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A Semi-Symmetric Non-Metric Connection on a Unified Structure Manifold

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Abstract

In this paper we have defined a new manifold $\overset{*}{M}_n$. It is interesting to note that $\overset{*}{M}_n$ is integrable with respect to semi-symmetric non-metric connexion $\overset{*}{\nabla}$ [3]. Further it has been shown that if a vector field V is contravariant almost analytic with respect to Riemannian connexion D then it is also contravariant almost analytic with respect to $\overset{*}{\nabla}$.

Key words: C^∞-manifold, semi-symmetric non-metric connexion, unified structure manifold, Nijenhuise tensor, contravariant almost analytic vector field.
Mathematics Subject classification No: 53

1. Introduction

Let M_n be an even dimensional differentiable manifold of differentiability class C^∞. If there exist in M_n a vector valued linear function Φ of differentiability class C^∞ and a Riemannian metric G such that for any vector field X

(1.1) $$\Phi^2(X) = a^2 X$$

(1.2) $$G(\Phi X, \Phi Y) = a^2 G(X, Y)$$

where

(1.3) $$\Phi(X) \underline{\underline{def}} \ \bar{X}$$, and a is a nonzero complex number.

Then M_n satisfying (1.1) and (1.2) is called an unified structure manifold [1].

Agreement 1.1: An unified structure manifold will be denoted by M_n.
In the sequel, arbitrary vector fields will be denoted by $X, Y, Z, ...$etc.

Remark 1.1: It may be noted that an unified structure manifold M_n will be an almost Norden manifold or an almost product Riemannian manifold according as $a = \pm i$ or $a = \pm 1$ respectively.

Let us consider in M_n, a tensor $`\Phi$ of the type (0, 2) defined by

(1.4) $$'\Phi(X,Y) \underline{\underline{def}} \ G(\overline{X},Y) = G(X,\overline{Y})$$

then it is easy to calculate that

(1.5) $$'\Phi(\overline{X},Y) = a^2 G(X,Y)$$

(1.6) $$'\Phi(\overline{X},\overline{Y}) = a^2 \ '\Phi(X,Y)$$

(1.7) $$'\Phi(X,Y) = '\Phi(Y,X)$$

Remark 1.2: From (1.6), we conclude that $`\Phi$ is pure or hybrid according as $a = \pm i$ or $a = \pm 1$ respectively.

Agreement 1.2: The C^∞ - manifold M_n satisfying

(1.8) $$(D_X \Phi)Y = 0$$

will be denoted by $\overset{*}{M}_n$

Remark 1.2: The C^∞ - manifold $\overset{*}{M}_n$ will be called a Norden Kahler manifold or an almost product and almost decomposable manifold according as $a = \pm i$ or $a = \pm 1$ respectively.

It is easy to see that (1.8) implies

(1.9) $$D_X \overline{Y} = \overline{D_X Y} \iff \overline{D_X \overline{Y}} = a^2 (D_X Y)$$

2. Semi-symmetric Non-metric Connexion

The relation between the semi-symmetric non-metric connexion $\overset{*}{\nabla}$ and the Riemannian connexion D is given by [3].

(2.1) $$\overset{*}{\nabla}_X Y = D_X Y + \xi(Y) X$$

where ξ is 1-form.

Using (2.1), the torsion tensor $\overset{*}{T}$ of $\overset{*}{M}_n$ with respect to the connexion $\overset{*}{\nabla}$ is given by

(2.2) $$\overset{*}{T}(X,Y) = \xi(Y) X - \xi(X) Y$$

where $$\overset{*}{T}(X,Y) \underset{\overline{\overline{}}}{def} \overset{*}{\nabla}_X Y - \overset{*}{\nabla}_Y X - [X,Y]$$

for arbitrary vector fields X and Y.

We define

(2.3) $$\xi(X) \underset{\overline{\overline{}}}{def} G(X,\eta)$$

where η is a vector field associated with ξ.

