + \frac{\epsilon^j}{2a}\|z_0\|^2. \tag{3.24}$$

So there is a constant γ depending only on $\|z_0\|^2,\ G_{thr},\ a\ b$ such as

$$\|\nabla z^{\epsilon,\nu}\|_{L^2([0,T),L^2(\mathbb{T}^2))} \leq \gamma. \tag{3.25}$$

Using the fact that $(g^\epsilon + \nu) > 0$ in (3.22) we get the following inequality

$$\frac{1}{2}\frac{d\|z^{\epsilon,\nu}\|_2^2}{dt} \leq \int_{\mathbb{T}^2}\frac{b}{\epsilon^i}|f^\epsilon||\nabla z^{\epsilon,\nu}|dx.$$

Using the hypothesis (3.12) and the inequality (3.25) we have

$$\frac{d\|z^{\epsilon,\nu}\|_2^2}{dt} \leq b\gamma. \tag{3.26}$$

By integrating this inequality from 0 to $t \in [0,T)$ we obtain the following inequality

$$\|z^{\epsilon,\nu}\|_2 \leq \sqrt{\|z_0\|_2^2 + b\gamma T}, \tag{3.27}$$

so what $\sup_{t\in[0,T)}\|z^{\epsilon,\nu}\|_2 \leq \sqrt{\|z_0\|_2^2 + b\gamma T}$, hence the inequality (3.19).
From (3.22) we have the following inequality

$$\frac{1}{2}\int_{\{t\in[0,T):\ |\mathbf{u}|\geq U_{thr}\}}\frac{d\|z^{\epsilon,\nu}\|_2^2}{dt}dt + \int_{\{t\in[0,T):\ |\mathbf{u}|\geq U_{thr}\}}\int_{\mathbb{T}^2}\frac{a}{\epsilon^j}((g^\epsilon + \nu)|\nabla z^{\epsilon,\nu}|^2dxdt \\ \leq \int_0^T\int_{\mathbb{T}^2}\frac{b}{\epsilon^i}|\nabla\cdot f^\epsilon||z^{\epsilon,\nu}|dxdt. \tag{3.28}$$

According to this last inequality, using (3.9), we have

$$G_{thr}\frac{a}{\epsilon^j}\int_{t\in[0,T):\ |\mathbf{u}|\geq U_{thr}}\int_{\mathbb{T}^2}|\nabla z^{\epsilon,\nu}|^2dxdt \\ \leq \int_0^T\int_{\mathbb{T}^2}\frac{b}{\epsilon^i}|\nabla\cdot(f^\epsilon)||z^{\epsilon,\nu}|dxdt + 2\|z_0\|_2 + b\gamma T. \tag{3.29}$$

it exists $t_0 \in \{t \in [0,T):\ |\mathbf{u}(t)| < U_{thr}\}$ such as

$$G_{thr}\int_{\mathbb{T}^2}|\nabla z^{\epsilon,\nu}(t_0)|^2dx \leq \frac{\epsilon^j}{a}\Big(2b\gamma + 2\|z_0\|_2^2\Big) \tag{3.30}$$

Which give

$$\|\nabla z^{\epsilon,\nu}(t_0,\cdot)\|_2^2 \le \frac{\epsilon^j}{aG_{thr}}\Big(2b\gamma + 2\|z_0\|_2^2\Big), j=0,1. \tag{3.31}$$

By multiplying (3.15) by $\dfrac{\partial z^{\epsilon,\nu}}{\partial t}$ and by integrating in $\mathbb{T}^2$, we have the inequality

$$\int_{\mathbb{T}^2}\Big|\frac{\partial z^{\epsilon,\nu}}{\partial t}\Big|^2 dx - \frac{a}{\epsilon^j}\int_{\mathbb{T}^2}\nabla\cdot\Big((g^\epsilon+\nu)\nabla z^{\epsilon,\nu}\Big)\frac{\partial z^{\epsilon,\nu}}{\partial t}dx = \frac{b}{\epsilon^i}\int_{\mathbb{T}^2}(\nabla\cdot f^\epsilon)\frac{\partial z^{\epsilon,\nu}}{\partial t}dx. \tag{3.32}$$

$$\int_{\mathbb{T}^2}\Big|\frac{\partial z^{\epsilon,\nu}}{\partial t}\Big|^2 dx + \frac{a}{\epsilon^j}\int_{\mathbb{T}^2}(g^\epsilon+\nu)\nabla z^{\epsilon,\nu}\nabla(\frac{dz^{\epsilon,\nu}}{dt})dx = \frac{b}{\epsilon^i}\int_{\mathbb{T}^2}(\nabla\cdot f^\epsilon)\frac{\partial z^{\epsilon,\nu}}{\partial t}dx \tag{3.33}$$

from where

$$\int_{\mathbb{T}^2}\Big|\frac{\partial z^{\epsilon,\nu}}{\partial t}\Big|^2 dx + \frac{a}{2\epsilon^j}\int_{\mathbb{T}^2}(g^\epsilon+\nu)|\nabla\frac{\partial z^{\epsilon,\nu}}{\partial t}|^2 dx \le \frac{b}{\epsilon^i}\int_{\mathbb{T}^2}|f^\epsilon|\Big|\nabla\frac{\partial z^{\epsilon,\nu}}{\partial t}\Big|dx, \tag{3.34}$$

The first term of the previous inequality being positive, we have

$$\frac{a}{\epsilon^j}\Big\|\sqrt{g^\epsilon+\nu}|\nabla\frac{\partial z^{\epsilon,\nu}}{\partial t}|\Big\|_2^2 \le b\gamma\Big\|\nabla\frac{\partial z^{\epsilon,\nu}}{\partial t}\Big\|_2. \tag{3.35}$$

By integrating in $\{t\in[0,T),\ |\mathbf{u}|<U_{thr}\}$, we get

$$\frac{a}{\epsilon^j}G_{thr}\int_{t\in[0,T),\ |\mathbf{u}|<Uthr}\Big\|\nabla\frac{\partial z^{\epsilon,\nu}}{\partial t}\Big\|_2^2 \le b\gamma\int_{t\in[0,T),\ |\mathbf{u}|<Uthr}\Big\|\nabla\frac{\partial z^{\epsilon,\nu}}{\partial t}\Big\|_2. \tag{3.36}$$

According to this last inequality, there are $t_0\in\{t\in[0,T),\ |\mathbf{u}|<Uthr\}$such as

$$\Big\|\nabla\frac{\partial z^{\epsilon,\nu}}{\partial t}(t_0)\Big\|_2 \le \epsilon^j\frac{b\gamma}{aG_{thr}}. \tag{3.37}$$

Using the Fourier series of $\frac{\partial z^{\epsilon,\nu}}{\partial t}$ we show that

$$\Big\|\frac{\partial z^{\epsilon,\nu}}{\partial t}(t_0)\Big\|_2 \le \epsilon^j\frac{b\gamma}{aG_{thr}}. \tag{3.38}$$

and taking the sup for $t\in[0,T]$, we have

$$\sup_{t\in[0,T)}\Big\|\frac{\partial z^{\epsilon,\nu}}{\partial t}(t_0)\Big\|_2 \le \epsilon^j\frac{b\gamma}{aG_{thr}}. \tag{3.39}$$

Which gives inequality (3.21).
Proof of the theorem 3.3.1 The existence of z^ϵ, solution of the equation (3.10), results from the proposition (3.3.3). Since the estimates (3.19),(3.20),(3.21) do not depend ν, when $\nu\to 0$, we get z^ϵ with the same properties.

3.4 Study of the existence and the uniqueness of solution in the general case

In this paragraph, we consider the general case, that is to say that we no longer assume (3.12) for $i = 1$. From (3.9) we prove that $\nabla \cdot f^\epsilon$ is bounded by a constant γ.
Since we want to study the asymptotic behavior of z^ϵ when ϵ tends to 0, we need estimates of z^ϵ which does not depend on ϵ or which are bounded when ϵ tends to zero. To find estimates of z^ϵ which do not depend on ϵ, we consider the periodic case, see [12] and [10]. As the wind frequency is considered to be periodic in the desert, see [12], [10] and [25], we can assume that

$$\left\{ \begin{array}{c} \mathbf{u}^\epsilon(t,x) = \mathcal{U}(t, \frac{t}{\epsilon}, x) \\ \text{et } \theta \longmapsto \mathcal{U}(t,\theta,x) \text{ is a periodic function of period 1.} \end{array} \right. \tag{3.40}$$

So $\theta \longmapsto g_a(|\mathbf{u}(t, \frac{t}{\epsilon}, x)|),\ g_c(|\mathbf{u}(t, \frac{t}{\epsilon}, x)|)\frac{\mathbf{u}(t,\frac{t}{\epsilon},x)}{|\mathbf{u}(t,\frac{t}{\epsilon},x)|}$ with g_a and g_c verifying the hypothesis (3.9), are also periodicals of period 1. Let

$$g^\epsilon(t,x) = g_\epsilon(t, \frac{t}{\epsilon}, x) \text{ et } f^\epsilon(t,x) = f_\epsilon(t, \frac{t}{\epsilon}, x). \tag{3.41}$$

with g^ϵ et f^ϵ are defined in(3.11)

Theorem 3.4.1. *Let $\epsilon > 0$, $a > 0$, b and c be reals. Under the hypotheses (3.9), (3.11) and (3.41), and $z_0 \in H^1(\mathbb{T}^2)$, for all $T > 0$, there is a unique solution $z^\epsilon \in L^2((0,T], L^2(\mathbb{T}^2))$ of (3.10) satisfactory*

$$\int_{\mathbb{T}^2} \frac{\partial z^\epsilon}{\partial t} dx = 0, \tag{3.42}$$

and

$$\|z^\epsilon\|_{L^2((0,T)\times\mathbb{T}^2)} \leq \gamma. \tag{3.43}$$

with γ a constant independent of ϵ.