We know that

$$X\left(G(Y,Z)\right) = (\overset{*}{\nabla}_X G)(Y,Z) + G(\overset{*}{\nabla}_X Y, Z) + G(Y, \overset{*}{\nabla}_X Z)$$

$$= G(D_X Y, Z) + G(Y, D_X Z),$$

Using (2.1) in the above equation, we get

(2.4) $$\left(\overset{*}{\nabla}_X G\right)(Y,Z) = -\xi(Y) G(X,Z) - \xi(Z) G(X,Y)$$

In view of (2.2) and (2.4), the linear connexion defined by (2.1) will be called a semi-symmetric non-metric connexion. It may be noted that a linear connexion $\overset{*}{\nabla}$ satisfying (2.2) and (2.4) is defined by (2.1)

3. Some theorems on $\overset{*}{M}_n$ equipped with connexion $\overset{*}{\nabla}$

Theorem 3.1: In $\overset{*}{M}_n$, we have

(3.1a) $$\left(\overset{*}{\nabla}_{\overline{X}}\Phi\right)\left(\overline{Y}\right) = -a^2\left(\overset{*}{\nabla}_X\Phi\right)\left(Y\right)$$

(3.1b) $$\left(\overset{*}{\nabla}_X\Phi\right)\left(Y\right) = 0 \text{ iff } \xi\left(\overline{Y}\right)X = \xi\left(Y\right)\overline{X}$$

Proof: Replacing Y by $\overline{Y}$ in (2.1), we get

(3.2) $$\overset{*}{\nabla}_X\overline{Y} = D_X\overline{Y} + \xi\left(\overline{Y}\right)X\ ,$$

which implies

(3.3) $$\left(\overset{*}{\nabla}_X\Phi\right)\left(Y\right) = \overline{D_XY} + \xi\left(\overline{Y}\right)X - \overline{\overset{*}{\nabla}_X Y}$$

Operating both sides of (2.1) by Φ, we have

(3.4) $$\overline{\overset{*}{\nabla}_X Y} = \overline{D_XY} + \xi\left(Y\right)\overline{X}$$

Using (3.4) in (3.3), we get

(3.5) $$(\overset{*}{\nabla}_X\Phi)\left(Y\right) = \xi\left(\overline{Y}\right)X - \xi\left(Y\right)\overline{X}$$

Barring X and Y in (3.5) and using (1.1), we get

(3.6) $$(\overset{*}{\nabla}_{\bar{X}}\Phi)(\bar{Y}) = a^2\left[\xi(Y)\bar{X} - \xi(\bar{Y})X\right]$$

From (3.5) and (3.6), we get (3.1a).

(3.1b) is obvious from (3.5).

Theorem 3.1: In $\overset{*}{M}_n$, we have

(3.7) $$\overset{*}{d}\,{}'\Phi(X,Y,Z) = -2\left[\xi(X)G(\bar{Y},Z) + \xi(Y)G(\bar{Z},X) + \xi(Z)G(\bar{X},Y)\right]$$

where

(3.8) $$\overset{*}{d}\,{}'\Phi(X,Y,Z) \underset{=\!=}{def} \left[(\overset{*}{\nabla}_X{}'\Phi)(Y,Z) + (\overset{*}{\nabla}_Y{}'\Phi)(Z,X) + (\overset{*}{\nabla}_Z{}'\Phi)(X,Y)\right]$$

Proof: From (1.4), we have

(3.9) $${}'\Phi(Y,Z) = G(\bar{Y},Z)$$

Differentiating the above equation covariantly with respect to connexion $\overset{*}{\nabla}$ and using (1.4), (2.1) and (2.4), we get

(3.10) $$(\overset{*}{\nabla}_X{}'\Phi)(Y,Z) = -\left[\xi(Y)G(\bar{Z},X) + \xi(Z)G(\bar{X},Y)\right]$$

Similarly we have

(3.11) $$(\overset{*}{\nabla}_Y{}'\Phi)(Z,X) = -\left[\xi(Z)G(\bar{X},Y) + \xi(X)G(\bar{Y},Z)\right]$$

and

(3.12) $$(\overset{*}{\nabla}_Z{}'\Phi)(X,Y) = -\left[\xi(X)G(\bar{Y},Z) + \xi(Y)G(\bar{Z},X)\right]$$

Using (3.10), (3.11) and (3.12) in (3.8), we get (3.7).