The existence of z^ϵ over a time interval as a function of ϵ is a direct consequence of the adaptations of the results of Ladyzhenskaja, Solonnikov and Ural 'Ceva [14] or Lions [16] . Our objective is to prove that z^ϵ solution of (3.10) is bounded independently of ϵ. For this, we will introduce the following regularized equation: for all $\nu > 0$, find $Z^\nu = Z^\nu(t, \theta, x)$, periodic of period 1 in θ solution of the following equation:

$$\frac{\partial Z^\nu}{\partial \theta} - \frac{a}{\epsilon^j}\nabla \cdot \Big((g_\epsilon + \nu)\nabla Z^\nu\Big) = \nabla \cdot f_\epsilon \ [0,T) \times \mathbb{T}^2,\ j = 0,1. \tag{3.44}$$

Under the hypotheses (3.9) and (3.41), the functions f_ϵ and g_ϵ verify the following hypothesis:

$$
\begin{cases}
\theta \mapsto (g_\epsilon, f_\epsilon) \text{ is a periodic function of periodic 1,} \\
x \mapsto (g_\epsilon, f_\epsilon) \text{ is set to } \mathbb{T}^2 \\
|g_\epsilon| \leq \gamma,\ |f_\epsilon| \leq \gamma,\ \left|\frac{\partial f_\epsilon}{\partial t}\right| \leq \epsilon^2 d,\ \left|\frac{\partial g_\epsilon}{\partial t}\right| \leq \epsilon d, \\
\left|\frac{\partial f_\epsilon}{\partial \theta}\right| \leq \epsilon d,\ \left|\frac{\partial \tilde{g}_\epsilon}{\partial \theta}\right| \leq \epsilon d \\
\left|\nabla g_\epsilon\right| \leq \epsilon d,\ \left|\nabla \tilde{\cdot} f_\epsilon\right| \leq \epsilon d, \left|\frac{\partial \nabla \cdot f_\epsilon}{\partial t}\right| \leq \epsilon^2 d,\ \ |g_\epsilon| \leq d f_\epsilon.
\end{cases}
\tag{3.45}
$$

$$
\begin{cases}
\exists\ \tilde{G}_{thr} > 0,\ \exists \theta_\alpha,\ \theta_\omega \in [0, 1] \\
suchas\, \forall \theta \in [\theta_\alpha, \theta_\omega] \Longrightarrow \tilde{G}_{thr} \leq g_\epsilon(t, \theta, x)
\end{cases}
\tag{3.46}
$$

Theorem 3.4.2. *Under the assumptions (3.9),(3.40),(3.41), (3.45), (3.46), for everyone $\epsilon > 0$ et $\nu > 0$, there is a unique solution $Z^\nu \in L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))$ de (3.44).*
In addition, this solution checks the following inequalities

$$
\int_{\theta_\alpha}^{\theta_\omega} \int_{\mathbb{T}^2} |\nabla Z^\nu|^2 dx\, d\theta \leq \epsilon^i \frac{\gamma}{G_{thr}},\ i = 0,\ 1,
\tag{3.47}
$$

$$
\|\nabla Z^\nu(\theta_0, \cdot)\|_2 \leq \frac{\gamma \epsilon^i}{\sqrt{\tilde{G}_{thr}}},\ i = 0, 1,
\tag{3.48}
$$

$$
\|Z^\nu(\theta_0, \cdot)\|_{L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))} \leq \frac{\gamma \epsilon^i}{\sqrt{\tilde{G}_{thr}}} + 2\gamma,
\tag{3.49}
$$

$$
\|\frac{\partial Z^\nu}{\partial t}(\theta_0, \cdot)\|_{L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))} \leq \frac{\gamma f(\epsilon)}{\sqrt{\tilde{G}_{thr}}},
\tag{3.50}
$$

with $\lim_{\epsilon \longrightarrow 0} f(\epsilon) = \lambda > 0$

Proof of the theorem By multiplying (3.44) Z^ν and by integrating on $\mathbb{T}^2$ and on $[0, 1]$, we obtain

$$
\int_0^1 \int_{\mathbb{T}^2} \frac{\partial Z^\nu}{\partial \theta} Z^\nu dx\, d\theta + \frac{a}{\epsilon^i} \int_0^1 \int_{\mathbb{T}^2} (g_\epsilon + \nu) |\nabla Z^\nu|^2 dx d\theta \leq \int_0^1 \int_{\mathbb{T}^2} |f_\epsilon| |\nabla Z^\nu| dx\, d\theta.
\tag{3.51}
$$

According to this inequality, we have

$$
\frac{a}{\epsilon^i} \int_0^1 \int_{\mathbb{T}^2} g_\epsilon |\nabla Z^\nu|^2 dx d\theta \leq \int_0^1 \int_{\mathbb{T}^2} |f_\epsilon| |\nabla Z^\nu| dx\, d\theta.
\tag{3.52}
$$

Using the hypothesis (3.46) we get

$$
\|\sqrt{\tilde{G}_{thr}} |\nabla Z^\nu| \|_{L^2(\mathbb{R}, L^2(\mathbb{T}^2))} \leq \frac{\epsilon^i \gamma}{a}.
\tag{3.53}
$$

By integrating (3.53) with respect to θ of θ_α at θ_ω, we have

$$\int_{\theta_\alpha}^{\theta_\omega}\int_{\mathbb{T}^2}|\nabla Z^\nu|^2 dx\,d\theta \le \epsilon^i\frac{\gamma}{a\sqrt{\tilde{G}_{thr}}},\ i=0,\,1. \tag{3.54}$$

According to the inequality (3.54), there is $\theta_0\ in[\theta_\alpha,\theta_\omega]$ such that

$$\|\nabla Z^\nu(\theta_0,\cdot)\|_2^2 \le \epsilon^i\frac{\gamma}{a\sqrt{\tilde{G}_{thr}}},\ i=0,\,1. \tag{3.55}$$

Using the Fourier series of Z^ν, we prove that

$$\|Z^\nu(\theta_0,\cdot)\|_2^2 \le \|\nabla Z^\nu(\theta_0,\cdot)\|_2^2 \le \epsilon^i\frac{\gamma}{a\sqrt{\tilde{G}_{thr}}},\ i=0,\,1. \tag{3.56}$$

By multiplying (3.44) by Z^ν, and by integrating into $\mathbb{T}^2$, we get

$$\frac{1}{2}\frac{d\|Z^\epsilon(\theta,\cdot)\|_2^2}{d\theta}+\frac{a}{\epsilon^i}\int_{\{x\in\mathbb{T}^2,\ g_a(|\mathcal{U}(t,\theta,x)|)=0\}}\nu|\nabla Z^\nu|^2dx+\frac{a}{\epsilon^i}\int_{\{x\in\mathbb{T}^2,\ g_a(|\mathcal{U}(t,\theta,x)|)\neq 0\}}(g_\epsilon+\nu)|\nabla Z^\nu|^2dx$$
$$\le \int_{\{x\in\mathbb{T}^2,\ g_a(|\mathcal{U}(t,\theta,x)|)\neq 0\}}|f^\epsilon\nabla\cdot Z^\nu|dx. \tag{3.57}$$

But we have

$$\int_{\{x\in\mathbb{T}^2,\ g_a(|\mathcal{U}(t,\theta,x)|)\neq 0\}}\left|f^\epsilon\nabla Z^\nu\right|dx \le \int_{\{x\in\mathbb{T}^2,\ g_a(|\mathcal{U}(t,\theta,x)|)\neq 0\}}\frac{g_\epsilon+\nu}{4}|\nabla Z^\nu|^2dx$$
$$+\int_{\{x\in\mathbb{T}^2,\ g_a(|\mathcal{U}(t,\theta,x)|)\neq 0\}}\frac{|f^\epsilon|^2}{g_\epsilon+\nu}dx. \tag{3.58}$$

Using (3.58) and passing the first term from the left member of (3.57) to the right member we get

$$\frac{d\|Z^\nu(\theta,\cdot)\|^2}{d\theta}\le 2\gamma. \tag{3.59}$$

By integrating this last inequality of θ_0 to another $\theta\in[\theta_0,\theta_0+1]$ and using (3.56) we get the next inequality

$$\|Z^\nu(\theta,\cdot)\|_2^2 \le \epsilon^i\frac{\gamma}{a\sqrt{\tilde{G}_{thr}}}+2\gamma, i=0,1 \tag{3.60}$$

and since Z^ν is periodic, we get the inequality(3.49).
$\frac{\partial Z^\nu}{\partial t}$ is solution of

$$\frac{\partial\left(\frac{\partial Z^\nu}{\partial t}\right)}{\partial\theta}-\frac{a}{\epsilon^j}\nabla\cdot\left((g_\epsilon+\nu)\nabla\frac{\partial Z^\nu}{\partial\theta}\right)=\frac{a}{\epsilon^j}\nabla\left(\frac{\partial g_\epsilon}{\partial t}\nabla Z^\nu\right)+\nabla\left(\frac{\partial f_\epsilon}{\partial t}\right), \tag{3.61}$$

Multiplying (3.61) by $\frac{\partial Z^\nu}{\partial t}$ and integrating on $\in \mathbb{T}^2$, we get

$$\frac{1}{2}\frac{\partial\Big\|\frac{\partial Z^\nu}{\partial t}\Big\|_2^2}{\partial\theta}+\frac{a}{\epsilon^j}\int_{\mathbb{T}^2}(g_\epsilon+\nu)\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|^2dx\leq\int_{\mathbb{T}^2}\Big|\frac{\partial f_\epsilon}{\partial t}\Big|\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|dx+\frac{a}{\epsilon^j}\int_{\mathbb{T}^2}\Big|\frac{\partial g_\epsilon}{\partial t}\Big||\nabla Z^\nu|\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|dx. \tag{3.62}$$

But the first term of the second member can be written using the hypothesis (3.45) and (3.9) it results

$$\begin{aligned}\int_{\mathbb{T}^2}\Big|\frac{\partial f_\epsilon}{\partial t}\Big|\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|dx&\leq\gamma\int_{\mathbb{T}^2}\sqrt{g_\epsilon}\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|dx\\&\leq\gamma\Big\|\sqrt{g_\epsilon}\nabla\frac{\partial Z^\nu}{\partial t}\Big\|_2.\end{aligned} \tag{3.63}$$

According to the hypothesis (3.45), we deduce

$$\begin{aligned}\int_{\mathbb{T}^2}\Big|\frac{\partial g_\epsilon}{\partial t}\Big||\nabla Z^\nu|\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|dx&\leq\Big\|\sqrt{\Big|\frac{\partial g_\epsilon}{\partial t}\Big|}|\nabla Z^\nu|\Big\|_2\Big\|\sqrt{\Big|\frac{\partial g_\epsilon}{\partial t}\Big|}\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|\Big\|_2\\&\leq\gamma^2\Big\|\sqrt{g_\epsilon}|\nabla Z^\nu|\Big\|_2\Big\|\sqrt{g_\epsilon}\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|\Big\|_2.\end{aligned} \tag{3.64}$$