4. Nijenhuis tensor equipped with connexion $\overset{*}{\nabla}$ in $\overset{*}{M}_n$

Theorem 4.1: In $\overset{*}{M}_n$, Nijenhuis tensor with respect to semi-symmetric non-metric connexion $\overset{*}{\nabla}$ vanishes i.e.

$$\overset{*}{N}\left(X,Y\right)=0 \tag{4.1}$$

Proof: The Nijenhuis tensor with respect to semi-symmetric non-metric connexion $\overset{*}{\nabla}$ is given by

$$\overset{*}{N}\left(X,Y\right)=(\overset{*}{\nabla}_{\overline{X}}\Phi)Y-(\overset{*}{\nabla}_{\overline{Y}}\Phi)X-\overline{\left(\overset{*}{\nabla}_X\Phi\right)Y}+\overline{\left(\overset{*}{\nabla}_Y\Phi\right)X} \tag{4.2}$$

From (3.5) and (1.1), we have

$$(\overset{*}{\nabla}_{\overline{X}}\Phi)Y=\xi\left(\overline{Y}\right)\overline{X}-a^2\xi\left(Y\right)X \tag{4.3}$$

Interchanging X and Y in the above equation, we get

$$(\overset{*}{\nabla}_{\overline{Y}}\Phi)X=\xi\left(\overline{X}\right)\overline{Y}-a^2\xi\left(X\right)Y \tag{4.4}$$

Operating Φ on both sides of (3.5) and using (1.1), we get

$$\overline{\left(\overset{*}{\nabla}_X\Phi\right)Y}=\xi\left(\overline{Y}\right)\overline{X}-a^2\xi\left(Y\right)X \tag{4.5}$$

Interchanging X and Y in the above equation, we have

$$\overline{\left(\overset{*}{\nabla}_Y\Phi\right)X}=\xi\left(\overline{X}\right)\overline{Y}-a^2\xi\left(X\right)Y \tag{4.6}$$

Using (4.3), (4.4), (4.5) and (4.6) in (4.2), we get (4.1).

Remark 4.1: Equation (4.1) implies that $\overset{*}{M}_n$ is integrable.

5. Contravariant almost analytic vector field in $\overset{*}{M}_n$

A vector field V is said to be contravariant almost analytic if the Lie-derivative of Φ with respect to V vanishes identically [2], [4], [5] i.e.

(5.1) $$(L_V \Phi) X = 0, \text{ for all } X .$$

Now (5.1) is equivalent to the equation

(5.2) $$[V, \overline{X}] = \overline{[V, X]}$$

In $\overset{*}{M}_n$ the above equation becomes

(5.3) $$D_{\overline{X}} V - \overline{D_X V} = 0$$

Theorem 5.1: If a vector field V is contravariant almost analytic in $\overset{*}{M}_n$ with respect to the connexion D, then it is also contravariant almost analytic in $\overset{*}{M}_n$ with respect to the semi-symmetric non-metric connexion $\overset{*}{\nabla}$.

Proof: From (2.1), we have

(5.4) $$\overset{*}{\nabla}_X V = D_X V + \xi(V) X$$

Replacing X by $\overline{X}$ in the above equation, we get

(5.5) $$\overset{*}{\nabla}_{\overline{X}} V = D_{\overline{X}} V + \xi(V) \overline{X}$$

Operating both the sides of (5.4), by Φ, we have

(5.6) $$\overline{\overset{*}{\nabla}_X V} = \overline{D_X V} + \xi(V)\,\overline{X}$$

Subtracting (5.6) from (5.5) and using (5.3), we get

$$\overset{*}{\nabla}_{\overline{X}} V - \overline{\overset{*}{\nabla}_X V} = 0$$

which proves the theorem.

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