From this inequality, according to (3.63) and (3.64) we obtain the following inequality

$$\frac{1}{2}\frac{\partial\Big\|\frac{\partial Z^\nu}{\partial t}\Big\|_2^2}{\partial\theta}+\frac{a}{\epsilon^j}\int_{\mathbb{T}^2}(g_\epsilon+\nu)\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|^2dx\leq\gamma\Big\|\sqrt{g_\epsilon}\nabla\frac{\partial Z^\nu}{\partial t}\Big\|_2+\frac{a}{\epsilon^j}\gamma^2\Big\|\sqrt{g_\epsilon}|\nabla Z^\nu|\Big\|_2\Big\|\sqrt{g_\epsilon}\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|\Big\|_2. \tag{3.65}$$

By integrating from 0 to 1, we have

$$\frac{a}{\epsilon^j}\int_0^1\int_{\mathbb{T}^2}g_\epsilon\Big|\nabla\frac{\partial Z\nu}{\partial t}\Big|^2dxd\theta\leq\int_0^1\gamma\Big\|\sqrt{g_\epsilon}\nabla\frac{\partial Z^\nu}{\partial t}\Big\|_2d\theta+\int_0^1\frac{a}{\epsilon^j}\gamma^2\Big\|\sqrt{g_\epsilon}|\nabla Z^\nu|\Big\|_2\Big\|\sqrt{g_\epsilon}\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|\Big\|_2d\theta. \tag{3.66}$$

By applying Holder's inequality to the second term, we obtain

$$\begin{aligned}&\frac{a}{\epsilon^j}\Big\|\sqrt{g_\epsilon}\nabla\frac{\partial Z^\nu}{\partial t}\Big\|^2_{L^2(\mathbb{R},L^2(\mathbb{T}^2))}\leq\\&\gamma\Big\|\sqrt{g_\epsilon}\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|\Big\|_2+\frac{a}{\epsilon^j}\gamma^2\Big\|\sqrt{g_\epsilon}|\nabla Z^\nu|\Big\|_{L^2(\mathbb{R},L^2(\mathbb{T}^2))}\Big\|\sqrt{g_\epsilon}\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|\Big\|_{L^2(\mathbb{R},L^2(\mathbb{T}^2))}.\end{aligned} \tag{3.67}$$

Which give

$$\frac{a}{\epsilon^j}\Big\|\sqrt{g_\epsilon}\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big|\Big\|_{L^2(\mathbb{R},L^2(\mathbb{T}^2))}\leq\gamma+\frac{a}{\epsilon^j}\gamma^2\Big\|\sqrt{g_\epsilon}|\nabla Z^\nu|\Big\|_{L^2(\mathbb{R},L^2(\mathbb{T}^2))}. \tag{3.68}$$

Hence according to the hypothesis (3.53) we have

$$\Big\|\sqrt{g_\epsilon}\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big\|_{L^2(\mathbb{R},L^2(\mathbb{T}^2))} \le \frac{\epsilon^j}{a}\gamma + \epsilon^j\gamma^3. \tag{3.69}$$

According to the hypothesis (3.45) we have

$$\Big\|\Big|\nabla\frac{\partial Z^\nu}{\partial t}\Big\|_{L^2(\mathbb{R},L^2(\mathbb{T}^2))} \le \gamma\frac{f(\epsilon)}{\sqrt{\tilde{G}_{thr}}}. \tag{3.70}$$

By integrating (ref rrr3) from θ_α to θ_ω, we get

$$\int_{\theta_\alpha}^{\theta_\omega}\Big\|\nabla\frac{\partial Z^\nu}{\partial t}\Big\|_{L^2(\mathbb{R},L^2(\mathbb{T}^2))}d\theta \le \gamma\frac{f(\epsilon)}{\sqrt{\tilde{G}_{thr}}}. \tag{3.71}$$

From (3.71), there is $\theta_0 \in [\theta_\alpha, \theta_\omega]$ such as

$$\Big\|\Big|\nabla\frac{\partial Z^\nu(\theta_0,.)}{\partial t}\Big\|_{L^2(\mathbb{R},L^2(\mathbb{T}^2))} \le \gamma\frac{f(\epsilon)}{\sqrt{\tilde{G}_{thr}}}. \tag{3.72}$$

Using the Fourier series development of Z^ν, we show that

$$\Big\|\frac{\partial Z^\nu(\theta_0,.)}{\partial t}\Big\|_{L^2(\mathbb{R},L^2(\mathbb{T}^2))} \le \Big\|\nabla\frac{\partial Z^\nu(\theta_0,.)}{\partial t}\Big\|_{L^2(\mathbb{R},L^2(\mathbb{T}^2))} \le \gamma\frac{f(\epsilon)}{\sqrt{\tilde{G}_{thr}}}, \tag{3.73}$$

Which give (3.50).

Theorem 3.4.3. *Under the assumptions (3.9), (3.40), (3.41), (3.45) and (3.46), there is only one solution* $Z = Z(t,\theta,x) \in L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2))$,

$$\frac{\partial Z}{\partial\theta} - \frac{a}{\epsilon^j}\nabla\cdot\Big(g_\epsilon\nabla Z\Big) = \nabla\cdot f_\epsilon\ [0,T)\times\mathbb{T}^2. \tag{3.74}$$

In addition, this solution checks the following inequalities:

$$\int_{\theta_\alpha}^{\theta_\omega}\int_{\mathbb{T}^2}|\nabla Z|^2 dx\, d\theta \le \epsilon^i\frac{\gamma}{G_{thr}},\ \ i = 0,\ 1, \tag{3.75}$$

$$\|\nabla Z(\theta_0,\cdot)\|_2 \le \frac{\gamma\epsilon^i}{\sqrt{\tilde{G}_{thr}}},\ \ i = 0, 1, \tag{3.76}$$

$$\|Z(\theta_0,\cdot)\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \frac{\gamma\epsilon^i}{\sqrt{\tilde{G}_{thr}}} + 2\gamma, \tag{3.77}$$

$$\|\frac{\partial Z}{\partial t}(\theta_0,\cdot)\|_{L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))} \le \frac{\gamma f(\epsilon)}{\sqrt{\tilde{G}_{thr}}}, \tag{3.78}$$

with $f(\epsilon) \to \lambda > 0$ *when* $\epsilon \to 0$.

Proof of the theorem The proof of this theorem flows from the theorem 3.4.2. Indeed, it suffices to make ν tend towards 0 and the inequalities flow from it.
Proof of the theorem 3.4.1 Let $Z^\epsilon(t,x) = Z(t,\frac{t}{\epsilon},x)$, therefore $z^\epsilon(t,x) - Z(t,\frac{t}{\epsilon},x)$ is solution of the equation

$$\begin{cases} \frac{\partial(z^\epsilon - Z^\epsilon)}{\partial t} - \frac{a}{\epsilon^j}\nabla\cdot\Big(g^\epsilon\nabla(z^\epsilon - Z^\epsilon)\Big) = 0 \\ z^\epsilon(0,x) - Z^\epsilon(0,x) = z_0(x) - Z(0,0,x). \end{cases} \tag{3.79}$$

By multiplying (3.79) by $(z^\epsilon - Z^\epsilon)$ and by integrating on $\mathbb{T}^2$, we have the following inequality

$$\|z^\epsilon(t,\cdot) - Z^\epsilon(t,\cdot)\|_2 \leq \gamma. \tag{3.80}$$

This last inequality shows that $z^\epsilon(t,x)$ is not very far from $Z^\epsilon(t,x)$and like $Z^\epsilon(t,x) = Z(t,\frac{t}{\epsilon},x)$ is bounded in $L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))$ we conclude thate z^ϵ is bounded in $L^\infty_\#(\mathbb{R},L^2(\mathbb{T}^2))$

Chapter 4

Study of the homogenization of the mathematical model obtained

The goal of homogenization theory is to establish the behavior of a system that is microscopically heterogeneous, in order to write some characteristics of the heterogeneous environment for example, its temperature or its electrical conductivity. This means that the heterogeneous material is replaced by a material homogeneous with good overall characteristics approximation of the initial material.

4.1 Reminder of some notions on convergence at two scales

Convergence on two scales is a branch of the theory of homogenization. The objective of two-scale convergence is to study the behavior when $\epsilon \longrightarrow 0$
of certain partial differential equations. These concepts have been well presented and detailed in [1] and [18].

Definition 4.1.1. *Let an open Ω de $\mathbb{R}^n$, $p \in]1, +\infty[$ et $Y = [0,1]^n$,we say that the sequence of functions $(u^\epsilon)_{\epsilon>0} \subset L^p(\Omega)$ converges on two scales to a function U in the space $L^p(\Omega, L^p_\#(Y))$, if for any function $\psi \in \mathcal{C}^\infty(\Omega, \boldsymbol{C}^\infty_\#(Y))$ we have*

$$\lim_{\epsilon \longrightarrow 0} \int_\Omega u^\epsilon(x)\psi(x, \frac{x}{\epsilon})dxdy = \int_Y \int_\Omega U(x,y)\psi(x,y)dxdy. \tag{4.1}$$

U is then called 2-scale limit of u^ϵ in $L^p(\Omega, L^p_\#(Y))$

Theorem 4.1.2. *Let Ω open of $\mathbb{R}^n$, Y be a block of the form $Y = [0,1]^n$ and $(u^\epsilon)_\epsilon$ a bounded sequence independently of ϵ in $L^p(\Omega)$. Then there is an extracted sequence still denoted $(u^\epsilon)_{\epsilon>0}$ of $(u^\epsilon)_\epsilon$ which converges on two scales to U. In addition $(u^\epsilon)_{\epsilon>0}$ converges weakly $-*$ in $L^p(\Omega)$ towards the function u defined by $x \longmapsto u(x) = \int_Y U(x,y)dy$.*

Definition 4.1.3. *Let X be a Banach space, $q \in [1, +\infty[$ and X' the dual X. $\langle ., . \rangle$ denotes the duality hook between X' and X et q^ϵ the conjugate exponent of q such that $\frac{1}{q} + \frac{1}{q^\epsilon} = 1$.*
Let $u_\epsilon \in L^{q'}(0, T; X')$ et $U^0 = U^0(t, \theta) \in L^{q'}((0, T) \times (0, \tau); X')$.
We say that
$u_\epsilon \longmapsto U^0$ at two scales when $\epsilon \longmapsto 0$
if for any function $\psi \in L^q(0, T; \boldsymbol{C}_\#(\mathbb{R}, X))$ on a

$$\lim_{\epsilon \longrightarrow 0} \int_0^T \langle u_\epsilon(t), \psi(t, \frac{t}{\epsilon}) \rangle dt = \frac{1}{\tau} \int_0^T \int_0^\tau \langle U^0(x, y), \psi(t, \theta) \rangle d\theta dt. \tag{4.2}$$

4.2 Homogenization results

Let us consider (3.10) with coefficients which are given by (3.11). We are interested in the behavior of z^ϵ when $\epsilon \to 0$ and this problem called problem of homogenization associated with (3.10). According to the theorem (3.3.1) we have shown that there exists a unique solution of (3.10) which is bounded independently of ϵ in $L^\infty(\mathbb{R}, L^2(\mathbb{T}^2))$.
It's obvious that,

$$g^\epsilon(t, x) \text{ converges on two scales towards } \tilde{g}(t, \theta, x) \in L^\infty([0, T), L^\infty(\mathbb{R}, L^2(\mathbb{T}^2))) \tag{4.3}$$

and

$$f^\epsilon(t, x) \text{ converges on two scales towards } \tilde{f}(t, \theta, x) \in L^\infty([0, T), L^\infty(\mathbb{R}, L^2(\mathbb{T}^2))) \tag{4.4}$$

where $\tilde{g}$ et $\tilde{f}$ are given by

$$\tilde{g}(t, \theta, x) = a g_a(|\mathcal{U}(t, \theta, x)|) \text{ and } \tilde{f}(t, \theta, x) = c g_c(|\mathcal{U}(t, \theta, x)|) \frac{\mathcal{U}(t, \theta, x)}{|\mathcal{U}(t, \theta, x)|}. \tag{4.5}$$

Theorem 4.2.1. *Under the assumptions (3.40), (3.41) et (3.45)-(3.46) et (4.3et 4.4) for all $T > 0$ not depending on ϵ, the sequence (z^ϵ) solution of (3.10), with coefficients given by (3.11) , converges on two scales to $U \in L^\infty([0, T), L^\infty_\#(\mathbb{R}, L^2(\mathbb{R}, L^(\mathbb{T}^2)))$ solution of*

$$\frac{\partial U}{\partial \theta} - \nabla \cdot \left(\tilde{g} \nabla U\right) = \nabla \cdot \tilde{f} \tag{4.6}$$

in the short and medium term dynamics,

$$-\nabla \cdot \left(\tilde{g} \nabla U\right) = 0 \tag{4.7}$$

in the long-term dynamics of the dunes
with $\tilde{g}$ et $\tilde{f}$ given by (4.5).

Proof of theorem Consider the test function $\psi^\epsilon(t,x) = \psi(t,\frac{t}{\epsilon},x)$ for all $\psi(t,\theta,x)$ regular function with compact support in $[0,T)\times\mathbb{T}^2$ and periodic with respect to the variable θ of period 1.
Multiplying (3.10) by ψ^ϵ and integrating into $[0,T)\times\mathbb{T}^2$, we get

$$\int_{\mathbb{T}^2}\int_0^T \frac{\partial z^\epsilon}{\partial t}\psi^\epsilon dtdx - \frac{a}{\epsilon^j}\int_{\mathbb{T}^2}\int_0^T \nabla\cdot\Big(g^\epsilon(t,x)\nabla z^\epsilon\Big)\psi^\epsilon dtdx$$
$$= \frac{b}{\epsilon^i}\int_{\mathbb{T}^2}\int_0^T \nabla\cdot f^\epsilon(t,x)\psi^\epsilon dtdx \tag{4.8}$$

By integrating by parts in the first integral on $[0,T)$ and using Green's formula on $\mathbb{T}^2$ in the second integral, we have

$$-\int_{\mathbb{T}^2} z_0^\epsilon(x)\psi(0,0,x)dx + \int_{\mathbb{T}^2}\int_0^T \frac{\partial\psi^\epsilon}{\partial t}z^\epsilon dtdx + \frac{a}{\epsilon^j}\int_{\mathbb{T}^2}\int_0^T g^\epsilon(t,x)\nabla z^\epsilon\nabla\psi^\epsilon dtdx$$
$$= \frac{b}{\epsilon^i}\int_{\mathbb{T}^2}\int_0^T \nabla\cdot f^\epsilon(t,x)\psi^\epsilon dtdx \tag{4.9}$$

Again using Green's formula in the third integral we get

$$-\int_{\mathbb{T}^2} z_0^\epsilon(x)\psi(0,0,x)dx + \int_{\mathbb{T}^2}\int_0^T \frac{\partial\psi^\epsilon}{\partial t}z^\epsilon dtdx - \frac{a}{\epsilon^j}\int_{\mathbb{T}^2}\int_0^T z^\epsilon\nabla\cdot\big(g^\epsilon(t,x)\nabla\psi^\epsilon\big)dtdx$$
$$= \frac{b}{\epsilon^i}\int_{\mathbb{T}^2}\int_0^T \nabla\cdot f^\epsilon(t,x)\psi^\epsilon dtdx \tag{4.10}$$

as
$$\frac{\partial\psi^\epsilon}{\partial t} = \Big(\frac{\partial\psi}{\partial t}\Big)^\epsilon + \frac{1}{\epsilon}\Big(\frac{\partial\psi}{\partial\theta}\Big)^\epsilon$$

then we have

$$-\int_{\mathbb{T}^2}\int_0^T \Big(\Big(\frac{\partial\psi}{\partial t}\Big)^\epsilon - \frac{1}{\epsilon}\Big(\frac{\partial\psi}{\partial\theta}\Big)^\epsilon + \frac{a}{\epsilon^j}\nabla\cdot\big(g^\epsilon(t,x)\nabla\psi^\epsilon\big)\Big)z^\epsilon dtdx$$
$$= \frac{b}{\epsilon^i}\int_{\mathbb{T}^2}\int_0^T \nabla\cdot f^\epsilon(t,x)\psi^\epsilon dtdx + \int_{\mathbb{T}^2} z_0^\epsilon(x)\psi(0,0,x)dx \tag{4.11}$$

Using the two-scale convergence due to Nguetseng [18] and Allaire [1], if a sequence f^ϵ is bounded in $L^\infty(0,T,L^2(\mathbb{T}^2))$, then there is a function $U(t,\theta,x)$ periodic in θ of period 1, such as for any function $\psi(t,\theta,x)$ regular with compact support compared to (t,x) and periodic in θ of period 1, we have

$$\int_{\mathbb{T}^2}\int_0^T f^\epsilon\psi^\epsilon dtdx \longrightarrow \int_{\mathbb{T}^2}\int_0^T\int_0^1 U\psi dtdxd\theta \tag{4.12}$$

In the case of short and medium term dune dynamics, the exponent of the parameter ϵ in the equation is $i = j = 1$ therefore by multiplying (4.11) by ϵ we have

$$-\int_{\mathbb{T}^2}\int_0^T \big(\epsilon(\frac{\partial\psi}{\partial t})^\epsilon - (\frac{\partial\psi}{\partial\theta})^\epsilon + \nabla\cdot\big(g^\epsilon(t,x)\nabla\psi^\epsilon)\big)z^\epsilon dtdx$$

$$= \int_{\mathbb{T}^2}\int_0^T \nabla\cdot f^\epsilon(t,x)\psi^\epsilon dtdx + \epsilon\int_{\mathbb{T}^2} z_0^\epsilon(x)\psi(0,0,x)dx \tag{4.13}$$

Since f^ϵ and g^ϵ are bounded (see(3.45)) and ψ^ϵ is regular function , $g^\epsilon(t,x)\nabla\psi^\epsilon$ and $\nabla\psi^\epsilon$ can be considered as test functions. Using(4.3) and (4.4)and making ϵ tend towards 0 in (4.13) we get

$$\frac{\partial U}{\partial\theta} - \nabla\cdot\big(\tilde{g}\nabla U\big) = \nabla\tilde{f}. \tag{4.14}$$

Multiplying (4.11) by ϵ^2 in the case where $i = 1$ et $j = 2$ we have

$$-\int_{\mathbb{T}^2}\int_0^T \big(\epsilon^2(\frac{\partial\psi}{\partial t})^\epsilon - \epsilon(\frac{\partial\psi}{\partial\theta})^\epsilon + \nabla\cdot\big(g^\epsilon(t,x)\nabla\psi^\epsilon)\big)z^\epsilon dtdx$$

$$= \epsilon\int_{\mathbb{T}^2}\int_0^T \nabla\cdot f^\epsilon(t,x)\psi^\epsilon dtdx + \epsilon^2\int_{\mathbb{T}^2} z_0^\epsilon(x)\psi(0,0,x)dx \tag{4.15}$$

By making ϵ tend towards 0 in (4.15)and using the results of convergence at two scales we have

$$-\nabla\cdot\big(\tilde{g}(t,x)\nabla U\big) = 0 \tag{4.16}$$

.

Conclusion

In this book we have studied the asymptotic analysis of sand transport model in the desert. First, we explained the framework.
Considering the short, medium and long term dynamics of dunes and sandbanks in the desert, we establish the following model:

$$\frac{\partial z^\varepsilon}{\partial t} - \frac{a}{\varepsilon^j}\nabla\cdot(g_a(|u^\varepsilon|)\nabla z^\epsilon) = \frac{b}{\varepsilon^i}\nabla\cdot(g_c(|u^\varepsilon|\ \frac{u^\epsilon}{|u^\epsilon|}) \tag{4.17}$$

With $i = 0, 1$, $j = 0, 1, 2$ and the initial condition associated with (4.17) is of the form $z^\varepsilon(x, 0) = z_0(x)$ with $x \in \mathbb{T}^2$.
The functions g_a et g_c are increasing, regular defined functions of: $\mathbb{R}_+ \longmapsto \mathbb{R}_+$
With a and b are positive real constants.

$z^\varepsilon = z^\varepsilon(t,x)$ defined from $[0.T] \times \mathbb{T}^2 \to \mathbb{R}$. The position $x = (x_1, x_2) \in \mathbb{T}^2$
and the function u: $[0.T] \times \mathbb{T}^2 \to \mathbb{R}_+$.
We show the existence and the uniqueness of the solution of the equation (4.17) under certain assumptions.
We also show that z^ϵ solution of the equation (4.17) converges on two scales to a profile $U \in L^\infty([0,T), L^\infty_\#(\mathbb{R}, L^2(\mathbb{R}, L^(\mathbb{T}^2)))$ when ϵ tends to 0 solution of

$$\frac{\partial U}{\partial \theta} - \nabla \cdot \big(\tilde{g} \nabla U\big) = \nabla \tilde{f} \tag{4.18}$$

In the case where $i = j = 1$
and

$$-\nabla \cdot \big(\tilde{g} \nabla U\big) = 0 \tag{4.19}$$

In the case where $i = 1$ and $j = 2$
and where the functions $\tilde{g}$ and $\tilde{f}$ are given by(4.5)..
This book has allowed us to build short, medium and long term sand transport models from the sand dunes in the desert.
The results open the door to several perspectives. We plan in our future work to build digital models capable of simulating the dynamics of dunes and sandbanks in the desert.

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Contents